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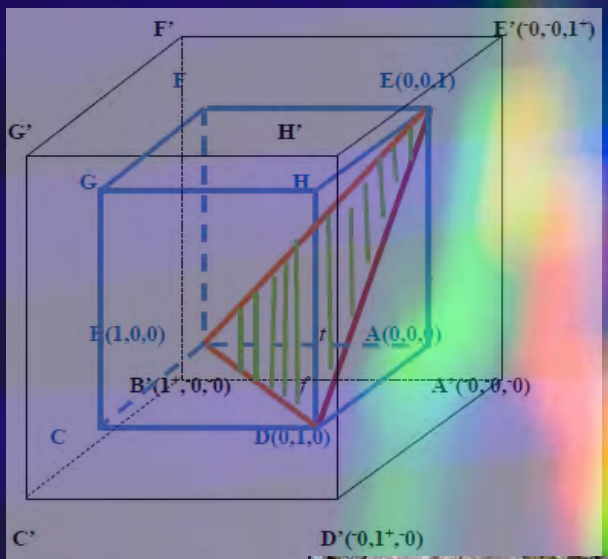

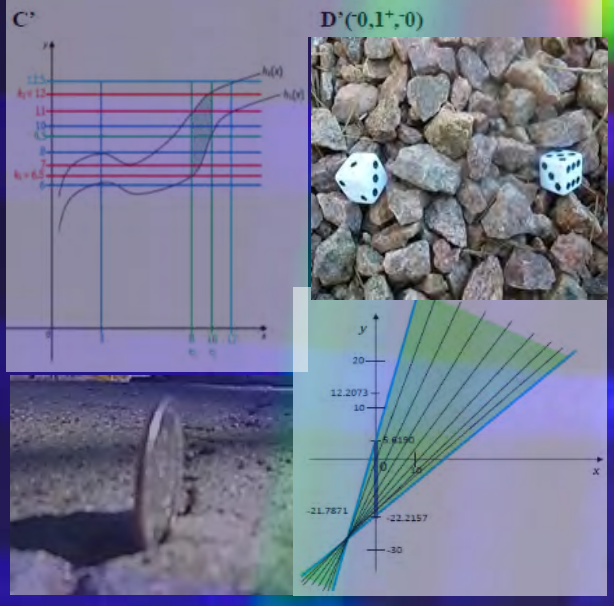

⟨A⟩ ⟨neutA⟩ ⟨antiA⟩

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The submitted papers should be professional, in good English, containing a brief review of a problem and obtained results.

Neutrosophy is a new branch of philosophy that studies the origin, nature, and scope of neutralities, as well as their interactions with different ideational spectra.

This theory considers every notion or idea <A> together with its opposite or negation <antiA> and with their spectrum of neutralities <neutA> in between them (i.e. notions or ideas supporting neither <A> nor <antiA>). The <neutA> and <antiA> ideas together are referred to as <nonA>.

Neutrosophy is a generalization of Hegel's dialectics (the last one is based on <A> and <antiA> only).

According to this theory every idea <A> tends to be neutralized and balanced by <antiA> and <nonA> ideas - as a state of equilibrium.

In a classical way <A>, <neutA>, <antiA> are disjoint two by two. But, since in many cases the borders between notions are vague, imprecise, Sorites, it is possible that <A>, <neutA>, <antiA> (and <nonA> of course) have common parts two by two, or even all three of them as well.

Neutrosophic Set and *Neutrosophic Logic* are generalizations of the fuzzy set and respectively fuzzy logic (especially of intuitionistic fuzzy set and respectively intuitionistic fuzzy logic). In neutrosophic logic a proposition has a degree of truth (*T*), a degree of indeterminacy (*I*), and a degree of falsity (*F*), where *T, I, F* are standard or non-standard subsets of $]^{-}0, 1^{+}[$.

Neutrosophic Probability is a generalization of the classical probability and imprecise probability.

Neutrosophic Statistics is a generalization of the classical statistics.

What distinguishes the neutrosophics from other fields is the <neutA>, which means neither <A> nor <antiA>.

<neutA>, which of course depends on <A>, can be indeterminacy, neutrality, tie game, unknown, contradiction, ignorance, imprecision, etc.

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University of New Mexico

Editors-in-Chief

Prof. Emeritus Florentin Smarandache, PhD, Postdoc, Mathematics, Physical and Natural Sciences Division, University of New Mexico, Gallup Campus, NM 87301, USA, Email: smarand@unm.edu.

Dr. Mohamed Abdel-Baset, Head of Department of Computer Science, Faculty of Computers and Informatics, Zagazig University, Egypt, Email: mohamedbasset@ieee.org.

Prof. Dr. Maikel Leyva-Vázquez, Instituto Superior Tecnológico Bolivariano de Tecnología, Ecuador: maikel.leyvav@ug.edu.ec.

Associate Editors

Dr. Said Broumi, Laboratory of Information Processing, Faculty of Science Ben M'Sik, University of Hassan II, Casablanca, Morocco, Email: s.broumi@flbenmsik.ma.

Assoc. Prof. Alok Dhital, Mathematics, Physical and Natural Sciences Division, University of New Mexico, Gallup Campus, NM 87301, USA, Email: adhital@unm.edu.

Dr. S. A. Edalatpanah, Department of Applied Mathematics, Ayandegan Institute of Higher Education, Tonekabon, Iran, Email: saedalatpanah@gmail.com.

Charles Ashbacker, Charles Ashbacher Technologies, Box 294, 118 Chaffee Drive, Hiawatha, IA 52233, United States, Email: cashbacher@prodigy.net.

Prof. Dr. Xiaohong Zhang, Department of Mathematics, Shaanxi University of Science &Technology, Xian 710021, China, Email: zhangxh@shmtu.edu.cn.

Prof. Dr. W. B. Vasantha Kandasamy, School of Computer Science and Engineering, VIT, Vellore 632014, India, Email: vasantha.wb@vit.ac.in.

Editors

Yanhui Guo, University of Illinois at Springfield, One University Plaza, Springfield, IL 62703, United States,Email: yguo56@uis.edu.

Giorgio Nordo, MIFT - Department of Mathematical and Computer Science, Physical Sciences and Earth Sciences, Messina University, Italy,Email: giorgio.nordo@unime.it.

Mohamed Elhoseny, American University in the Emirates,Dubai,UAE,
Email: mohamed.elhoseny@aue.ae.

Le Hoang Son, VNU Univ. of Science, Vietnam National Univ. Hanoi, Vietnam, Email: sonlh@vnu.edu.vn.

Huda E. Khalid, Head of Scientific Affairs and Cultural Relations Department, Nineveh Province, Telafer University, Iraq, Email: dr.huda-ismael@uotelafer.edu.iq.

A. A. Salama, Dean of the Higher Institute of Business and Computer Sciences, Arish, Egypt, Email: ahmed_salama_2000@sci.psu.edu.eg.

Young Bae Jun, Gyeongsang National University, South Korea, Email: skywine@gmail.com.

Yo-Ping Huang, Department of Computer Science and Information, Engineering National Taipei University, New Taipei City, Taiwan, Email: yphuang@ntut.edu.tw.

Tarek Zayed, Department of Building and Real Estate, The Hong Kong Polytechnic University, Hung Hom, 8 Kowloon, Hong Kong, China, Email: tarek.zayed@polyu.edu.hk.

Vakkas Ulucay, Kilis 7 Aralık University, Turkey, Email: vulucay27@gmail.com.

Peide Liu, Shandong University of Finance and Economics, China, Email: peide.liu@gmail.com.

Jun Ye, Ningbo University, School of Civil and Environmental Engineering, 818 Fenghua Road, Jiangbei District, Ningbo City, Zhejiang Province, People's Republic of China, Email: yejun1@nbu.edu.cn.

Memet Şahin, Department of Mathematics, Gaziantep University, Gaziantep 27310, Turkey, Email: mesahin@gantep.edu.tr.

Muhammad Aslam & Mohammed Alshumrani, King Abdulaziz Univ., Jeddah, Saudi Arabia, Emails magmuhammad@kau.edu.sa, maalshmrani @kau.edu.sa.

Mutaz Mohammad, Department of Mathematics, Zayed University, Abu Dhabi 144534, United Arab Emirates. Email: Mutaz.Mohammad@zu.ac.ae.

Abdullahi Mohamud Sharif, Department of Computer Science, University of Somalia, Makka Al-mukarrama Road, Mogadishu, Somalia, Email: abdullahi.shariif@uniso.edu.so.

.

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Katy D. Ahmad, Islamic University of Gaza, Palestine,Email: katyon765@gmail.com.
NoohBany Muhammad, American University of Kuwait,Kuwait,Email: noohmuhammad12@gmail .com.
Soheyb Milles, Laboratory of Pure and Applied Mathematics, University of Msila, Algeria, Email: soheyb.milles@univ-msila.dz.
Pattathal Vijayakumar Arun, College of Science and Technology,Phuentsholing,Bhutan,
Email: arunpv2601@gmail.com.
Endalkachew Teshome Ayele, Department of Mathematics, Arbaminch University, Arbaminch, Ethiopia,Email: endalkachewteshome83@yahoo. com.
A. Al-Kababji, College of Engineering, Qatar University,Doha,Qatar,
Email: ayman.alkababji@ieee.org.
Xindong Peng, School of Information Science and Engineering, Shaoguan University, Shaoguan 512005, China, Email: 952518336@qq.com.
Xiao-Zhi Gao, School of Computing, University of Eastern Finland, FI-70211 Kuopio, Finland, xiao-zhi.gao@uef.fi.
Madad Khan, Comsats Institute of Information Technology,Abbottabad,Pakistan,
Email: madadmath@yahoo.com.
G. Srinivasa Rao, Department of Statistics, The University of Dodoma, Dodoma, PO. Box: 259, Tanzania,Email: gaddesrao@gmail.com.
Ibrahim El-henawy, Faculty of Computers and Informatics,ZagazigUniversity,Egypt,
Email: henawy2000@yahoo.com.
Muhammad Saeed, Department of Mathematics, University of Management and Technology, Lahore, Pakistan,
Email: muhammad.saeed@umt.edu.pk.
A. A. A. Agboola, Federal University of Agriculture, Abeokuta, Nigeria,
Email: agboolaaaa@funaab.edu.ng.
Abduallah Gamal, Faculty of Computers and Informatics, Zagazig University, Egypt,
Email: abduallahgamal@zu.edu.eg.
Ebenezer Bonyah, Department of Mathematics Education, Akenten Appiah-Menka University of Skills Training and Entrepreneurial Development, Kumasi 00233, Ghana,
Email: ebbonya@gmail.com.
Roan Thi Ngan, Hanoi University of Natural Resources and Environment, Hanoi, Vietnam,
Email: rtngan@hunre.edu.vn.
Sol David Lopezdomínguez Rivas, Universidad Nacional de Cuyo, Argentina.
Email: sol.lopezdominguez@fce.uncu.edu.ar.
Maikel Yelandi Leyva Vázquez, Universidad Regional Autónoma de los Andes (UNIANDES), Avenida Jorge Villegas, Babahoyo, Los Ríos, Ecuador,
Email: ub.c.investigacion@uniandes.edu.ec.
Arlen Martín Rabelo, Exxis, Avda. Aviadores del Chaco N° 1669 c/ San Martin, Edif. Aymac I, 4to. piso, Asunción, Paraguay,
Email: arlen.martin@exxis-group.com.
Carlos Granados, Estudiante de Doctorado en Matematicas, Universidad del Antioquia, Medellın, Colombia,
Email: carlosgranadosortiz@outlook.es.
Tula Carola Sanchez Garcia, Facultad de Educacion de la Universidad Nacional Mayor de San Marcos, Lima, Peru,
Email: tula.sanchez1@unmsm.edu.pe.
Carlos Javier Lizcano Chapeta, Profesor - Investigador de pregrado y postgrado de la Universidad de Los Andes, Mérida 5101, Venezuela, Email: lizcha_4@hotmail.com.
Noel Moreno Lemus, Procter & Gamble International Operations S.A., Panamá,
Email: nmlemus@gmail.com.
Asnioby Hernandez Lopez, Mercado Libre, Montevideo, Uruguay.
Email: asnioby.hernandez@mercadolibre.com.
Muhammad Akram, University of the Punjab, New Campus, Lahore, Pakistan,
Email: m.akram@pucit.edu.pk.
Tatiana Andrea Castillo Jaimes, Universidad de Chile, Departamento de Industria, Doctorado en Sistemas de Ingeniería, Santiago de Chile, Chile,
Email: tatiana.a.castillo@gmail.com.
Irfan Deli, Muallim Rifat Faculty of Education, Kilis 7 Aralik University, Turkey,
Email: irfandeli@kilis.edu.tr.
Ridvan Sahin, Department of Mathematics, Faculty of Science, Ataturk University, Erzurum 25240, Turkey, Email: mat.ridone@gmail.com.
Ibrahim M. Hezam, Department of computer, Faculty of Education, Ibb University, Ibb City,

Yemen, Email: ibrahizam.math@gmail.com.
Moddassir khan Nayeem, Department of Industrial and Production Engineering, American International University-Bangladesh, Bangladesh; nayeem@aiub.edu.
Aiyared Iampan, Department of Mathematics, School of Science, University of Phayao, Phayao 56000, Thailand, Email: aiyared.ia@up.ac.th.
Ameirys Betancourt-Vázquez, 1 Instituto Superior Politécnico de Tecnologias e Ciências (ISPTEC), Luanda, Angola, Email: ameirysbv@gmail.com.
H. E. Ramaroson, University of Antananarivo, Madagascar, Email: erichansise@gmail.com.
G. Srinivasa Rao, Department of Mathematics and Statistics, The University of Dodoma, Dodoma PO. Box: 259, Tanzania.
Onesfole Kuramaa, Department of Mathematics, College of Natural Sciences, Makerere University, P.O Box 7062, Kampala, Uganda, Email: onesfole.kurama@mak.ac.ug.
Karina Pérez-Teruel, Universidad Abierta para Adultos (UAPA), Santiago de los Caballeros, República Dominicana, Email: karinaperez@uapa.edu.do.
Neilys González Benítez, Centro Meteorológico Pinar del Río, Cuba, Email: neilys71@nauta.cu.
Jesus Estupinan Ricardo, Centro de Estudios para la Calidad Educativa y la Investigation Cinetifica, Toluca,Mexico,Email: jestupinan2728@gmail.com
Victor Christianto, Malang Institute of Agriculture (IPM), Malang, Indonesia, Email: victorchristianto@gmail.com.
Wadei Al-Omeri, Department of Mathematics, Al-Balqa Applied University, Salt 19117, Jordan, Email: wadeialomeri@bau.edu.jo.
Ganeshsree Selvachandran, UCSI University, Jalan Menara Gading, Kuala Lumpur, Malaysia, Email: Ganeshsree@ucsiuniversity.edu.my.
Ilanthenral Kandasamy, School of Computer Science and Engineering (SCOPE), Vellore Institute of Technology (VIT), Vellore 632014, India, Email: ilanthenral.k@vit.ac.in
Kul Hur, Wonkwang University, Iksan, Jeollabukdo, South Korea, Email: kulhur@wonkwang.ac.kr.
Kemale Veliyeva & Sadi Bayramov, Department of Algebra and Geometry, Baku State University, 23 Z. Khalilov Str., AZ1148, Baku, Azerbaijan, Email: kemale2607@mail.ru,
Email: baysadi@gmail.com.
Irma Makharadze & Tariel Khvedelidze, Ivane Javakhishvili Tbilisi State University, Faculty of Exact and Natural Sciences, Tbilisi, Georgia.
Inayatur Rehman, College of Arts and Applied Sciences, Dhofar University Salalah, Oman, Email: irehman@du.edu.om.
Mansour Lotayif, College of Administrative Sciences, Applied Science University, P.O. Box 5055, East Al-Ekir, Kingdom of Bahrain.
Riad K. Al-Hamido, Math Department, College of Science, Al-Baath University, Homs, Syria, Email: riad-hamido1983@hotmail.com.
Saeed Gul, Faculty of Economics, Kardan University, Parwan-e- Du Square, Kabil, Afghanistan, Email: s.gul@kardan.edu.af.
Faruk Karaaslan, Çankırı Karatekin University, Çankırı, Turkey, Email: fkaraaslan@karatekin.edu.tr.
Morrisson Kaunda Mutuku, School of Business, Kenyatta University, Kenya
Surapati Pramanik, Department of Mathematics, Nandalal Ghosh B T College, India, Email: drspramanik@isns.org.in.
Suriana Alias, Universiti Teknologi MARA (UiTM) Kelantan, Campus Machang, 18500 Machang, Kelantan, Malaysia, Email: suria588@kelantan.uitm.edu.my.
Arsham Borumand Saeid, Dept. of Pure Mathematics, Faculty of Mathematics and Computer, Shahid Bahonar University of Kerman, Kerman, Iran, Email: arsham@uk.ac.ir.
Ahmed Abdel-Monem, Department of Decision support, Zagazig University, Egypt, Email: aabdelmounem@zu.edu.eg.
Çağlar Karamasa, Anadolu University, Faculty of Business,
Turkey, Email: ckaramasa@anadolu.edu.tr.
Mohamed Talea, Laboratory of Information Processing, Faculty of Science Ben M'Sik, Morocco, Email: taleamohamed@yahoo.fr.
Assia Bakali, Ecole Royale Navale, Casablanca, Morocco, Email: assiabakali@yahoo.fr.
V.V. Starovoytov, The State Scientific Institution «The United Institute of Informatics Problems of the National Academy of Sciences of Belarus»,

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Minsk, Belarus, Email: ValeryS@newman.bas-net.by.
E.E. Eldarova, L.N. Gumilyov Eurasian National University, Nur-Sultan, Republic of Kazakhstan, Email: Doctorphd_eldarova@mail.ru. Mukhamed iyeva Dilnoz Tulkunovna & Egamberdiev Nodir Abdunazarovich, Science and innovation center for information and communication technologies, Tashkent University of Information Technologies (named after Muhammad Al-Khwarizmi), Uzbekistan.
Mohammad Hamidi, Department of Mathematics, Payame Noor University (PNU), Tehran, Iran. Email: m.hamidi@pnu.ac.ir.
Lemnaouar Zedam, Department of Mathematics, Faculty of Mathematics and Informatics, University Mohamed Boudiaf, M'sila, Algeria, Email: l.zedam@gmail.com.
M. Al Tahan, Department of Mathematics, Lebanese International University, Bekaa, Lebanon, Email: madeline.tahan@liu.edu.lb.
Mohammad Abobala, Tishreen University, Faculty of Science, Department of Mathematics, Lattakia, Syria,Email" mohammad.abobala@tishreen.edu.sy
Rafif Alhabib, AL-Baath University, College of Science, Mathematical Statistics Department, Homs, Syria, Email: ralhabib@albaath-univ.edu.sy.
R. A. Borzooei, Department of Mathematics, Shahid Beheshti University, Tehran, Iran, borzooei@hatef.ac.ir.
Selcuk Topal, Mathematics Department, Bitlis Eren University, Turkey, Email: s.topal@beu.edu.tr.
Qin Xin, Faculty of Science and Technology, University of the Faroe Islands, Tórshavn, 100, Faroe Islands.
Sudan Jha, Pokhara University,Kathmandu, Nepal, Email: jhasudan@hotmail.com.
Mimosette Makem and Alain Tiedeu, Signal, Image and Systems Laboratory, Dept. of Medical and Biomedical Engineering, Higher Technical Teachers' Training College of EBOLOWA, PO Box 886, University of Yaoundé, Cameroon, E-mail: alain_tiedeu@yahoo.fr.
Mujahid Abbas, Department of Mathematics and Applied Mathematics, University of Pretoria Hatfield 002, Pretoria, South Africa, Email: mujahid.abbas@up.ac.za.
Željko Stević, Faculty of Transport and Traffic Engineering Doboj, University of East Sarajevo, Lukavica, East Sarajevo, Bosnia and Herzegovina, Email: zeljko.stevic@sf.ues.rs.ba.
Michael Gr. Voskoglou, Mathematical Sciences School of Technological Applications, Graduate Technological Educational Institute of Western Greece, Patras, Greece, Email: voskoglou@teiwest.gr.
Saeid Jafari, College of Vestsjaelland South, Slagelse, Denmark, Email: sj@vucklar.dk.
Angelo de Oliveira, Ciencia da Computacao, Universidade Federal de Rondonia, Porto Velho - Rondonia, Brazil, Email: angelo@unir.br.
Valeri Kroumov, Okayama University of Science, Okayama, Japan, Email: val@ee.ous.ac.jp.
Rafael Rojas, Universidad Industrial de Santander, Bucaramanga, Colombia, Email: rafael2188797@correo.uis.edu.co.
Walid Abdelfattah, Faculty of Law, Economics and Management, Jendouba, Tunisia, Email: abdelfattah.walid@yahoo.com.
Akbar Rezaei, Department of Mathematics, Payame Noor University, P.O.Box 19395-3697, Tehran, Iran, Email: rezaei@pnu.ac.ir.
John Frederick D. Tapia, Chemical Engineering Department, De La Salle University - Manila, 2401 Taft Avenue, Malate, Manila, Philippines, Email: john.frederick.tapia@dlsu.edu.ph.
Darren Chong, independent researcher, Singapore, Email: darrenchong2001@yahoo.com.sg.
Galina Ilieva, Paisii Hilendarski, University of Plovdiv, 4000 Plovdiv, Bulgaria, Email: galili@uni-plovdiv.bg.
Paweł Pławiak, Institute of Teleinformatics, Cracow University of Technology, Warszawska 24 st., F-5, 31-155 Krakow, Poland, Email: plawiak@pk.edu.pl.
E. K. Zavadskas, Vilnius Gediminas Technical University, Vilnius, Lithuania, Email: edmundas.zavadskas@vgtu.lt.
Darjan Karabasevic, University Business Academy, Novi Sad, Serbia, Email: darjan.karabasevic@mef.edu.rs.
Dragisa Stanujkic, Technical Faculty in Bor, University of Belgrade, Bor, Serbia, Email: dstanujkic@tfbor.bg.ac.rs.
Katarina Rogulj, Faculty of Civil Engineering,

Architecture and Geodesy, University of Split, Matice Hrvatske 15, 21000 Split, Croatia; Email: katarina.rogulj@gradst.hr.
Luige Vladareanu, Romanian Academy, Bucharest, Romania, Email: luigiv@arexim.ro.
Hashem Bordbar, Center for Information Technologies and Applied Mathematics, University of Nova Gorica, Slovenia, Email: Hashem.Bordbar@ung.si.
N. Smidova, Technical University of Kosice, SK 88902, Slovakia, Email: nsmidova@yahoo.com.
Quang-Thinh Bui, Faculty of Electrical Engineering and Computer Science, VŠB-Technical University of Ostrava, Ostrava-Poruba, Czech Republic, Email: qthinhbui@gmail.com.
Mihaela Colhon & Stefan Vladutescu, University of Craiova, Computer Science Department, Craiova, Romania,Emails: colhon.mihaela@ucv.ro, vladutescu.stefan@ucv.ro.
Philippe Schweizer, Independent Researcher, Av. de Lonay 11, 1110 Morges, Switzerland, Email: flippe2@gmail.com.
Madjid Tavanab, Business Information Systems Department, Faculty of Business Administration and Economics University of Paderborn, D-33098 Paderborn, Germany, Email: tavana@lasalle.edu.
Rasmus Rempling, Chalmers University of Technology, Civil and Environmental Engineering, Structural Engineering, Gothenburg, Sweden.
Fernando A. F. Ferreira, ISCTE Business School, BRU-IUL, University Institute of Lisbon, Avenida das Forças Armadas, 1649-026 Lisbon, Portugal, Email: fernando.alberto.ferreira@iscte-iul.pt.
Julio J. Valdés, National Research Council Canada, M-50, 1200 Montreal Road, Ottawa, Ontario K1A 0R6, Canada, Email: julio.valdes@nrc-cnrc.gc.ca.
Tieta Putri, College of Engineering Department of Computer Science and Software Engineering, University of Canterbury, Christchurch, New Zeeland.
Phillip Smith, School of Earth and Environmental Sciences, University of Queensland, Brisbane, Australia, phillip.smith@uq.edu.au.
Sergey Gorbachev, National Research Tomsk State University, 634050 Tomsk, Russia, Email: gsv@mail.tsu.ru.
Sabin Tabirca, School of Computer Science, University College Cork, Cork, Ireland, Email: tabirca@neptune.ucc.ie.
Umit Cali, Norwegian University of Science and Technology, NO-7491 Trondheim, Norway, Email: umit.cali@ntnu.no.
Willem K. M. Brauers, Faculty of Applied Economics, University of Antwerp, Antwerp, Belgium, Email: willem.brauers@uantwerpen.be.
M. Ganster, Graz University of Technology, Graz, Austria, Email: ganster@weyl.math.tu-graz.ac.at.
Ignacio J. Navarro, Department of Construction Engineering, Universitat Politècnica de València, 46022 València, Spain, Email: ignamar1@cam.upv.es.
Francisco Chiclana, School of Computer Science and Informatics, De Montfort University, The Gateway, Leicester, LE1 9BH, United Kingdom, Email: chiclana@dmu.ac.uk.
Jean Dezert, ONERA, Chemin de la Huniere, 91120 Palaiseau, France, Email: jean.dezert@onera.fr.

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University of New Mexico

Contents

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Study of Knowledge, Attitude, and Practice of Self-medication in Riobamba, Ecuador based on an Indeterminate Likert Scale

Lizbeth Geovanna Silva-Guayasamín [1], Zully Mayra Romero-Orellana [2], Lisbeth Josefina Reales-Chacón [3], Isaac Germán Pérez-Vargas [4], Sulema de la Caridad Hernández-Batista [5], Carlos Antonio Escobar-Suárez [6], María José López-Pino [7] and Josselyn Gabriela Bonilla-Ayala [8]

[1] Universidad Nacional de Chimborazo, Riobamba, Ecuador. Email: lizbethg.silva@unach.edu.ec
[2] Universidad Nacional de Chimborazo, Riobamba, Ecuador. Email: zully.romero@unach.edu.ec
[3] Universidad Nacional de Chimborazo, Riobamba, Ecuador. Email: lisbeth.reales@unach.edu.ec
[4] Universidad Nacional de Chimborazo, Riobamba, Ecuador. Email: iperez@unach.edu.ec
[5] Escuela Superior Politécnica de Chimborazo, Riobamba. Ecuador. Email: sulemadelacaridad@gmail.com
[6] Universidad Regional Autónoma De Los Andes, Ambato, Ecuador. Email: caes2050@hotmail.com
[7] Escuela Superior Politécnica de Chimborazo, Riobamba, Ecuador. Email: mariajose.lopez@espoch.edu.ec
[8] Universidad Técnica de Ambato, Ambato, Ecuador. Email: jg.bonilla@uta.edu.ec

Abstract. Self-medication (SM) is a significant public health problem in Riobamba, Ecuador, characterized by the use of drugs without a prescription. This practice carries risks such as inappropriate drug use and antimicrobial resistance. This study assesses the prevalence, associated factors, and attitudes towards SM in Riobamba. To do so, data were collected from 383 city residents, who also responded to a survey. To ensure the accuracy of the results, we asked respondents to answer some questions based on an Indeterminate Likert Scale. In this way, people can evaluate each of the different opinions, thus obtaining a more reliable result of the person's opinion, compared to the classic Likert scale. In addition, the Neutrosophic Similarity theory is used to convert the responses to a nominal scale.

Keywords: Self-medication, public health, Indeterminate Likert Scale, triple refined indeterminate neutrosophic set (TRINS), Neutrosophic Similarity.

1 Introduction

Self-medication refers to the use of medicinal products by the consumer to treat self-diagnosed disorders or symptoms, or the intermittent or continued use of a medication previously prescribed by a physician. This also includes the use of drugs for other family members, especially children and older adults.

Throughout the history of human care, people have stood out for their knowledge and skill in maintaining life and health. These people, such as Hippocrates, Galen, Avicenna, and Paracelsus, among others, were recognized for their mastery of plants and the creation of the first drugs to treat diseases. Over time, the constant evolution and success of drugs have made them an essential component of therapeutic health management.

The World Health Organization (WHO) includes self-medication as an important part of self-care and considers it a public health resource, especially in primary care within the health system. However, it is necessary to consider that in recent years, this practice has been increasing and has become established as a real public health problem, both in developing and developed countries. This practice entails several risks and many consequences, such as delayed diagnosis of diseases and their prolongation, due to masking symptoms; resistance to medications, especially antibiotics; appearance of comorbidities and even death.

Approximately 71% of men and 82% of women in the United States have resorted to self-medication at least once in their lives. In Latin America, it has been observed that 62.6% of the population self-medicates. There is no representative data on this practice in Ecuador, and even worse for the city of Riobamba, therefore, the local panorama regarding self-medication is unclear.

The type of disease, the social, cultural, and economic status of patients, national laws that regulate the use and sale of drugs, and even the illegal sale of medicines, are factors that give rise to self-medication. Specifically in middle- and low-income countries, the prevalence of self-medication is high, due to limited access to medical care and its high cost, poor conditions in health facilities, and inappropriate behaviors in seeking medical assistance in the general population.

In short, self-medication is a social phenomenon that arises from the human desire to take care of one's health. However, this practice has become so common that public health organizations cannot control it. Self-medication

can become a worldwide problem because it carries more risks than benefits. It has a multifactorial origin, with both its prevalence and its causes varying depending on the geographical area.

Regarding the factors associated with self-medication in Ecuador, self-medication is related to social, cultural, and economic factors, highlighting the commercial practices of the pharmaceutical industry, non-compliance with commercial regulations, lack of patient education, and poor accessibility to medical consultation. This last aspect is also mentioned by some authors, who indicate that other factors associated with self-medication are the high cost of medical care, poor conditions of health centers, and inappropriate behaviors in seeking medical care in the general population.

Although self-medication has been established as a public health problem, this problem has not been studied in depth in Ecuador, since the literature on this topic is very limited, and the available data does not provide relevant data. This is confirmed by those who claim that there has been no local information available in the last 10 years and that the studies that address this topic have not been carried out with a significant sample that would allow these results to be generalized to the population.

Also noteworthy is the limited literature available about this practice in Riobamba. There are few studies that provide relevant data about the prevalence of self-medication and the factors that predispose its initiation and maintenance. From this perspective, this research aims to determine the prevalence of self-medication and the factors associated with self-medication in Riobamba.

For the study, 383 city residents were surveyed using a questionnaire containing questions about self-medication. In addition, the surveyors collected some data from the people interviewed.

In the study, the accuracy of the results was prioritized. This is achieved by capturing people's opinions. These opinions can be defined as apparently contradictory feelings that coexist with each other. That is why respondents are asked to give their opinion based on an Indeterminate Likert Scale [1-7]. This type of scale gives respondents the possibility of scoring for each of the possible values of the Likert scale, instead of selecting a single value.

The Indeterminate Likert Scale is based on the concept of a triple refined indeterminate neutrosophic set (TRINS) [8, 9]. This is a refined single-valued neutrosophic set where the indeterminacy component I is split into two further components to obtain greater precision about the type of indeterminacy. The values obtained from the Indeterminate Likert Scale are converted into a determinate value concerning a predetermined scale. The conversion is done with the help of Neutrosophic Similarity, like in [1]. Consult about this topic in [10-15] .

To carry out the study, this paper is divided into a section called Materials and Methods, where the main concepts of the Indeterminate Likert Scale and Neutrosophic Similarity appear. The section called Results contains the details and results of the study carried out. We finish with the conclusions.

2 Materials and Methods

This section contains the basic notions of the Indeterminate Likert Scale [1].

2.1 Basic Notions on Indeterminate Likert Scale

Definition 1 ([1, 16]). The *Single-Valued Neutrosophic Set* (SVNS) N over U is $A = \{< x; T_A(x), I_A(x), F_A(x) > : x \in U\}$, where $T_A: U \to [0, 1]$, $I_A: U \to [0, 1]$, and $F_A: U \to [0, 1]$, $0 \leq T_A(x) + I_A(x) + F_A(x) \leq 3$.

Definition 2 ([1, 16]). The *refined neutrosophic logic* is defined such that: a truth T is divided into several types of truths: $T_1, T_2, \ldots, T_p$, I into various indeterminacies: $I_1, I_2, \ldots, I_r$ and F into various falsities: $F_1, F_2, \ldots, F_s$, where all $p, r, s \geq 1$ are integers, and $p + r + s = n$.

Definition 3 ([1, 8, 9]). A *triple refined indeterminate neutrosophic set* (TRINS) A in X is characterized by positive $P_A(x)$, indeterminacy $I_A(x)$, negative $N_A(x)$, positive indeterminacy $I_{P_A}(x)$ and negative indeterminacy $I_{N_A}(x)$ membership functions. Each of them has a weight $w_m \in [0, 1]$ associated with it. For each $x \in X$, there are $P_A(x), I_{P_A}(x), I_A(x), I_{N_A}(x), N_A(x) \in [0,1]$,
$w_P^m(P_A(x)), w_{I_P}^m(I_{P_A}(x)), w_I^m(I_A(x)), w_{I_N}^m(I_{N_A}(x)), w_N^m(N_A(x)) \in [0, 1]$ and $0 \leq P_A(x) + I_{P_A}(x) + I_A(x) + I_{N_A}(x)(x) + N_A(x) \leq 5$. Therefore, a TRINS A can be represented by $A = \{\langle x; P_A(x), I_{P_A}(x), I_A(x), I_{N_A}(x), N_A(x)\rangle | x \in X\}$.

Let A and B be two TRINS in a finite universe of discourse $X = \{x_1, x_2, \ldots, x_n\}$, which are denoted by:

$A = \{\langle x; P_A(x), I_{P_A}(x), I_A(x), I_{N_A}(x), N_A(x)\rangle | x \in X\}$ and $B = \{\langle x; P_B(x), I_{P_B}(x), I_B(x), I_{N_B}(x), N_B(x)\rangle | x \in X\}$,

Where $P_A(x_i), I_{P_A}(x_i), I_A(x_i), I_{N_A}(x_i), N_A(x_i), P_B(x_i), I_{P_B}(x_i), I_B(x_i), I_{N_B}(x_i), N_B(x_i) \in [0, 1]$, for every $x_i \in X$. Let $w_i (i = 1,2,...,n)$ be the weight of an element x_i $(i = 1,2,...,n)$, with $w_i \geq 0$ $(i = 1,2,...,n)$ and $\sum_{i=1}^{n} w_i = 1$.

The *generalized TRINS weighted distance* is ([1, 8, 9]):

$$d_\lambda(A,B) = \left\{\frac{1}{5}\sum_{i=1}^{n} w_i \left[|\mathrm{P_A}(\mathrm{x_i}) - \mathrm{P_B}(\mathrm{x_i})|^\lambda + \left|\mathrm{I_{P_A}}(\mathrm{x_i}) - \mathrm{I_{P_B}}(\mathrm{x_i})\right|^\lambda + |\mathrm{I_A}(\mathrm{x_i}) - \mathrm{I_B}(\mathrm{x_i})|^\lambda + \left|\mathrm{I_{N_A}}(\mathrm{x_i}) - \mathrm{I_{N_B}}(\mathrm{x_i})\right|^\lambda + |\mathrm{N_A}(\mathrm{x_i}) - \mathrm{N_B}(\mathrm{x_i})|^\lambda\right]\right\}^{1/\lambda} \quad (1)$$

Where $\lambda > 0$.

The Indeterminate Likert Scale is formed by the following five elements:

– Negative membership,
– Indeterminacy leaning towards negative membership,
– Indeterminate membership,
– Indeterminacy leaning towards positive membership,
– Positive membership.

These values substitute the classical Likert scale with values:

–Strongly disagree,
– Disagree,
–Neither agree or disagree,
– Agree,
–Strongly agree.

Respondents are asked to give their opinion on a scale of 0-5 about their agreement in each of the possible degrees, which are "Very disagree", "Disagree", "Neutral", "Agree", and "Very agree". For this, they were provided with a visual scale like the one shown in Figure 1.

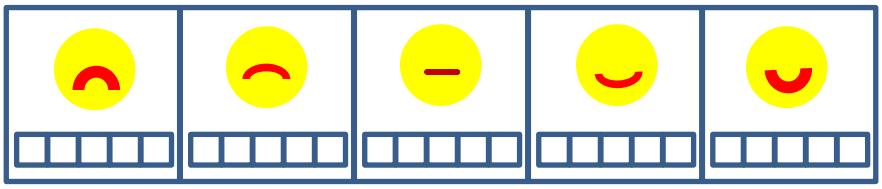

Figure 1. Graphic representation of the proposed Indeterminate Likert Scale. Source: [1].

2.2 Neutrosophic Similarity

In order to reduce the number of respondents' evaluations to a more specific scale, the Neutrosophic Similarity theory is used as done in [1].

Definition 4: ([1]) The *degree of similarity* between two single-valued neutrosophic sets A and B is a mapping S: $\mathcal{N}(X) \times \mathcal{N}(X) \to [0,1]^3$, where $\mathcal{N}(X)$ is the set of all single-valued neutrosophic sets in $X = \{x_1, x_2, \cdots, x_n\}$, such that $S(A,B) = \big(S_T(A,B), S_I(A,B), S_F(A,B)\big)$ satisfies conditions (S1)-(S4).

(S1) $S(A,B) = S(B,A)$,

(S2) $S(A,B) = \underline{1} = (1,0,0)$ if and only if $A = B$, $A, B \in \mathcal{N}(X)$,

(S3) $S_T(A,B) \geq 0, S_I(A,B) \geq 0, S_F(A,B) \geq 0$,

(S4) If $A \subseteq B \subseteq C$, then $S(A,B) \geq S(A,C)$ is satisfied as well as $S(B,C) \geq S(A,C)$.

Definition 5: ([1]) Let $A, B \in \mathcal{N}(X)$ in $X = \{x_1, x_2, \cdots, x_n\}$, then a measure of similarity between A and B is calculated by $S(A,B) = \big(S_T(A,B), S_I(A,B), S_F(A,B)\big)$, where $S_T(A,B)$ is the degree of similarity of truthfulness, $S_I(A,B)$ is the degree of similarity of indeterminacy, and $S_F(A,B)$ is the degree of similarity of falsity. The formulas for similarity are the following:

$$S_T(A,B) = \left(\sum_{i=1}^{n}\left[\frac{min\,(T_A(x_i), T_B(x_i))}{max\,(T_A(x_i), T_B(x_i))}\right]\right)/n \quad (2a)$$

$$S_I(A,B) = 1 - \left(\sum_{i=1}^{n}\left[\frac{min\,(I_A(x_i), I_B(x_i))}{max\,(I_A(x_i), I_B(x_i))}\right]\right)/n \quad (2b)$$

$$S_F(A,B) = 1 - \left(\sum_{i=1}^{n}\left[\frac{min\,(F_A(x_i), F_B(x_i))}{max\,(F_A(x_i), F_B(x_i))}\right]\right)/n \quad (2C)$$

$\forall x_i \in X$.

Definition 6: ([1]) Suppose that for each $x_i \in X = \{x_1, x_2, \cdots, x_n\}$ a weight is associated $w_i \in [0,1]$ such that $\sum_{i=1}^{n} w_i = 1$. Given $A, B \in \mathcal{N}(X)$, then a weighted similarity measure between A and B is calculated by

$S_w(A,B) = \left(S_w^T(A,B), S_w^I(A,B), S_w^F(A,B)\right)$, where $S_w^T(A,B)$ is the degree of similarity of veracity, $S_w^I(A,B)$ is the degree of similarity of indeterminacy, and $S_w^F(A,B)$ is the degree of similarity of the falsehood. The formulas for similarity are the following:

$$S_w^T(A,B) = \sum_{i=1}^n w_i \left[\frac{min\,(T_A(x_i),T_B(x_i))}{max\,(T_A(x_i),T_B(x_i))}\right] \quad (3a)$$

$$S_w^I(A,B) = 1 - \sum_{i=1}^n w_i \left[\frac{min\,(I_A(x_i),I_B(x_i))}{max\,(I_A(x_i),I_B(x_i))}\right] \quad (3b)$$

$$S_w^F(A,B) = 1 - \sum_{i=1}^n w_i \left[\frac{min\,(F_A(x_i),F_B(x_i))}{max\,(F_A(x_i),F_B(x_i))}\right] \quad (3c)$$

$\forall x_i \in X$.

Definition 7: ([1]) Let $A, B \in \mathcal{N}(X)$ in $X = \{x_1, x_2, \cdots, x_n\}$, then a measure of similarity between A and B is calculated by $L(A,B) = \left(L_T(A,B), L_I(A,B), L_F(A,B)\right)$, where $L_T(A,B)$ is the degree of similarity of truthfulness, $L_I(A,B)$ is the degree of similarity of indeterminacy, and $L_F(A,B)$ is the degree of similarity of falsity. The formulas for similarity are the following:

$$L_T(A,B) = 1 - \frac{\sum_{i=1}^n |T_A(x_i) - T_B(x_i)|}{\sum_{i=1}^n |T_A(x_i) + T_B(x_i)|} \quad (4a)$$

$$L_I(A,B) = \frac{\sum_{i=1}^n |I_A(x_i) - I_B(x_i)|}{\sum_{i=1}^n |I_A(x_i) + I_B(x_i)|} \quad (4b)$$

$$L_F(A,B) = \frac{\sum_{i=1}^n |F_A(x_i) - F_B(x_i)|}{\sum_{i=1}^n |F_A(x_i) + F_B(x_i)|} \quad (4c)$$

$\forall x_i \in X$.

Definition 8: ([1]) Let $A, B \in \mathcal{N}(X)$ in $X = \{x_1, x_2, \cdots, x_n\}$, then a measure of similarity between A and B is calculated by $M(A,B) = \left(M_T(A,B), M_I(A,B), M_F(A,B)\right)$, where $M_T(A,B)$ is the degree of similarity of truthfulness, $M_I(A,B)$ is the degree of similarity of indeterminacy, and $M_F(A,B)$ is the degree of similarity of falsity. The formulas for similarity are the following:

$$M_T(A,B) = \frac{1}{n}\sum_{i=1}^n \left(1 - \frac{|T_A(x_i) - T_B(x_i)|}{2}\right) \quad (5a)$$

$$M_I(A,B) = \frac{1}{n}\sum_{i=1}^n \left(\frac{|I_A(x_i) - I_B(x_i)|}{2}\right) \quad (5b)$$

$$M_F(A,B) = \frac{1}{n}\sum_{i=1}^n \left(\frac{|F_A(x_i) - F_B(x_i)|}{2}\right) \quad (5c)$$

$\forall x_i \in X$.

Definition 9: ([1]) Let $A, B \in \mathcal{N}(X)$ and $X = \{x_1, x_2, \cdots, x_n\}$, then a similarity measure based on the distance between A and B is calculated by:

$$S^1(A,B) = \frac{1}{1 + d(A,B)} \quad (6)$$

Such that $d(A,B)$ is a distance function between the two single-valued neutrosophic sets.

Let us remember that the distance function satisfies the following axioms $\forall A, B, C \in \mathcal{N}(X)$:

(1) $d(A,B) \geq 0$ and $d(A,B) = 0$ if and only if $A = B$,

(2) $d(A,B) = d(B,A)$,

(3) $d(A,C) \leq d(A,B) + d(B,C)$.

3 Results

In our research, we set the following objectives:

1. To assess the level of knowledge about self-medication among the inhabitants of Riobamba.
2. To analyze the attitudes of the inhabitants of Riobamba towards self-medication.
3. To identify the age groups with the highest prevalence of self-medication in the city of Riobamba.
4. To determine the prevalence of self-medication by sex in the population of Riobamba.
5. To identify the types of medications most frequently used in self-medication in Riobamba during the last three months.

6. To quantify the frequency of self-medication in the population of Riobamba in the last three months.
7. To evaluate the correlation between self-medication and factors such as attitude, knowledge, age group, and educational level in the inhabitants of Riobamba.

For the study, we selected 383 inhabitants of Riobamba who were visited. The survey measures the following aspects, as shown in Table 1 about the data of the respondents:

Variables	Dimension	Indicators/ coding
Socio-demographic characteristics	Age	12-17 (1); 18-59 (2); Over 60 (3)
	Sex	Male (1); Female (2)
	City of habitual residence	Yes (1); No (0)
	Parish of habitual residence	Lizarzaburu (1); Maldonado (2); Velasco (3); Veloz (4); Yaruquies (5)
	Marital status	Single (1); Married (2); Widowed (3); Divorced (4); Free Union (5)
	Level of Education	Basic (1); High School (2); Superior (3); Fourth Level (4); Illiterate (5)
	Occupation	Not working (1); Public Sector (2); Private Sector (3); Other (4); Retired (5)
	Field of occupation	Health Sector (1); Other Sector (2); I do not work or study (3)
	Family income	It is barely enough (1); It is not enough (2); It is more than enough (3)
	Health Insurance	Yes (1); No (0)
	Drug allergy	Yes (1); No (2); I'm not sure (3)
	Chronic or Congenital Diseases	Cardiovascular diseases (e.g. hypertension) (1) Musculoskeletal disorders (e.g., rheumatoid arthritis, osteoarthritis, osteoporosis) (2) Endocrine disorders (e.g., diabetes mellitus, hypothyroidism) (3) Chronic headache (4) Respiratory conditions (e.g., asthma) (5) Blood diseases (e.g. anemia, leukemia) (6) Otorhinolaryngologic disorders (e.g., sinusitis) (7) Kidney diseases (8) Liver diseases (9) Gastrointestinal conditions (e.g., diarrhea, constipation, gastritis) (10) None (11)

Table 1. Questions regarding the sociodemographic characteristics of the respondents.

Variables	Dimension	Indicators/ coding
Knowledge about self-medication	Self-medication is taking medications without a prescription or supervision from a specialized doctor.	Indeterminate Likert Scale
	Taking medicines from unknown sources or origins, such as medicinal plants, is always safe.	Indeterminate Likert Scale
	If any side effects occur, we should consult a doctor immediately.	Indeterminate Likert Scale
	Taking medications without medical supervision can hide serious symptoms or conditions and lead to their exaggeration.	Indeterminate Likert Scale
Attitude towards self-medication	Self-medication is part of self-care.	Indeterminate Likert Scale
	The general population can prescribe medications appropriately without medical training.	Indeterminate Likert Scale

	Some people can properly recognize and diagnose illnesses without consulting a doctor.	Indeterminate Likert Scale
	Some people may take medications on their own without consulting a doctor.	Indeterminate Likert Scale
	Self-medication is safe and I recommend it to my friends and family.	Indeterminate Likert Scale

Table 2. Questions regarding knowledge and attitude towards self-medication. These questions use the Indeterminate Likert Scale.

Variables	**Dimension**	**Indicators/ coding**
Self-medication practice	Have you taken any medications on your own without supervision or prescription during the last three months?	Yes (1); No (0)
	How many times have you taken those medications in the last three months?	Less than 3 times (1); 3 to 6 times (2); More than 6 times (3)
	Do these medications help you feel better?	I feel worse (1); There is no difference/I rarely feel better (2); Sometimes I feel better (3); I always feel better (4)
	Where do you get these over-the-counter medications?	From friends (1); From family or relatives (2); On the recommendation of a pharmacist (3); Reusing old prescriptions (5); From old medications at home (6); Others (7)
	What medications have you self-medicated within the last 3 months?	Allergy medications (1); Acne or skin problem medications (2); Fever reducers (3); Vitamins and dietary supplements (4); Sleep aids (5); Steroids or Corticosteroids (6); Birth control (contraceptives, the morning after pill, etc.) (7); Antibiotics (8) Herbal Medicines (9); Eye Drops/Ointments (10); Heartburn Medicines (11); Cough Medicines (12); Laxatives (13); Analgesics (Pain Relievers) (14); Digestive Problem Medicines (15); Others (16)
	Which of the following characteristics do you consider about medications before taking them?	Indications for use (1); Name of the medicine (2); How to take the medicine (3); Possible side effects (4); Duration of use or intake (5); Contraindications (6); Methods of proper storage at home (7); None (8)
	Why did you decide to take the medications without consulting a health professional?	More privacy than going to the doctor (1); Lack of sufficient time to visit a doctor (2); Needed a quick response (3); Lack of nearby hospitals, clinics, or health centers (4); Sufficient knowledge about my health problems from past experiences (5); Mild or minor symptoms (6); Embarrassed to visit and allow examination by a doctor (7); Fear of hospital infections (8); Financial (economic) constraints or lack of health insurance to visit a doctor (9); Others (10)
	How did the COVID-19 pandemic affect your self-medication habits?	Increased since the COVID-19 pandemic (1); No difference (The COVID-19 pandemic did not affect this habit) (2); Decreased (or stopped) since the COVID-19 pandemic (3); I don't remember (4)

Table 3. Questions regarding self-medication practices by respondents.

The results in Table 2 are converted into more specific results by the following method:

1. The respondent answers the question with an answer that is a five-fold $A = \langle a_1, a_2, a_3, a_4, a_5 \rangle$ based $a_1, a_2, a_3, a_4, a_5 \in \{0, 1, 2, 3, 4, 5\}$ on the scale graphed in Figure 1.
2. $A_N = \langle x_1, x_2, x_3, x_4, x_5 \rangle$ becomes from A, where $x_i = \frac{A}{5}$, $i = 1, 2, 3, 4, 5$. This makes A_N a TRINS.

3. If the question is of increasing connotation then it is $\bar{A}_N = A_N$, that is, when the answer "agree" rather than "disagree" is the desired one. Otherwise, its negation is taken as $\bar{A}_N = \langle x_5, x_4, x_3, x_2, x_1 \rangle$.
4. Each value $\bar{A}_N$ is then compared with a more approximate qualitative value. To do this, the most similar results are calculated using Neutrosophic similarity, comparing each $\bar{A}_N$ with B_N. To do this, we must be guided by Table 4.

Linguistic value	Equivalent TRINS
Very bad	$\langle 0,0,0,0,1 \rangle$
Bad	$\langle 0,0,0,1,0 \rangle$
Neutral	$\langle 0,0,1,0,0 \rangle$
Good	$\langle 0,1,0,0,0 \rangle$
Very good	$\langle 1,0,0,0,0 \rangle$

Table 4. Proposed scale to concentrate the results of the responses. Association between a linguistic value and a TRINS.

In practice, the largest $S^1(\bar{A}_N, B_N)$ using Equation 6 is calculated, where B_N is a TRINS that appears in Table 4. That is, the TRINS that appear in Table 4 with the greatest similarity between the response given by the interviewee and the TRINS is selected. Then, the linguistic value associated with the selected TRINS is assigned to the response.

Taking into account all these steps, some of the results obtained are shown in Tables 5-11:

		Frequency	Percentage
Valid	Less than 3 times	143	37.3
	From 3 to 6 times	79	20.6
	More than 6 times	26	6.8
	Total	248	64.8
No valid	System	135	35.2
Total		383	100.0

Table 5. AM frequency of survey participants.

		Frequency	Percentage
Valid	From friends	45	11.7
	From family members or relatives	68	17.8
	Old medicines at home	10	2.6
	Others	125	32.6
	Total	248	64.8
No valid		135	35.2
Total		383	100.0

Table 6. Source of obtaining medications with which survey participants self-medicate.

			AM practice in the last 3 months		Total
			No	Yes	
Gender	Male	Count	66	109	175
		% within Sex	37.7%	62.3%	100.0%
	Female	Count	69	139	208
		% within Sex	33.2%	66.8%	100.0%
Total		Count	135	248	383
		% within Sex	35.2%	64.8%	100.0%

Table 7. Self-medication practice based on the gender of survey participants.

			AM practice in the last 3 months		Total
			No	Yes	
Age Group	Teenagers	Count	11	16	27
		% within Age Group	40.7%	59.3%	100.0%
	Adults	Count	119	206	325
		% within Age Group	36.6%	63.4%	100.0%
	Older adults	Count	5	26	31
		% within Age Group	16.1%	83.9%	100.0%
Total		Count	135	248	383
		% within Age Group	35.2%	64.8%	100.0%

Table 8. AM practice based on the age group of survey participants.

			AM practice in the last 3 months		Total
			No	Yes	
Level of Knowledge about AM	Good knowledge (From "Neutral" to "Very good" in Table 4)	Count	73	107	180
		% within AM Knowledge Level	40.6%	59.4%	100.0%
	Poor knowledge ("Very bad" or "Bad" in Table 4)	Count	62	141	203
		% within AM Knowledge Level	30.5%	69.5%	100.0%
Total		Count	135	248	383
		% within AM Knowledge Level	35.2%	64.8%	100.0%

Table 9. AM practice as a function of survey participants' level of AM knowledge.

			AM practice in the last 3 months		Total
			No	Yes	
Attitude towards AM	Favorable Attitude (From "Neutral" to "Very good" in Table 4)	Count	68	166	234
		% within Attitude toward AM	29.1%	70.9%	100.0%
	Unfavorable attitude ("Very bad" or "Bad" in Table 4)	Count	67	82	149
		% within Attitude toward AM	45.0%	55.0%	100.0%
Total		Count	135	248	383
		% within Attitude toward AM	35.2%	64.8%	100.0%

Table 10. AM practice as a function of survey participants' attitudes towards AM.

			AM practice in the last 3 months		Total
			No	Yes	
The highest level of education achieved	Basic	Count	10	13	23
		% within the Maximum level of education achieved	43.5%	56.5%	100.0%
	Baccalaureate	Count	45	106	151
		% within the Maximum level of education achieved	29.8%	70.2%	100.0%
	Superior	Count	71	106	177
		% within the Maximum level of education achieved	40.1%	59.9%	100.0%
	Fourth Level	Count	9	21	30
		% within the Maximum level of education achieved	30.0%	70.0%	100.0%
	Illiterate	Count	0	2	2
		% within the Maximum level of education achieved	0.0%	100.0%	100.0%
Total		Count	135	248	383
		% within the Maximum level of education achieved	35.2%	64.8%	100.0%

Table 11. AM practice as a function of the educational level of survey participants

Conclusion

Analyzing the results obtained and in contrast to previous studies, within the vast panorama of public health, self-medication emerges as a significant challenge in the city of Riobamba. With a prevalence of 64.75% among respondents in the last three months, this practice highlights the urgent need to implement educational and regulatory strategies to mitigate the associated risks and promote responsible use of medications. The study revealed that self-medication is an alarmingly common practice in Riobamba. It was observed that 53% of the population has

poor knowledge about self-medication. This lack of knowledge contributes to the inappropriate use of medications, increasing the risk of adverse effects and health complications. Attitudes towards self-medication in Riobamba are predominantly favorable. This cultural acceptance could be contributing to the high prevalence observed. Self-medication is significantly correlated with factors such as attitude, knowledge, age group, and educational level. Young adults and those with lower educational levels are more likely to self-medicate. Furthermore, health insurance coverage does not appear to be a protective factor, as even those with insurance show a high prevalence of self-medication.

In short, self-medication is a significant public health problem in Riobamba. The results of this study highlight the urgent need to intervene through educational and regulatory strategies. Improving knowledge about self-medication and controlling the sale of medicines are essential steps to reduce the associated risks and promote safer and more responsible use of medicines in the community of Riobamba.

Part of the results obtained is due to the use we made of the Indeterminate Likert Scale, which allowed us to obtain more reliable results than those surveyed thought. We also used the Neutrosophic similarity theory to convert the original data to a more understandable scale.

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Neutrosophic Formulation of the Quadratic Transmuted Generalized Exponential Distribution: Properties and Applications

Kumarapandiyan Gnanasegaran[1], Benitta Susan Aniyan[2,*] and Abdul Kani Jabarali[3]

[1,3]Assistant Professor, Department of Statistics, University of Madras, Madras Christian College, Chennai, Tamilnadu 600059, India; [1]kumarapandiyan@mcc.edu.in, [3]jabarali@mcc.edu.in

[2]Research Scholar, Department of Statistics, University of Madras, Madras Christian College, Chennai, Tamilnadu 600059, India; benittasusananiyan2014@gmail.com

*Correspondence: Benitta Susan Aniyan: benittasusananiyan2014@gmail.com

Abstract. The Quadratic Transmuted Generalized Exponential Distribution (QTGED) enhances the generalized exponential distribution, making it a significant development for handling complex decision-making contexts. Traditional statistical distributions often focus on representing degrees of truth or membership in fuzzy sets, yet they struggle to capture situations involving incomplete, vague, or contradictory data accurately. This study introduces the Neutrosophic Quadratic Transmuted Generalized Exponential Distribution (NQTGED), specifically designed to address indeterminacy and transmuted data. Neutrosophic theory is essential here, as it overcomes the limitations of classical and fuzzy set theories by effectively managing uncertainty, indeterminacy, and inconsistency in data. By simultaneously representing truth, indeterminacy, and falsity, neutrosophic sets offer a comprehensive framework for modeling uncertainty. Traditional distributions lack the adaptability needed for evolving data complexities, often falling short when faced with non-standard data distributions or outliers. Addressing these challenges requires innovative approaches that incorporate advanced mathematical models for uncertainty. This is especially valuable in real-world situations, where data is frequently incomplete, imprecise, or contradictory, and sometimes transmuted. The study derives various mathematical properties of the model, assesses parameter estimation using maximum likelihood and simulation, and demonstrates practical applications with cancer remission data. Simulation results reveal that Neutrosophic Average Biases (NABs) and Neutrosophic Mean Square Errors (NMSEs) decrease as sample sizes increase, indicating strong and accurate parameter estimation. NQTGED provides superior fit and performance, offering significant insights for applications in reliability engineering and biomedical sciences.

Keywords: Neutrosophic; indeterminacy; transmutation; generalized exponential distribution; maximum likelihood estimator; Monte Carlo simulation.

1. Introduction

Neutrosophic theory emerges as a significant advancement over classical and fuzzy set theories by offering a robust framework to model uncertainty, indeterminacy, and inconsistency in data. Florentin Smarandache's [1] introduces Neutrosophic Sets represents a significant advancement in modeling uncertainty by integrating three core elements: truth, falsity, and indeterminacy. Classical sets operate on binary logic, representing data as either true or false, while fuzzy sets extend this by allowing degrees of truth. However, both approaches fall short in scenarios involving indeterminate or contradictory information.This triadic framework allows Neutrosophic Sets to more effectively address the complexities and subtleties found in real-world systems compared to classical and fuzzy sets. Neutrosophic sets address this gap by enabling the simultaneous representation of truth, indeterminacy, and falsity, thus providing a more nuanced and flexible approach to data analysis. This capability is particularly crucial in fields such as engineering, medical sciences, and social sciences, where data is frequently incomplete or imprecise, and decision-making processes require a comprehensive understanding of all aspects of uncertainty. The introduction of neutrosophic theory into these domains enhances the ability to interpret complex systems accurately, making it an indispensable tool for modern data analysis and informed decision-making.

In real-world scenarios, ambiguity and uncertainty often hinder the precise assignment of statistical values, making classical probability inadequate for accurate results [2, 3]. Neutrosophic statistics, with its framework for managing ambiguous and inconsistent data, addresses these challenges effectively. This approach has been advanced through applications of fuzzy logic and neutrosophy [4–8], with significant contributions from Smarandache's pioneering work [9]. On top of this, extensions of fuzzy sets, such as Picture Fuzzy Graphs (PFGs) and Picture Fuzzy Soft Graphs (PFSGs), enhance the modeling of vagueness by introducing parameters like cardinality and domination [10]. In software engineering, hybrid models combining Interval Type-2 Fuzzy Logic Systems with Artificial Neural Networks improve reliability predictions [11]. Moreover, recent studies on fuzzy functions reveal that somewhat fuzzy continuous functions are equivalent to somewhat fuzzy semicontinuous functions, while being weaker than fuzzy semicontinuous functions [12]. Recent studies in urban sustainability emphasize structured frameworks for addressing challenges like water scarcity and waste management using advanced fuzzy methodologies [13]. Neutrosophic philosophy acknowledges three key factors: truth membership, indeterminacy membership, and falsity membership, each representing the degree of truth, ambiguity, or untruth associated with an observation or hypothesis [6]. This makes them highly useful in fields such as decision-making, artificial intelligence, and risk assessment. Furthermore, the foundational principles of revolutionary

topologies highlight how Neutrosophic Sets excel at representing contradictions and ambiguities in diverse scenarios [14].

While the application of neutrosophic probability theory holds practical relevance, it remains an area that has not received significant attention. However, recent years have witnessed an increasing focus on the neutrosophic statistics approach and its diverse applications across various fields. Neutrosophic probability distributions, such as the Neutrosophic Weibull distribution proposed by Alhasan and Smarandache [15], have been developed to address indeterminacy in real-life scenarios, often surpassing classical statistics in problem-solving effectiveness. Additionally, distributions such as Neutrosophic Uniform, Neutrosophic Exponential, and Neutrosophic Poisson have been numerically solved [7–10], [16]. Furthermore, the neutrosophic interpretation of normal and binomial distributions has been extensively explored through various examples [17]. Aslam and Ahtisham [3] introduced the neutrosophic variant of the Rayleigh distribution, while Nayana et al. [18] developed a novel neutrosophic model using the DUS-Weibull transformation. Rao et al. [19] provided insights into the neutrosophic generalized exponential distribution, elucidating its properties and applications. Recent studies [20–25] showcase advancements in neutrosophical probability distributions.

Neutrosophic statistics finds applications across diverse fields, including decision-making, pattern identification, data mining, and image processing in various industries [26–29]. The Box and Muller Technique for generating neutrosophic random variables with a normal distribution exemplifies the effective use of neutrosophic methods in operations research by enhancing decision-making and minimizing risk. This technique integrates goal functions, constraints, and optimal solutions within a neutrosophic framework, enabling simulations to tackle complex problems that are not easily represented mathematically [30]. The neutrosophic theory offers numerous applications, including the treatment of static models, integration of renewable energy sources such as photovoltaic panels, wind turbines, and addressing challenges related to COVID-19 and its Omicron mutation. The neutrosophic approach has recently been applied to tackle the complex challenges posed by the COVID-19 pandemic, utilizing multi-criteria decision-making (MCDM) methods like the Analytic Hierarchy Process (AHP) within a neutrosophic framework to manage uncertain data [31]. Recently, neutrosophic methods have expanded into Industry 5.0, introducing technology-driven inventory models where demand and cost parameters adapt to technological advancements, thereby supporting production optimization in dynamic manufacturing settings [32]. In parallel, plithogenic forest hypersoft sets (PFHS) have been developed to enhance multi-criteria decision-making, allowing complex attribute handling for applications like manufacturing plant site selection [33]. Moreover, generating gamma-distributed neutrosophic variables broadens the applicability of neutrosophic simulations, converting random numbers to improve precision in engineering and

other fields [34]. Neutrosophic methods are employed to evaluate sustainable road transport systems, focusing on reducing carbon emissions and improving energy efficiency, while using MCDM models to balance various sustainability criteria. The composition method facilitates the generation of Poisson-distributed neutrosophic variables, enhancing fidelity in representing real-world environments [35]. The integration of neutrosophic sets with Decision-Making Trial and Evaluation Laboratory (DEMATEL) techniques in this research provides a structured approach to assess and enhance the sustainability of road transport systems [36].In cryptography, neutrosophic integers contribute to innovative encryption and decryption schemes, showcasing their potential in secure information processing [37]. Furthermore, neutrosophic stratified sampling applied to climate data yields reliable estimators with lower bias and improved mean square error (MSE), demonstrating the superior performance of neutrosophic methods over traditional approaches [38]. In addition, advancements in group decision-making models have highlighted the limitations of traditional neutrosophic AHP approaches. Recent developments have introduced enhanced models using neutrosophic trapezoidal numbers to address these limitations, providing more robust solutions for complex decision-making scenarios [39]. Furthermore, neutrosophic logic aids in supply chain risk management by modeling uncertainties and ranking risks using a hybrid AHP and Technique for Order of Preference by Similarity to Ideal Solution (TOPSIS) approach, providing valuable insights for risk mitigation [40]. While traditional mathematics prioritizes precision, real-world problems often involve ambiguous data, necessitating the use of mathematical concepts grounded in uncertainty. Despite its practical utility, neutrosophic probability theory has not received widespread attention, although it has been the subject of some studies. Recent research efforts have focused on various aspects of neutrosophic statistics, including correlation, regression analysis, test procedures, and probability distributions [24].

An influential extension of the exponential distribution, known as the Generalized Exponential Distribution (GED), was introduced by Gupta and Kundu [41]- [42]. Widely acknowledged in the literature, this two-parameter model has gained popularity for analyzing lifetime data. Serving as a versatile alternative to traditional distributions like Weibull or Gamma, the generalized exponential distribution finds applications in various fields including reliability analysis, hydrology, quality control, and medicine [41]- [48].

The groundbreaking research conducted by Shaw and Buckley [49] introduced the quadratic transmuted family of distributions, opening a new avenue for enhancing existing probability models to better capture the quadratic patterns inherent in data.

Khan et al. [50] introduced the Transmuted Generalized Exponential Distributions (TGED), stemming partly from advancements made in the transmuted family of lifetime distributions. They utilized the Quadratic Rank Transmutation Map (QRTM) technique and designated G(x)

as the Cumulative Distribution Function (CDF) of the GED, serving as the baseline model. Additionally, they provided a comprehensive exploration of the mathematical properties associated with the three-parameter TGED, with the anticipation of its potential applications in reliability engineering and the biomedical sciences.

The study of transmuted neutrosophic distributions has garnered significant attention in recent years. Jabarali et al. [51] proposed Generalized p-Transmuted Neutrosophic Distributions (GTND), which delve into the theory and applications of various distributions, including exponential, Weibull, and Rayleigh, in both quadratic and cubic transmuted cases. This research emphasizes the theoretical foundations and practical implications of GTND, contributing to a deeper understanding of its potential applications across various fields. Additionally, the reliability and hazard functions of these distributions are analyzed, shedding light on their performance and reliability. This investigation provides valuable insights for researchers and practitioners.

The introduction of the Neutrosophic Quadratic Transmuted Generalized Exponential Distribution (NQTGED) extends the existing Quadratic Transmuted Generalized Exponential Distribution by incorporating neutrosophic elements, addressing uncertainties, vagueness, and indeterminacy in the data. This new distribution offers enhanced flexibility and versatility in modeling complex real-world scenarios, potentially leading to more accurate and robust statistical analyses. Motivated by the need to handle ambiguity and uncertainty, advance neutrosophic statistical theory, and provide practical applications for data characterized by indeterminacy, the researchers have developed this new distribution. The studies referenced and the reviews of the existing literature have inspired the researchers to embark on the development of a neutrosophic quadratic transmuted generalized exponential distribution, along with an exploration of its associated properties.

In the realm of statistical distribution theory, the accurate modeling of uncertainty, indeterminacy, and inconsistency in data remains a persistent challenge. Traditional statistical distributions, such as the GED and its extensions like the QTGED, primarily focus on capturing degrees of truth or membership in fuzzy sets. However, they often struggle to effectively represent scenarios where data is incomplete, vague, or contradictory.

The limitations of existing frameworks become particularly pronounced in fields such as engineering, medical sciences, and social sciences, where datasets frequently exhibit complex and nuanced patterns that defy simple categorization into true or false states. For instance, in biomedical research, the dynamics of disease progression or treatment outcomes may vary significantly among individuals, leading to data that is inherently uncertain or indeterminate.

Moreover, traditional distributions lack the flexibility to adapt to these diverse and evolving data complexities. They may fail to provide robust estimations or accurate predictions when

faced with non-standard data distributions or outliers. This gap underscores the need for novel approaches that can integrate advanced mathematical frameworks capable of handling the full spectrum of uncertainty.

The proposed work is motivated by the need to better handle uncertainty, indeterminacy, and inconsistency in data analysis, areas where traditional statistical distributions and classical or fuzzy set theories fall short. Real-world data often exhibits complexity, vagueness, and contradictions that existing models cannot adequately capture. Specifically, the limitations of the QTGED in addressing indeterminate data highlight the necessity for a more comprehensive approach. This research addresses gaps such as the inadequate handling of indeterminacy by classical and fuzzy set theories, limited flexibility in existing models to manage imprecise and inconsistent data, and the need for enhanced robustness in real-world applications. The objectives of the proposed work include developing the NQTGED to integrate neutrosophic logic, deriving its mathematical properties, establishing parameter estimation methods using the maximum likelihood approach, and validating its practical applicability with real-world data. This includes comparing its performance against existing distributions using goodness-of-fit tests. The justification for introducing NQTGED lies in its potential to provide more accurate and reliable models for data analysis, particularly in contexts where data is uncertain or indeterminate. By incorporating neutrosophic logic, the proposed distribution offers significant advancements over existing models, contributing to more informed and accurate decision-making processes in various fields.

This paper is structured into seven sections. Section 2 outlines the preliminaries and research framework for the proposed distribution. Subsequent sections delve into properties and discuss various mathematical characteristics (Section 3). Section 4 presents a simulation study assessing the flexibility of the estimates. The penultimate section 5 includes a real-life data analysis, followed by a discussion of the study's implications in the section 6. Finally, the paper concludes with a summary in the last section.

2. **Preliminaries and Research Framework**

This section discusses the preliminaries and research framework of the study.

2.1. *Preliminaries*

In classical data analysis, information is typically represented by precise values. However, in neutrosophic statistics, data can manifest in various forms due to the potential for indeterminacy across different problem contexts. Neutrosophic statistics extends the principles of classical statistics. Neutrosophic numbers adhere to a standard format derived from classical

statistics:

$$X_N = E + I$$

Here, the data is divided into two components: 'E' represents the exact or determined data, while 'I' signifies the uncertain, imprecise, or indeterminate part. Both 'E' and 'I' can take on any real number value. This format is equivalent to $X_N \in [X_L, X_U]$. To distinguish the neutrosophic random variable, it is denoted with the subscript N, as seen in the notation X_N.

Definition 1. Generalized Exponential distribution

[39] Let X be a continuous random variable, the Probability Density Function (PDF) of GED is given as,

$$g(x) = \begin{cases} \frac{\alpha}{\theta} exp\left(-\frac{x}{\theta}\right)\left(1 - exp\left(-\frac{x}{\theta}\right)\right)^{\alpha-1} & ; x > 0, \alpha > 0, \text{ and } \theta > 0 \\ 0 & otherwise \end{cases} \tag{1}$$

where, α is the shape parameter and θ is the scale parameter.

The Cumulative Density Function (CDF) of GED is given as,

$$G(x) = \begin{cases} \left[1 - exp\left(-\frac{x}{\theta}\right)\right]^{\alpha} & ; x > 0, \alpha > 0, \text{ and } \theta > 0 \\ 0 & otherwise \end{cases} \tag{2}$$

Definition 2. Quadratic Transmuted Distribution

[47] According to the QRTM approach, the CDF and probability density function (PDF) satisfy the following relationship:

$$F(x) = (1 + \lambda)G(x) - \lambda[G(x)]^2 \tag{3}$$

and

$$f(x) = g(x)[1 + \lambda - 2\lambda G(x)]; \ x > 0; |\lambda| \leq 1 \tag{4}$$

where, λ is the transmuted parameter.

G(x) and g(x) are the CDF and PDF of the baseline distribution respectively.

Definition 3. Transmuted Generalized Exponential Distribution

[48] If a random variable X follows the TGED with parameters α , θ and λ, then its PDF and CDF is as follows:

$$f(x) = \begin{cases} \frac{\alpha}{\theta} exp\left(-\frac{x}{\theta}\right)\left(1 - exp\left(-\frac{x}{\theta}\right)\right)^{\alpha-1}\left[1 + \lambda - 2\lambda\left(1 - exp\left(-\frac{x}{\theta}\right)\right)^{\alpha}\right] & ; x > 0, \alpha > 0, \theta > 0 \text{ and } |\lambda| \leq 1 \\ 0 & otherwise \end{cases} \tag{5}$$

and

$$F(x) = \begin{cases} \left[1 - exp\left(-\frac{x}{\theta}\right)\right]^{\alpha}\left[1 + \lambda - \lambda\left(1 - exp\left(-\frac{x}{\theta}\right)\right)^{\alpha}\right] & ; x > 0, \alpha > 0, \theta > 0 \text{ and } |\lambda| \leq 1 \\ 0 & otherwise \end{cases} \tag{6}$$

If $\alpha = 1$, TGED reduces to Transmuted Exponential Distribution.

2.2. *Conceptual Framework*

The conceptual framework, where the core concept is the NQTGED. This central arrow represents the innovative contribution of the paper. Surrounding NQTGED are elements such as traditional statistical distributions or QTGED, illustrating how NQTGED builds upon and modifies existing frameworks. Connected to NQTGED is another labeled neutrosophic logic, indicating how neutrosophic logic is integrated into the distribution. Neutrosophic logic allows NQTGED to handle uncertainty, indeterminacy, and inconsistency in data, capabilities that traditional distributions may not fully address. Arrows extend from various data sources, such as real-world data (e.g., cancer patient remission times), pointing towards NQTGED, demonstrating its application and validation with empirical data. Comparative arrows between NQTGED and traditional frameworks highlight NQTGED's advancements: it offers greater flexibility, improved accuracy in modeling real-world phenomena, and employs advanced methodological techniques like Bayesian estimation and robust methods. This diagram visually captures how NQTGED expands upon traditional frameworks through neutrosophic integration, making it a powerful tool for analyzing complex datasets with enhanced precision and applicability (Figure 1).

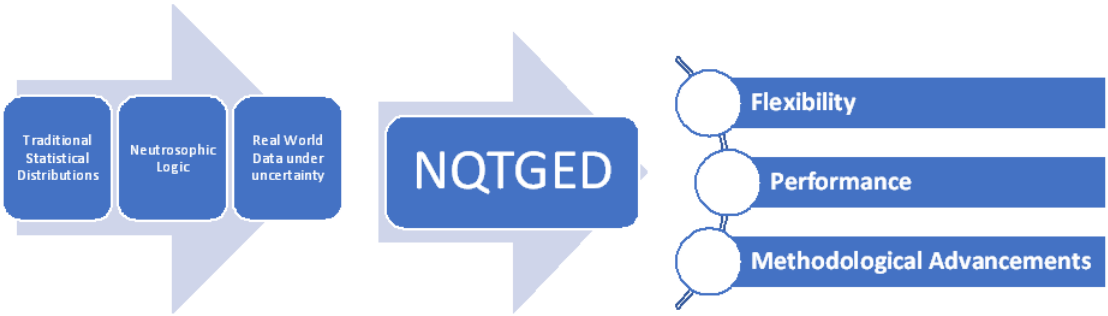


FIGURE 1. Conceptual Framework of the Proposed Work

The primary contributions of this paper include:

- **Development of NQTGED:** Introducing a novel distribution that integrates neutrosophic logic to handle uncertainty and indeterminacy.
- **Mathematical Properties:** Deriving key mathematical properties of NQTGED, such as moments, reliability measures, and order statistics.

- **Methodological Advancements:** Proposing and validating parameter estimation methods using advanced techniques like maximum likelihood estimation under neutrosophic uncertainty.
- **Application in Real-World Scenarios:** Demonstrating the practical applicability of NQTGED through case studies, such as analyzing remission times in cancer patients, to showcase its superiority over traditional distributions.

2.3. *Theoretical Framework*

Definition 4. *A neutrosophic random variable X follows a NQTGED, characterized by neutrosophic shape parameter α_N, neutrosophic scale parameter θ_N, and a neutrosophic transmuted parameter λ_N. The Neutrosophic Probability Density Function (NPDF) of the proposed distribution is defined as:*

$$f_N(x) = \begin{cases} \frac{\alpha_N}{\theta_N} exp\left(-\frac{x}{\theta_N}\right)\left(1-exp\left(-\frac{x}{\theta_N}\right)\right)^{\alpha_N-1} \\ \quad \times \left[1+\lambda_N - 2\lambda_N\left(1-exp\left(-\frac{x}{\theta_N}\right)\right)^{\alpha_N}\right] & \text{if } x>0,\ \alpha_N>0,\ \theta_N>0,\ \text{and}\ \ |\lambda_N|\le 1 \\ 0 & \text{otherwise} \end{cases} \tag{7}$$

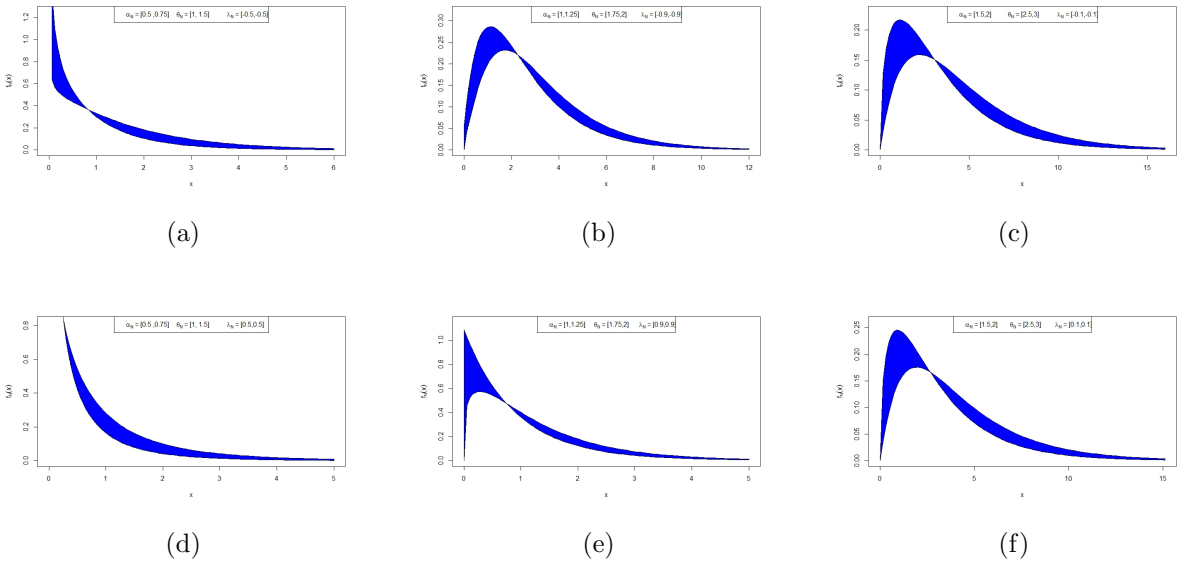
The Neutrosophic Cumulative Distribution Function (NCDF) of NQTGED is,

$$F_N(x) = \begin{cases} \left[1-exp\left(-\frac{x}{\theta_N}\right)\right]^{\alpha_N} \\ \quad \times \left[1+\lambda_N - \lambda_N\left(1-exp\left(-\frac{x}{\theta_N}\right)\right)^{\alpha_N}\right] & \text{if } x>0,\ \alpha_N>0,\ \theta_N>0,\ \text{and}\ \ |\lambda_N|\le 1 \\ 0 & \text{otherwise} \end{cases} \tag{8}$$

Also, the neutrosophic baseline distribution is the Neutrosophic Generalized Exponential Distribution (NGED).

Figure 2 illustrates the probability density curves for the NQTGED, maintaining a constant transmuted parameter while varying the shape parameter and scale parameter within specified intervals. In the neutrosophic framework, the curve is depicted as a thick layer rather than a single curve. The thick layers indicate the range of possible densities, reflecting the uncertainty and variability captured by the neutrosophic parameters. The curves show a noticeable distortion towards the right side. They reveal an exponential decline, commencing from an infinite point when α_N<1. On the other hand, if $\alpha_N \ge 1$, the exponential decline persists, but it initiates from a specific position on the y-axis and also shows a unimodal behavior.

The NCDF curves of the proposed model are presented in Figure 3. In each panel depicted in Figure 3, the NCDF curve exhibits a non-decreasing pattern, covering the range from 0 to

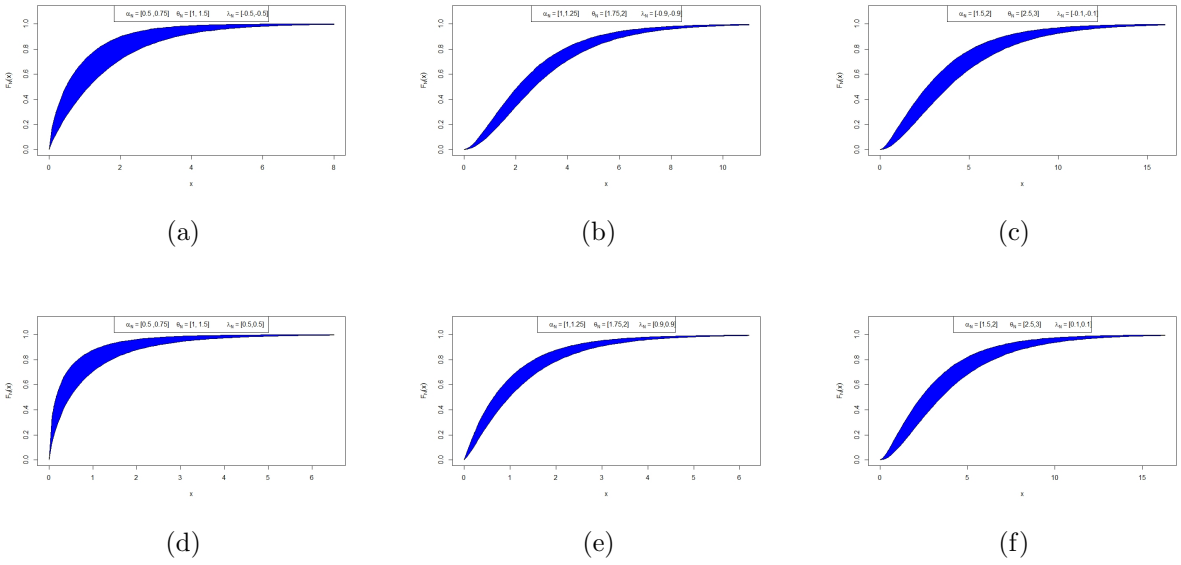


(a) (b) (c)

(d) (e) (f)

FIGURE 2. NPDF Plot of NQTGED

1. The inclusion of neutrosophic parameters ensures that the distribution can model a variety of real-world scenarios where data uncertainty is significant.



(a) (b) (c)

(d) (e) (f)

FIGURE 3. NCDF Plot of NQTGED

When analyzing the panels (a), (b) and (c), we observe that lower values of α_N lead to a gradual increase in the NCDF, suggesting a slower accumulation of probability and a more spread-out distribution. Higher values of α_N result in a steeper NCDF, indicating rapid probability accumulation and a more peaked distribution. The scale parameter θ_N stretches or compresses the NCDF horizontally, reflecting wider or tighter clustering of values, respectively.

In panels (d), (e) and (f), variations in the transmuted parameter λ_N show that positive values steepen the NCDF, concentrating probability in a smaller range, while negative values result in a flatter NCDF, spreading the probability mass more broadly.

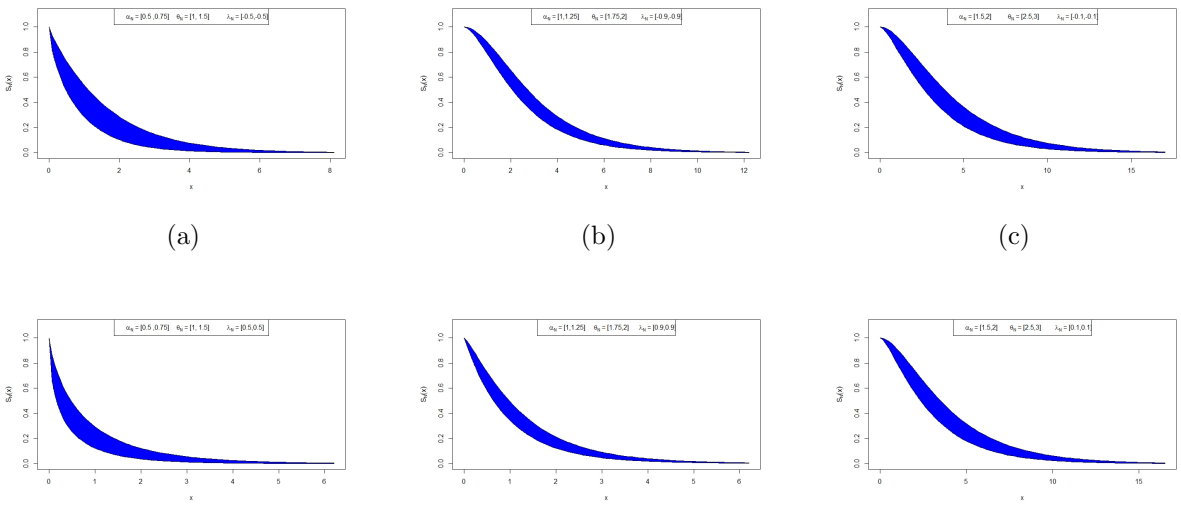
Definition 5. *Surival and Hazard Function*
The Neutrosophic Survival Function (NSF) and Neutrosophic Hazard Function (NHF) of the NQTGED are respectively formulated as follows:

$$S_N(x) = 1 - \left[1 - exp\left(-\frac{x}{\theta_N}\right)\right]^{\alpha_N} \left[1 + \lambda_N - \lambda_N \left(1 - exp\left(-\frac{x}{\theta_N}\right)\right)^{\alpha_N}\right] \tag{9}$$

and

$$h_N(x) = \frac{\frac{\alpha_N}{\theta_N} exp\left(-\frac{x}{\theta_N}\right)\left(1 - exp\left(-\frac{x}{\theta_N}\right)\right)^{\alpha_N - 1}\left[1 + \lambda_N - 2\lambda_N \left(1 - exp\left(-\frac{x}{\theta_N}\right)\right)^{\alpha_N}\right]}{1 - \left[1 - exp\left(-\frac{x}{\theta_N}\right)\right]^{\alpha_N}\left[1 + \lambda_N - \lambda_N \left(1 - exp\left(-\frac{x}{\theta_N}\right)\right)^{\alpha_N}\right]} \tag{10}$$

Figure 4 showcases the NSF Plot of NQTGED, illustrating survival functions under different parameter settings to highlight the distribution's robustness. The panels (a), (b) and (c), with negative λ_N values, shows that increasing α_N leads to a steeper decline in survival probabilities, while increasing θ_N results in a more gradual decline. Negative λ_N values generally slow the decline, extending survival times. The panels (d), (e) and (f), with positive λ_N values, reveals similar trends where higher α_N steepens the decline, and higher θ_N makes it more gradual. Positive λ_N values tend to quicken the decline, shortening survival times.



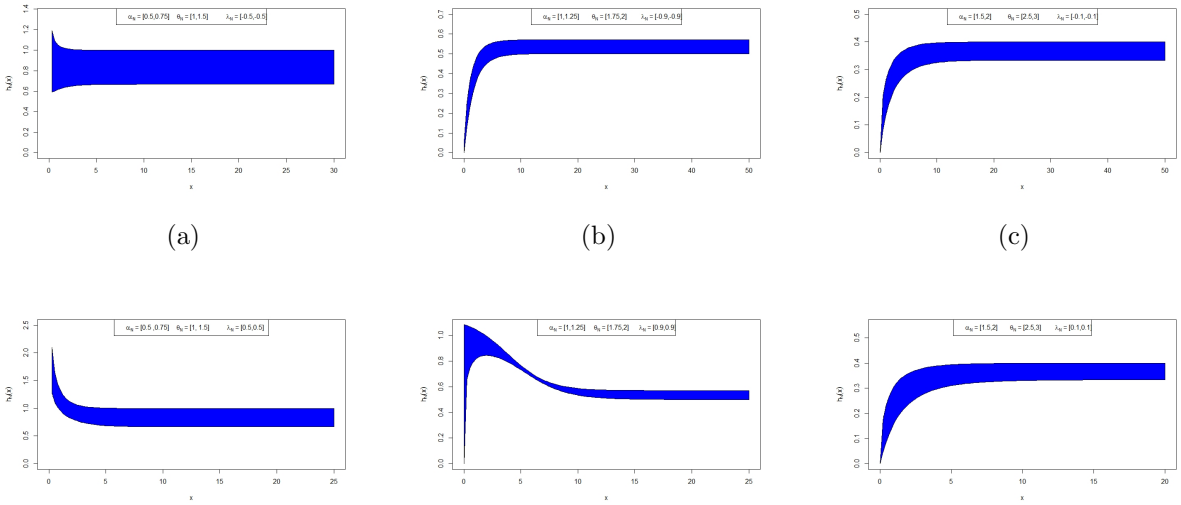
(a) (b) (c)

(d) (e) (f)

FIGURE 4. NSF Plot of NQTGED

Notably, Figure 5 highlights the distinctive bathtub-shaped and increasing behavior of the hazard rate in the NQTGED. Panels (a), (b) and (c) illustrates that negative λ_N values

show that as α_N increases, the initial hazard rate rises sharply, while higher θ_N results in a slower increase in the hazard rate over time. Negative λ_N values generally lower the hazard rate, delaying its significant rise. In panels (d), (e), and (f), positive λ_N values, shows that increasing α_N leads to a more pronounced initial increase in the hazard rate, while higher θ_N causes a slower rise. Positive λ_N values quickly elevate the hazard rate, reducing the time before it peaks. These insights illustrate the substantial effects of α_N, θ_N, and λ_N on hazard rate functions, emphasizing the adaptability of the NQTGED distribution in modeling different hazard scenarios. The bathtub-shaped hazard rate benefits practical applications by identifying early failures for quick fixes, offering a stable period for routine maintenance, and predicting end-of-life replacements. This approach improves resource allocation and reliability. It accommodates uncertainty and indeterminacy, making the model versatile and realistic.



(a) (b) (c)

(d) (e) (f)

FIGURE 5. NHRF Plot of NQTGED

3. **Properties**

Theorem 1. *If $X \sim NQTGED(x;\alpha_N,\theta_N,\lambda_N)$, then the neutrosophic r^{th} moment of X is:*

$$\mu'_{Nr} = \sum_{j=0}^{\infty}(-1)^j \alpha_N \frac{\Gamma(r+1)}{(j+1)^{r+1}}\theta_N^r \left((1+\lambda_N)\binom{\alpha_N-1}{j} - 2\lambda_N\binom{2\alpha_N-1}{j}\right)$$

Proof. The r^{th} moment associated with the NQTGED are detailed as follows,

$$\mu'_{Nr} = \int_0^{\infty} x^r \frac{\alpha_N}{\theta_N} exp\left(-\frac{x}{\theta_N}\right)\left(1-exp\left(-\frac{x}{\theta_N}\right)\right)^{\alpha_N-1}\left[1+\lambda_N-2\lambda_N\left(1-exp\left(-\frac{x}{\theta_N}\right)\right)^{\alpha_N}\right]dx \tag{11}$$

Employing the binomial expansion equation (11) simplifies to

$$\mu_{Nr}^{'} = (1+\lambda_N)\sum_{j=0}\infty\frac{\alpha_N}{\theta_N}(-1)^j\binom{\alpha_N-1}{j}exp-\frac{xj}{\theta_N}\int_0^\infty x^r exp-\frac{xj}{\theta_N}dx$$

$$-2\lambda_N\sum_{j=0}\infty\frac{\alpha_N}{\theta_N}(-1)^j\binom{2\alpha_N-1}{j}exp-\frac{xj}{\theta_N}\int_0^\infty x^r exp-\frac{xj}{\theta_N}dx$$

$$-2\lambda_N\sum_{j=0}\infty\frac{\alpha_N}{\theta_N}(-1)^j\binom{2\alpha_N-1}{j}exp-\frac{xj}{\theta_N}\int_0^\infty x^r exp-\frac{xj}{\theta_N}dx$$

Ultimately, we acquire

$$\mu_{Nr}^{'} = \sum_{j=0}^{\infty}(-1)^j\alpha_N\frac{\Gamma(r+1)}{(j+1)^{r+1}}\theta_N^r\left((1+\lambda_N)\binom{\alpha_N-1}{j}-2\lambda_N\binom{2\alpha_N-1}{j}\right) \tag{12}$$

□

Theorem 2. *If $X \sim NQTGED(x;\alpha_N,\theta_N,\lambda_N)$, then the neutrosophic Moment Generating Function of X is:*

$$M_{NX}(t) = \sum_{r=0}^{\infty}\sum_{j=0}^{\infty}(-1)^j\frac{t^r}{r!}\alpha_N\theta_N^r\frac{\Gamma(r+1)}{(j+1)^{r+1}}\left((1+\lambda_N)\binom{\alpha_N-1}{j}-2\lambda_N\binom{2\alpha_N-1}{j}\right)$$

Proof.

$$\begin{aligned}M_{NX}(t) &= E\left[e^{tx}\right]\\ &= \int_0^\infty e^{tx}f_N(x)dx\\ &= \int_0^\infty\left(\sum_{r=0}^{\infty}\frac{t^r}{r!}x^r\right)f_N(x)dx\\ &= \sum_{r=0}^{\infty}\frac{t^r}{r!}\mu_{Nr}^{'}\end{aligned}$$

Using the equation (13), it yields,

$$M_{NX}(t) = \sum_{r=0}^{\infty}\sum_{j=0}^{\infty}(-1)^j\frac{t^r}{r!}\alpha_N\theta_N^r\frac{\Gamma(r+1)}{(j+1)^{r+1}}\left((1+\lambda_N)\binom{\alpha_N-1}{j}-2\lambda_N\binom{2\alpha_N-1}{j}\right) \tag{13}$$

□

Theorem 3. *The neutrosophic MGF of NQTGED can be expressed in terms of gamma function as:*

$$(1+\lambda_N)\alpha_N\frac{\Gamma(1-t\theta_N)\Gamma\alpha_N}{\Gamma(\alpha_N-t\theta_N+1)}-2\lambda_N\alpha_N\frac{\Gamma(1-t\theta_N)\Gamma 2\alpha_N}{\Gamma(2\alpha_N-t\theta_N+1)}$$

Proof.

$$\begin{aligned} M_{NX}(t) &= E\left[e^{tx}\right] \\ &= \int_0^\infty e^{tx} f_N(x)dx \\ &= \int_0^\infty e^{tx} \frac{\alpha_N}{\theta_N} exp\left(-\frac{x}{\theta_N}\right)\left(1-exp\left(-\frac{x}{\theta_N}\right)\right)^{\alpha_N-1} \\ &\left[1+\lambda_N-2\lambda_N\left(1-exp\left(-\frac{x}{\theta_N}\right)\right)^{\alpha_N}\right]dx \end{aligned} \tag{14}$$

Setting $y = exp\left(-\frac{x}{\theta_N}\right)$, $x = -\theta_N lny$ and $dx = \frac{-\theta_N}{y}dy$. Making substitution in equation (14),

$$\begin{aligned} M_{NX}(t) &= \alpha_N(1+\lambda_N)\int_0^1 y^{-t\theta-N}[1-y]^{\alpha_N-1}dy - 2\alpha_N\lambda_N\int_0^1 y^{-t\theta-N}[1-y]^{2\alpha_N-1}dy \\ &= \alpha_N(1+\lambda_N)\frac{\Gamma\alpha_N\Gamma(1-t\theta_N)}{\Gamma(\alpha_N-t\theta_N+1)} - 2\lambda_N\alpha_N\frac{\Gamma 2\alpha_N\Gamma(1-t\theta_N)}{\Gamma(2\alpha_N-t\theta_N+1)} \end{aligned} \tag{15}$$

□

Upon taking the derivative of $ln(M_{NX}(t))$ and assessing it at t = 0, we derive the mean and variance of X as

$$E_N(x) = \alpha_N\theta_N(1+\lambda_N)\left[\psi(\alpha_N+1)-\psi(1)\right] - 2\alpha_N\theta_N\lambda_N\left[\psi(2\alpha_N+1)-\psi(1)\right] \tag{16}$$

and

$$V_N(x) = \alpha_N\theta_N^2(1+\lambda_N)\left[\psi^{'}(1)-\psi^{'}(\alpha_N+1)\right] - 2\alpha_N\theta_N^2\lambda_N\left[\psi^{'}(1)-\psi^{'}(2\alpha_N+1)\right] \tag{17}$$

The qth quantile x_q of the NQTGED can be determined from equation (8) as follows;

$$x_q = -\theta_N ln\left[1-\left[\frac{(\lambda_N+1)-\sqrt{(\lambda_N+1)^2-4\lambda_N q}}{2\lambda_N}\right]^{\frac{1}{\alpha_N}}\right] \tag{18}$$

Specifically, the median of the distribution is;

$$x_{0.5} = -\theta_N ln\left[1-\left[\frac{(\lambda_N+1)-\sqrt{(\lambda_N+1)^2-8\lambda_N}}{2\lambda_N}\right]^{\frac{1}{\alpha_N}}\right]$$

3.1. *Random Number Generation and Estimation of Parameters of NQTGED*

By employing the inversion method, random numbers can be generated from the NQTGED according to the equation as

$$\left[1-exp\left(-\frac{x}{\theta_N}\right)\right]^{\alpha_N}\left[1+\lambda_N-\lambda_N\left(1-exp\left(-\frac{x}{\theta_N}\right)\right)^{\alpha_N}\right] = u \tag{19}$$

where, $u \sim U(0,1)$. By solving equation (19) for x becomes,

$$x = -\theta_N ln\left[1 - \left[\frac{(\lambda_N+1) - \sqrt{(\lambda_N+1)^2 - 4\lambda_N u}}{2\lambda_N}\right]^{\frac{1}{\alpha_N}}\right] \tag{20}$$

Equation (20) can be utilized to generate random numbers provided the parameters α_N, θ_N and λ_N are known.

3.2. *Order Statistics*

Consider a set of n independent and identically distributed (i.i.d) random variables denoted as $X_{N1}, X_{N2}, ..., X_{Nn}$, each associated with their respective order statistics $X_{N(1)}, X_{N(2)}, ..., X_{N(n)}$, derived from NQTGED with density function $f_N(x)$ and distribution function $F_N(x)$. Consequently, the j^{th} order statistics for the PDF and the CDF are expressed as:

$$f_{x_j}(x) = \frac{n!}{(j-1)!(n-j)!} f_N(x)(F_N(x))^{j-1}(1-F_N(x))^{n-j} \ \forall\ j = 1, 2, ..., n$$

The PDF of largest order statistic of NQTGED is given by,

$$f_{x_{(n)}}(x) = n\left(\frac{\alpha_N}{\theta_N} exp\left(-\frac{x}{\theta_N}\right)\left(1 - exp\left(-\frac{x}{\theta_N}\right)\right)^{\alpha_N - 1}\left[1 + \lambda_N - 2\lambda_N\left(1 - exp\left(-\frac{x}{\theta_N}\right)\right)^{\alpha_N}\right]\right)$$

$$\left(\left[1 - exp\left(-\frac{x}{\theta_N}\right)\right]^{\alpha_N}\left[1 + \lambda_N - \lambda_N\left(1 - exp\left(-\frac{x}{\theta_N}\right)\right)^{\alpha_N}\right]\right)^{n-1}$$

The PDF of the smallest order statistic of NQTGED is given by,

$$f_{x_{(1)}}(x) = n\left(\frac{\alpha_N}{\theta_N} exp\left(-\frac{x}{\theta_N}\right)\left(1 - exp\left(-\frac{x}{\theta_N}\right)\right)^{\alpha_N - 1}\left[1 + \lambda_N - 2\lambda_N\left(1 - exp\left(-\frac{x}{\theta_N}\right)\right)^{\alpha_N}\right]\right)$$

$$\left(1 - \left[1 - exp\left(-\frac{x}{\theta_N}\right)\right]^{\alpha_N}\left[1 + \lambda_N - \lambda_N\left(1 - exp\left(-\frac{x}{\theta_N}\right)\right)^{\alpha_N}\right]\right)^{n-1}$$

3.3. *Maximum Likelihood Estimates of NQTGED*

The maximum likelihood estimates (MLEs) of the parameters inherent in the neutrosophic quadratic transmuted exponential distribution function are as follows: Let X_{N1}, X_{N2},...,X_{Nn} be a random sample from NQTGED. Then, the log-likelihood function is provided as:

$$logL = nlog\alpha_N - nlog\theta_N - \sum_{i=1}^{n}\frac{x_i}{\theta_N} + (\alpha_N - 1)\sum_{i=1}^{n} log\left(1 - exp\left(-\frac{x_i}{\theta_N}\right)\right)$$
$$+ \sum_{i=1}^{n} log\left(1 + \lambda_N - 2\lambda_N\left(1 - exp\left(-\frac{x_i}{\theta_N}\right)\right)_N^{\alpha}\right) \tag{21}$$

The MLEs of α_N, θ_N and λ_N are indicated as $\hat{\alpha}_N$, $\hat{\theta}_N$ and $\hat{\lambda}_N$ respectively, and are derived by optimizing equation (21). And these are the solutions of the following equations:

$$\frac{\partial LogL}{\partial \alpha_N} = \frac{n}{\alpha_N} + \sum_{i=1}^{n} log\left(1 - exp\left(-\frac{x_i}{\theta_N}\right)\right) + \sum_{i=1}^{n} \frac{-2\lambda_N \left(1 - exp\left(-\frac{x_i}{\theta_N}\right)\right)^{\alpha_N} log\left(1 - exp\left(-\frac{x_i}{\theta_N}\right)\right)}{1 + \lambda_N - 2\lambda_N \left(1 - exp\left(-\frac{x_i}{\theta_N}\right)\right)^{\alpha_N}} \tag{22}$$

$$\frac{\partial LogL}{\partial \theta_N} = \frac{-n}{\theta_N} + \sum_{i=1}^{n} \frac{x_i}{\theta_N^2} + (\alpha_N - 1) \sum_{i=1}^{n} \frac{x_i exp\left(\frac{x_i}{\theta_N}\right)}{1 - exp\left(\frac{x_i}{\theta_N}\right)} - \sum_{i=1}^{n} \frac{2\lambda_N \alpha_N \left(1 - exp\left(-\frac{x_i}{\theta_N}\right)\right)^{\alpha_N - 1} x_i exp\left(-\frac{x_i}{\theta_N}\right)}{1 + \lambda_N - 2\lambda_N \left(1 - exp\left(-\frac{x_i}{\theta_N}\right)\right)^{\alpha_N}} \tag{23}$$

and

$$\frac{\partial LogL}{\partial \lambda_N} = \sum_{i=1}^{n} \frac{1 - 2\left(1 - exp\left(-\frac{x_i}{\theta_N}\right)\right)^{\alpha_N}}{1 + \lambda_N - 2\lambda_N \left(1 - exp\left(-\frac{x_i}{\theta_N}\right)\right)^{\alpha_N}} \tag{24}$$

Due to the complexity in computing the nonlinear equations, solving equations (22), (23) and (24) can be accomplished using the inversion method through statistical software R. This results in obtaining the maximum likelihood estimates of α_N, θ_N and λ_N, starting from a selected initial value.

4. **Computational Study of NQTGED**

In this section, to assess the performance of the proposed NGE distribution model, a simulation study is conducted. A Monte Carlo simulation is performed using R software across various sample sizes: specifically, n = 20, 50, 100, and 150, along with different neutrosophic parameter settings: (1) with $\alpha \in$ [0.5, 0.75], $\theta \in$ [1.5, 1.5], and $\lambda \in$ [-0.5, -0.5], (2) with $\alpha \in$ [0.5, 0.75], $\theta \in$ [1.5, 1.5], and $\lambda \in$ [0.5, 0.5], and (3) with $\alpha \in$ [1.5, 1.5], $\theta \in$ [0.5, 0.75], and $\lambda \in$ [-0.5, -0.5]. The simulation is repeated 1000 times. The estimated parameters of the NQTGED and their performance are evaluated in terms of Neutrosophic Average Estimates (NAEs), Neutrosophic Average Biases (NABs), and Neutrosophic Mean Square Errors (NMSEs) through simulation analysis. The simulation results for NAEs, NABs, and NMSEs are presented in Tables 1-3.

TABLE 1. α_N=**[0.5,0.75]**,θ_N=**[1.5,1.5]** **and** λ_N=**[-0.5,-0.5]**

	NAE			NAB			NMSE		
n	$\hat{\alpha}_N$	$\hat{\theta}_N$	$\hat{\lambda}_N$	$\hat{\alpha}_N$	$\hat{\theta}_N$	$\hat{\lambda}_N$	$\hat{\alpha}_N$	$\hat{\theta}_N$	$\hat{\lambda}_N$
20	[0.5967,0.7898]	1.5964	-0.4484	[0.0967,0.0398]	0.0964	0.0516	[0.0619,0.1027]	0.2992	0.2181
50	[0.5496,0.7734]	1.5785	-0.4669	[0.0496,0.0233]	0.0785	0.0331	[0.0365,0.0495]	0.1202	0.1873
100	[0.5198,0.7610]	1.5661	-0.5073	[0.0198,0.0110]	0.0662	-0.0073	[0.0212,0.0358]	0.0647	0.1565
150	[0.5064,0.7572]	1.5618	-0.5065	[0.0064,0.0072]	0.0618	-0.0065	[0.0169,0.0323]	0.0571	0.1364

TABLE 2. α_N=**[0.5,0.75]**,θ_N=**[1.5,1.5]** **and** λ_N=**[0.5,0.5]**

	NAE			NAB			NMSE		
n	$\hat{\alpha}_N$	$\hat{\theta}_N$	$\hat{\lambda}_N$	$\hat{\alpha}_N$	$\hat{\theta}_N$	$\hat{\lambda}_N$	$\hat{\alpha}_N$	$\hat{\theta}_N$	$\hat{\lambda}_N$
20	[0.5299,0.7951]	1.5881	0.5299	[0.0299,0.0451]	0.0881	0.0299	[0.0223,0.0675]	0.4570	0.2067
50	[0.5112,0.7565]	1.5754	0.5175	[0.0112,0.0065]	0.0754	0.0175	[0.0081,0.0194]	0.1801	0.1971
100	[0.5097,0.7534]	1.5177	0.5155	[0.0097,0.0034]	0.0177	0.0155	[0.0029,0.0067]	0.1610	0.1613
150	[0.5001,0.7529]	1.5047	0.5095	[0.001,0.0029]	0.0047	0.0095	[0.0026,0.0056]	0.1299	0.1327

TABLE 3. α_N=**[1.5,1.5]**,θ_N=**[0.5,0.75]** **and** λ_N=**[-0.5,-0.5]**

	NAE			NAB			NMSE		
n	$\hat{\alpha}_N$	$\hat{\theta}_N$	$\hat{\lambda}_N$	$\hat{\alpha}_N$	$\hat{\theta}_N$	$\hat{\lambda}_N$	$\hat{\alpha}_N$	$\hat{\theta}_N$	$\hat{\lambda}_N$
20	1.5227	[0.5218,0.7725]	-0.5280	0.0227	[0.0218,0.0225]	-0.0280	0.4255	[0.0236,0.0426]	0.2147
50	1.5163	[0.5213,0.7694]	-0.5191	0.0163	[0.0213,0.0194]	-0.0191	0.1554	[0.0106,0.0181]	0.1917
100	1.5112	[0.5181,0.7688]	-0.5090	0.0112	[0.0181,0.0188]	-0.0090	1.0923	[0.0065,0.0101]	0.1702
150	1.5022	[0.5147,0.7674]	-0.5079	0.0022	[0.0470,0.0174]	-0.0058	0.1036	[0.0042,0.0063]	0.1435

5. **Application**

In this section, to assess the interest in the NQTGED, a practical application utilized a real-world dataset comprising remission times of 128 cancer patients, measured in months, in Table 4. These remission times, sourced from subsets of bladder cancer studies referenced in [52], serve descriptive purposes. In a neutrosophic context, [19, 22] utilized the remission periods dataset to formulate the neutrosophic generalized exponential distribution. Furthermore, [51] conducted a comparison between the Neutrosophic Rayleigh Distribution and the Quadratic Transmuted Neutrosophic Rayleigh Distribution (QTNRD), adding depth to the study of neutrosophic statistical models. According to their research, the remission periods of cancer patients exhibit a good fit with the NGED model. The dataset's historical application in developing neutrosophic statistical models positions it as an excellent benchmark for showcasing the efficacy of the NQTGED. The analysis focused on evaluating how well the NQTGED represents the remission times by employing a range of goodness-of-fit measures, including the Akaike Information Criterion (AIC), Bayesian Information Criterion (BIC), Hannan-Quinn Criterion (HQC), and the Kolmogorov-Smirnov (KS) test, which are furnished for comparison across QTNRD, Neutrosophic Exponential Distribution (NED) and NGED, as presented in Table 5. This thorough evaluation underscores the potential of the NQTGED for modeling complex datasets within the realm of medical statistics and beyond.

TABLE 4. **Remission periods of 128 cancer patients**

0.08	2.09	3.48	4.87	6.94	8.66	13.11	23.63	0.2	2.23	3.52	4.98	6.97
9.02	13.29	0.4	2.26	3.57	5.06	7.09	9.22	13.8	25.74	0.5	2.46	3.64
5.09	**[7.26, 8.2]**	9.47	14.24	25.82	0.51	2.54	3.7	5.17	7.28	9.74	14.76	**[5.3, 7.1]**
0.81	2.62	3.82	5.32	7.32	10.06	**[12, 14.77]**	32.15	2.64	3.88	5.32	7.39	10.34
14.83	34.26	0.9	2.69	4.18	5.34	7.59	10.66	15.96	36.66	1.05	2.69	4.23
5.41	7.62	10.75	16.62	43.01	1.19	2.75	4.26	5.41	7.63	**[15, 17.2]**	46.12	1.26
2.83	4.33	5.49	7.66	11.25	17.14	**[75.02, 81]**	1.35	2.87	5.62	7.87	11.64	17.36
1.4	3.02	4.34	5.71	7.93	11.79	18.1	1.46	4.4	5.85	8.26	11.98	19.13
1.76	3.25	4.5	6.25	8.37	12.02	**[1.5, 3.2]**	3.31	4.51	6.54	**[7.5, 8.2]**	12.03	
20.28	2.02	3.36	6.76	12.07	21.73	2.07	3.36	6.93	8.65	12.63	22.69	

The indeterminant/ imprecise values are represented as bold values.

TABLE 5. **Estimation of Parameters and model adequacy measures of remission periods of cancer patients**

	QTNRD	NED	NGED	NQTGED
Parameter	$\hat{\theta}_N$=[11.0574,11.3365] $\hat{\lambda}_N$=[0.7949,0.8052]	$\hat{\theta}_N$=[0.1096,0.1081]	$\hat{\theta}_N$=[7.9506,8.0568] $\hat{\alpha}_N$=[1.2390,1.2397]	$\hat{\theta}_N$=[10.9445,11.2182] $\hat{\alpha}_N$=[1.3256,1.3236] $\hat{\lambda}_N$=[0.7006,0.7169]
Log Liklihood	[-463.1498,-466.6273]	[-410.9358,-412.6880]	[-409.4565,-411.2037]	[-407.2067,-408.8213]
AIC	[930.2996,937.2546]	[825.8715,829.3760]	[822.9129,826.4074]	[820.4134,823.6425]
BIC	[936.0037,942.9587]	[841.2796,844.7842]	[838.3210,841.8155]	[828.9695,832.1987]
HQC	[932.6172,939.5722]	[825.0304,828.5348]	[825.2306,828.7250]	[823.8898,827.1190]
KS-Value	[0.2805,0.2807]	[0.0815,0.0869]	[0.0752,0.0759]	[0.0551,0.055114]
P-value	[3x10^{-}9,3x10^{-}9]	[0.3631,0.2886]	[0.4642, 0.3736]	[0.8323,0.8317]

6. **Result Comparison**

The computational study yielded significant insights into the performance of the NQTGED. As anticipated, the results demonstrated a clear relationship between sample size and estimation accuracy. Specifically, from Table 1-3, as the sample size increased from n=20 to n=150, the NABs and NMSEs consistently decreased across all parameter settings. This trend indicates that the NQTGED's parameter estimates become more reliable with larger samples, showcasing its robustness in handling varying neutrosophic parameters. The results demonstrate that the NQTGED can effectively provide accurate estimates, making it a valuable tool for statistical analysis where uncertainty is inherent.

The goodness-of-fit tests, as shown in Table 5, revealed that the NQTGED fitted the remission times data well, surpassing the fits of alternative distributions. From Table 5, the NQTGED exhibited the lowest AIC, BIC and HQC values, indicating a more parsimonious model while

still accurately capturing the underlying patterns in the data. Likewise, KS test statistic further corroborated the superior fit of the NQTGED.

An algorithm for constructing the NQTGED as well as the flow chart of this study continues as follows:

Step 1: Define GED as the baseline distribution as given in **Definition 1**.

Step 2: Apply the Quadratic Transmutation to transform GED into TGED, following **Definition 3**.

Step 3: Introduce neutrosophic parameters to incorporate indeterminacy, forming the NQTGED, as specified in **Definition 4**.

Step 4: Plot the NPDF, NCDF, NSF and NHRF of NQTGED.

Step 5: Derive key mathematical properties for NQTGED.

Step 6: Run Monte Carlo simulations across different sample sizes and neutrosophic parameter settings to test model reliability.

Step 7: Evaluate model fit using real data and compare criteria like AIC, BIC, HQC, and the KS test with other models.

Step 8: Summarize the results to highlight NTGED's effectiveness in fitting data ith indeterminacy.

7. **Discussion and Implication of the Study**

In the realm of statistical distribution theory, effectively modeling uncertainty, indeterminacy, and inconsistency in data remains a formidable challenge. Traditional statistical distributions, such as the GED and its extensions like the QTGED, primarily focus on capturing degrees of truth or membership in fuzzy sets. However, these frameworks often struggle to accurately represent scenarios where data is incomplete, vague, or contradictory.

The limitations of existing methodologies become particularly evident in fields such as engineering, medical sciences, and social sciences, where datasets frequently exhibit complex and nuanced patterns that defy simple categorization into true or false states. For instance, in biomedical research, understanding disease progression or treatment outcomes requires models that can adeptly handle uncertainties inherent in clinical data.

Moreover, traditional distributions lack the flexibility to adapt to diverse and evolving data complexities. They may falter in providing robust estimations or accurate predictions when faced with non-standard data distributions or outliers. Addressing these challenges necessitates innovative approaches capable of integrating advanced mathematical frameworks to comprehensively model uncertainty.

To bridge these gaps, this paper introduces the NQTGED. This distribution extends the QTGED by incorporating neutrosophic logic, which allows for the simultaneous representation

of truth, indeterminacy, and falsity. Neutrosophic logic provides a more nuanced and flexible approach compared to traditional methods, as it can quantitatively capture uncertainties that classical and fuzzy set theories struggle to accommodate.

Firstly, the introduction of the NQTGED represents a significant advancement in probability modeling, particularly in handling situations characterized by ambiguity and indeterminacy. By incorporating neutrosophic principles into the quadratic transmuted generalized exponential distribution, the NQTGED offers a more flexible framework for modeling real-world data that may exhibit uncertain or imprecise characteristics.

The simulation studies conducted to assess the performance of the NQTGED model under different sample sizes and neutrosophic parameter settings provide valuable insights into its robustness and accuracy. The findings indicate that the neutrosophic maximum likelihood estimation (MLE) for the NQTGED yields accurate parameter estimation, particularly with larger sample sizes in the uncertainty scenario. This suggests the potential practical applicability of the NQTGED model in various fields, including reliability engineering and biomedical sciences during uncertain transmuted cases.

Moreover, the practical application of the NQTGED model to a real-world dataset of cancer patient remission times highlights its relevance and effectiveness in capturing complex data patterns. The Log-likelihood and the goodness-of-fit tests, including the AIC, BIC, HQC and Kolmogorov–Smirnov (KS) test, suggest that the NQTGED emerges as a plausible fit for the remission times, outperforming other potential distributions such as the QTNRD, NED and NGED.

Overall, the study's findings underscore the importance and potential impact of neutrosophic probability modeling in addressing the challenges posed by uncertain and imprecise data. The development and exploration of the NQTGED model offer researchers and practitioners a powerful tool for analyzing and interpreting complex datasets, with implications for various fields ranging from reliability engineering to biomedical sciences. Further research and applications of neutrosophic probability theory are warranted to continue advancing our understanding and utilization of probabilistic models in the face of ambiguity and uncertainty.

8. **Conclusion**

In conclusion, the NQTGED introduced in this study offers a flexible and robust framework for modeling uncertain and imprecise data. Through thorough exploration and practical application, the NQTGED has shown promising results in accurately capturing complex data patterns. Its development signifies a significant advancement in probabilistic modeling, with implications for various fields including reliability engineering and biomedical sciences. Further research and application of the NQTGED model are warranted to continue advancing

our understanding and utilization of neutrosophic probability theory in real-world scenarios. Future plans include deepening theoretical foundations by exploring additional mathematical properties and extending methodological advancements beyond maximum likelihood estimation to include Bayesian approaches and handling censored data. The application scope will expand beyond cancer patient remission times to encompass diverse fields like finance and environmental sciences, validating NQTGED's robustness across different domains.

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A Multi-Criteria Decision-Making Approach for Teaching Quality Evaluation in University Music Programs Using Plithogenic Sets and Weighted Decision Models

Mingjiao Liu[1*], Hongdan Wang[2]

[1,2]College of Education, Longdong University, Qingyang, 745000, Gansu, China

*Corresponding author, E-mail: mjliu@alu.scu.edu.cn

Abstract: Chinese colleges and universities have been studying and implementing quality monitoring systems for music art instruction as a key teaching management project. How to fairly and impartially evaluate the teaching skills of professors of art and music is a crucial problem in college and university administration. Evaluation of university music program have various criteria affecting the decision-making process. So, we used the multi-criteria decision-making methodology (MCDM) to deal with these criteria and select best program. We used the Plithogenic sets with the MCDM methodology to deal with uncertainty information. We used the operators of Plithogenic to combine the Plithogenic numbers. We use the weighted product model (WPM) to rank the alternatives. The weights of criteria are computed using the normalized crips values. The results show the Musical Performance Proficiency of Teachers criterion has the highest rate and Engagement with Music Industry and Professional Development criterion has the lowest rate. We compare the proposed methodology with other MCDM methods. The results show the proposed methodology is effective.

Keywords: Teaching Quality Evaluation; University Music Programs; Multi-criteria Decision Making; WPM Model; Uncertainty.

1. Introduction

A major teaching management initiative in Chinese colleges and universities over the past ten years has been the study and application of quality monitoring methods for music art instruction. The assessment and evaluation of the teaching state and effect of music art are crucial components of quality monitoring management. The quality

evaluation of music and art training is the process of collecting a lot of sample data, analyzing, summarizing, and evaluating it[1], [2].

Every college must conduct a standard teaching assessment each academic year to determine the quality of instruction in music and art courses. It is crucial for college and university administration to assess the effectiveness of music and art instructors and promptly comprehend the actual educational environment. With the growth of the school and college, it is challenging to administer the evaluation questionnaire by hand. The new teaching situation's requirements can no longer be met by the manual evaluation. Among its many shortcomings are high ongoing capital costs, a significant organizational burden, difficult data statistics, low efficiency, a high error rate, and a lack of intuitive statistical analysis, querying, and browsing[3], [4].

Therefore, it is imperative that a practical, effective, rational, and scientific evaluation system for the quality of instruction in music and art courses be established at the technical level. This will have a significant impact on assessing the quality of instruction and promoting the teaching of art and music.

There are still certain issues and restrictions with the music art teaching quality assessment method currently in use at colleges and universities, which need to be resolved:

- There is no targeting of the system function. Every system is autonomous.
- The evaluation program only offers rudimentary statistical features. It does not fully utilize the vast amount of fundamental data that has been gathered over time to conduct an exhaustive assessment of the quality of instruction[5].

Smarandache created plithogenic sets as a generalization of intuitionistic, fuzzy, and crisp sets and advanced the concept of plithogeny[6], [7]. With set P, dominating attribute a, set of attribute values V, degree of appurtenance d, and degree of contradiction c, the plithogenic sets are represented as a quintuple of the form (P, a, V, d, c). The traits and attribute values that help resolve complex decision-making processes are used to characterize lithogenic sets[8], [9].

There are frequently several criteria that must be taken into account simultaneously while making decisions. It can be difficult to evaluate and position decisions based on several factors, whether you're choosing an individual option, a venture technique, or an unneeded item to ship. Strategies for Multi-Criteria Decision Making (MCDM) provide a structured method for addressing such problems[10], [11].

The Weighted Product Method (WPM) is one such tactic that provides a systematic approach to combining preferences and weighting criteria to arrive at the final decision. We will go into the intricacies of the Weighted Product Method in this piece, examining its essential elements, inclinations, challenges, and practical uses[12], [13].

The proposed approach is carefully applied to address the problem of evaluating teaching quality in university music programs. It includes a detailed explanation of the methodology and its implementation. Moving to Section 3, the results of the proposed model are thoroughly analyzed and compared, offering insights into its effectiveness. Section 4 builds on this by providing a comparative analysis, showing how the results align with or differ from other established methodologies. Finally, the conclusion wraps up the study by summarizing the main findings, discussing their significance, and suggesting areas for further research.

1.2 Importance of Plithogenic Sets

Plithogenic sets represent a significant advancement in the field of decision-making, providing a flexible and precise framework for handling uncertainty, contradiction, and complexity. Unlike traditional fuzzy, crisp, or intuitionistic sets, plithogenic sets allow for a more nuanced representation of data by incorporating degrees of appurtenance and contradiction simultaneously. Their importance can be summarized as follows:

- Plithogenic sets are uniquely equipped to model situations where data is uncertain or conflicting. This capability is critical in complex decision-making scenarios, such as teaching quality evaluation, where subjective opinions often vary.
- By extending intuitionistic and fuzzy sets, plithogenic sets offer a broader range of applications, making them versatile tools for addressing a variety of multi-criteria decision-making (MCDM) problems.
- By accounting for contradictory attributes and degrees of relevance, plithogenic sets improve the accuracy of decisions, leading to more reliable outcomes.
- Plithogenic sets have found applications in various fields, including education, engineering, healthcare, and business, demonstrating their adaptability and effectiveness in tackling diverse problems.

2. Methodology of the Proposed Decision Approach

A few definitions relevant to our work are presented in this section. Smarandache's writings provide the fundamental definitions of plithogenic sets. A generalized definition of the plithogenic accuracy is also provided. P $\subseteq \mathcal{U}$, the universal set, 'a' is the attribute, V is the set of attribute values, d is the degree of appurtenance, and c is the degree of

contradiction. A quintuple of this kind is called a plithogenic set. Only attributes are used to create the plithogenic sets[14].

The steps in the suggested approach, which was developed by combining both combined plithogenic sets with MCDM techniques, are shown in this section. There are two stages to this decision-making process. In the first stage, a plithogenic accuracy matrix is created using a generalized plithogenic accuracy function and coupled plithogenic degrees of appurtenance. In the second stage, the choices are ranked using a WPM method[15], [16]. Figures. 1 and 2, respectively, provide a diagram of the procedures involved in each of the two phases.

Stage 1. Formulation of decision-making matrix

The first stage in the proposed methodology is the formulation of the decision-making matrix, a foundational step that structures the evaluation process by organizing data into a systematic format. This matrix is a vital tool in Multi-Criteria Decision-Making (MCDM) as it provides a clear and organized representation of the relationships between the alternatives under consideration and the criteria used for evaluation.

In this stage, the decision problem is defined by identifying two key components:

Criteria: These are the factors or attributes against which the alternatives are assessed. In the context of evaluating university music programs, criteria might include factors such as "Musical Performance Proficiency of Teachers," "Integration of Music Theory and Practice," or "Engagement with the Music Industry."
Alternatives: These represent the different options or entities being evaluated. For example, alternatives could be ten universities offering music programs.

Step 1. Problem definition

The first step involves identifying the criteria (e.g., teaching quality metrics) and alternatives (e.g., universities or music programs). The decision matrix is defined based on a set of criteria and alternatives.

Step 2. Evaluate the criteria and alternatives.

Experts assess each alternative against the criteria using plithogenic terms. These terms are then converted into plithogenic numbers, capturing degrees of appurtenance and contradiction. (we replace these terms with plithogenic numbers).

$$X^k = \begin{bmatrix} x_{11} & x_{12} & x_{13} & & x_{1j} & x_{1j} & x_{1n} \\ x_{21} & x_{22} & x_{23} & \cdots & x_{2n} & x_{2j} & x_{2n} \\ x_{31} & x_{32} & x_{33} & & x_{3n} & x_{3n} & x_{3n} \\ & \vdots & & \ddots & & \vdots & \\ x_{i1} & x_{ij} & x_{ij} & & x_{ij} & x_{ij} & x_{in} \\ x_{i2} & x_{ij} & x_{ij} & \cdots & x_{ij} & x_{ij} & x_{in} \\ x_{m1} & x_{mj} & x_{mj} & & x_{mj} & x_{mj} & x_{mn} \end{bmatrix} \quad (1)$$

Where k refers to the number of experts

Step 3. Combine the decision matrix.

We combine the plithogenic numbers using the plithogenic operators.

$$X = \begin{bmatrix} x_{11} & \cdots & x_{1n} \\ \vdots & \ddots & \vdots \\ x_{m1} & \cdots & x_{mn} \end{bmatrix} \quad (2)$$

Step 4. Obtain crisp values.

We convert the plithogenic numbers into a crisp value.

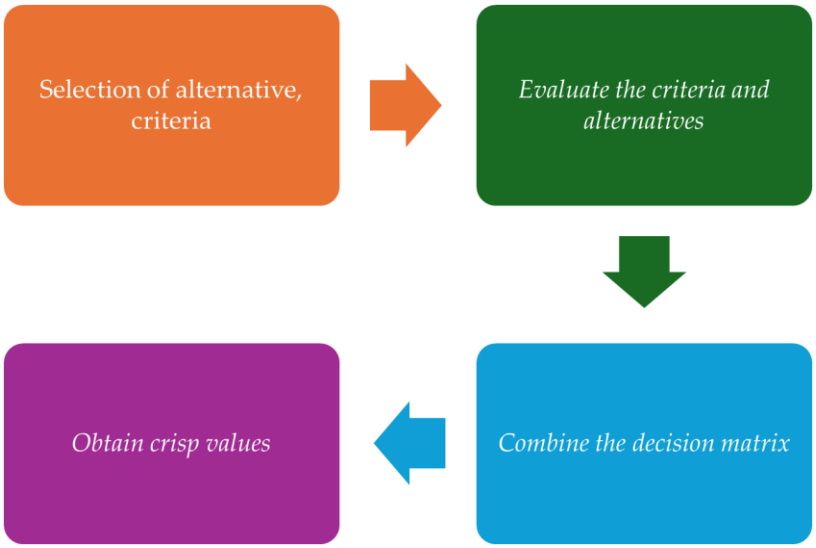
Selection of alternative, criteria
Evaluate the criteria and alternatives
Combine the decision matrix
Obtain crisp values

Figure 1. Steps of stage 1.

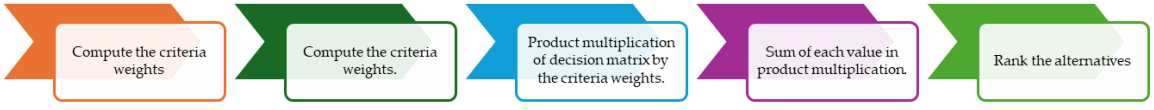
Compute the criteria weights
Compute the criteria weights.
Product multiplication of decision matrix by the criteria weights.
Sum of each value in product multiplication.
Rank the alternatives

Figure 2. Steps of stage 2.

Stage 2. Ranking the alternatives.

In this stage, we applied the WPM methodology to rank the alternatives.

Step 5. Compute the criteria weights.

$$w = \sum_{j=1}^{n} w_j = 1 \tag{3}$$

Step 6. Product multiplication of decision matrix by the criteria weights.

$$q_{ij} = w_j x_{ij} \tag{4}$$

Step 7. Sum of each value in product multiplication.

$$r_i = \sum_{j=1}^{n} q_{ij} \tag{5}$$

Step 8. Rank the alternatives.

$$T_i = \max r_i \tag{6}$$

2.1 Advantages of Methodology

1. The use of plithogenic sets allows the framework to adapt to various types of data, including subjective and contradictory inputs.
2. By combining plithogenic sets with WPM, the methodology ensures precise and reliable rankings, even in complex scenarios.
3. The final rankings and weighted scores provide clear and actionable insights for decision-makers.

To demonstrate the methodology, the study applied it to evaluate the teaching quality of ten university music programs based on seven criteria. The results showed that "Musical Performance Proficiency of Teachers" was he most significant criterion, while "Engagement with the Music Industry" was the least impactful. Alternative A7 emerged as the best-performing program.

3. Illustration for Decision Making on Teaching Quality Evaluation in University Music Programs

This section shows the results of the proposed methodology to compute the criteria weights and rank the alternatives. Three experts collected the criteria and alternatives from previous studies. These experts and decision makers have expertise in the field of evaluation the university music programs. Seven criteria and ten universities are gathered in this study as shown in Table 1.

Table 1. The criteria and alternatives.

	Criteria		Alternatives
C_1	Individualized Instruction and Mentorship of Music	A_1	University 1
C_2	Integration of Music Theory and Practice	A_2	University 2
C_3	Musical Performance Proficiency of Teachers	A_3	University 3
C_4	Use of Technology in Music Education	A_4	University 4
C_5	Curriculum Design and Repertoire Selection of Music	A_5	University 5
C_6	Performance Opportunities and Ensemble Leadership of Music	A_6	University 6
C_7	Engagement with Music Industry and Professional Development	A_7	University 7
		A_8	University 8
		A_9	University 9
		A_{10}	University 10

Three experts have built the decision matrix table in Table 2. Then we combine these numbers using the plithogenic operators. Then we obtain the crisp values. Then we compute the criteria weights as shown in Figure 3.

Table 2. The decision matrix.

	C_1	C_2	C_3	C_4	C_5	C_6	C_7
A_1	(0.65, 0.30, 0.45)	(0.25, 0.60, 0.80)	(0.40, 0.70, 0.50)	(0.10, 0.75, 0.85)	(0.65, 0.30, 0.45)	(0.80, 0.10, 0.30)	(0.25, 0.60, 0.80)
A_2	(0.50, 0.40, 0.60)	(0.10, 0.75, 0.85)	(0.25, 0.60, 0.80)	(0.40, 0.70, 0.50)	(0.50, 0.40, 0.60)	(0.25, 0.60, 0.80)	(0.10, 0.75, 0.85)
A_3	(0.40, 0.70, 0.50)	(0.95, 0.05, 0.05)	(0.10, 0.75, 0.85)	(0.25, 0.60, 0.80)	(0.25, 0.60, 0.80)	(0.10, 0.75, 0.85)	(0.95, 0.05, 0.05)
A_4	(0.25, 0.60, 0.80)	(0.95, 0.05, 0.05)	(0.95, 0.05, 0.05)	(0.25, 0.60, 0.80)	(0.10, 0.75, 0.85)	(0.95, 0.05, 0.05)	(0.95, 0.05, 0.05)
A_5	(0.10, 0.75, 0.85)	(0.25, 0.60, 0.80)	(0.25, 0.60, 0.80)	(0.10, 0.75, 0.85)	(0.95, 0.05, 0.05)	(0.95, 0.05, 0.05)	(0.95, 0.05, 0.05)
A_6	(0.40, 0.70, 0.50)	(0.10, 0.75, 0.85)	(0.10, 0.75, 0.85)	(0.95, 0.05, 0.05)	(0.95, 0.05, 0.05)	(0.10, 0.75, 0.85)	(0.50, 0.40, 0.60)
A_7	(0.25, 0.60, 0.80)	(0.95, 0.05, 0.05)	(0.95, 0.05, 0.05)	(0.95, 0.05, 0.05)	(0.95, 0.05, 0.05)	(0.95, 0.05, 0.05)	(0.40, 0.70, 0.50)
A_8	(0.10, 0.75, 0.85)	(0.95, 0.05, 0.05)	(0.95, 0.05, 0.05)	(0.95, 0.05, 0.05)	(0.10, 0.75, 0.85)	(0.95, 0.05, 0.05)	(0.25, 0.60, 0.80)
A_9	(0.95, 0.05, 0.05)	(0.95, 0.05, 0.05)	(0.65, 0.30, 0.45)	(0.95, 0.05, 0.05)	(0.40, 0.70, 0.50)	(0.25, 0.60, 0.80)	(0.10, 0.75, 0.85)
A_{10}	(0.95, 0.05, 0.05)	(0.65, 0.30, 0.45)	(0.95, 0.05, 0.05)	(0.65, 0.30, 0.45)	(0.25, 0.60, 0.80)	(0.10, 0.75, 0.85)	(0.25, 0.60, 0.80)
	C_1	C_2	C_3	C_4	C_5	C_6	C_7
A_1	(0.40, 0.70, 0.50)	(0.50, 0.40, 0.60)	(0.25, 0.60, 0.80)	(0.50, 0.40, 0.60)	(0.65, 0.30, 0.45)	(0.40, 0.70, 0.50)	(0.25, 0.60, 0.80)
A_2	(0.25, 0.60, 0.80)	(0.40, 0.70, 0.50)	(0.95, 0.05, 0.05)	(0.50, 0.40, 0.60)	(0.25, 0.60, 0.80)	(0.25, 0.60, 0.80)	(0.95, 0.05, 0.05)
A_3	(0.10, 0.75, 0.85)	(0.25, 0.60, 0.80)	(0.40, 0.70, 0.50)	(0.40, 0.70, 0.50)	(0.95, 0.05, 0.05)	(0.10, 0.75, 0.85)	(0.40, 0.70, 0.50)
A_4	(0.95, 0.05, 0.05)	(0.10, 0.75, 0.85)	(0.25, 0.60, 0.80)	(0.25, 0.60, 0.80)	(0.40, 0.70, 0.50)	(0.25, 0.60, 0.80)	(0.40, 0.70, 0.50)
A_5	(0.95, 0.05, 0.05)	(0.25, 0.60, 0.80)	(0.40, 0.70, 0.50)	(0.40, 0.70, 0.50)	(0.25, 0.60, 0.80)	(0.40, 0.70, 0.50)	(0.40, 0.70, 0.50)
A_6	(0.40, 0.70, 0.50)	(0.25, 0.60, 0.80)	(0.25, 0.60, 0.80)	(0.25, 0.60, 0.80)	(0.10, 0.75, 0.85)	(0.25, 0.60, 0.80)	(0.25, 0.60, 0.80)
A_7	(0.40, 0.70, 0.50)	(0.95, 0.05, 0.05)	(0.10, 0.75, 0.85)	(0.10, 0.75, 0.85)	(0.95, 0.05, 0.05)	(0.10, 0.75, 0.85)	(0.10, 0.75, 0.85)
A_8	(0.50, 0.40, 0.60)	(0.40, 0.70, 0.50)	(0.10, 0.75, 0.85)	(0.40, 0.70, 0.50)	(0.40, 0.70, 0.50)	(0.50, 0.40, 0.60)	(0.40, 0.70, 0.50)
A_9	(0.50, 0.40, 0.60)	(0.40, 0.70, 0.50)	(0.95, 0.05, 0.05)	(0.50, 0.40, 0.60)	(0.40, 0.70, 0.50)	(0.40, 0.70, 0.50)	(0.40, 0.70, 0.50)
A_{10}	(0.65, 0.30, 0.45)	(0.50, 0.40, 0.60)	(0.95, 0.05, 0.05)	(0.50, 0.40, 0.60)	(0.50, 0.40, 0.60)	(0.50, 0.40, 0.60)	(0.50, 0.40, 0.60)
	C_1	C_2	C_3	C_4	C_5	C_6	C_7

A_1	(0.25, 0.60, 0.80)	(0.40, 0.70, 0.50)	(0.25, 0.60, 0.80)	(0.50, 0.40, 0.60)	(0.65, 0.30, 0.45)	(0.80, 0.10, 0.30)	(0.25, 0.60, 0.80)
A_2	(0.25, 0.60, 0.80)	(0.25, 0.60, 0.80)	(0.25, 0.60, 0.80)	(0.25, 0.60, 0.80)	(0.50, 0.40, 0.60)	(0.40, 0.70, 0.50)	(0.25, 0.60, 0.80)
A_3	(0.95, 0.05, 0.05)	(0.25, 0.60, 0.80)	(0.95, 0.05, 0.05)	(0.25, 0.60, 0.80)	(0.25, 0.60, 0.80)	(0.25, 0.60, 0.80)	(0.95, 0.05, 0.05)
A_4	(0.40, 0.70, 0.50)	(0.95, 0.05, 0.05)	(0.10, 0.75, 0.85)	(0.95, 0.05, 0.05)	(0.25, 0.60, 0.80)	(0.25, 0.60, 0.80)	(0.10, 0.75, 0.85)
A_5	(0.40, 0.70, 0.50)	(0.65, 0.30, 0.45)	(0.25, 0.60, 0.80)	(0.65, 0.30, 0.45)	(0.95, 0.05, 0.05)	(0.25, 0.60, 0.80)	(0.25, 0.60, 0.80)
A_6	(0.65, 0.30, 0.45)	(0.65, 0.30, 0.45)	(0.95, 0.05, 0.05)	(0.65, 0.30, 0.45)	(0.65, 0.30, 0.45)	(0.25, 0.60, 0.80)	(0.25, 0.60, 0.80)
A_7	(0.65, 0.30, 0.45)	(0.10, 0.75, 0.85)	(0.50, 0.40, 0.60)	(0.10, 0.75, 0.85)	(0.65, 0.30, 0.45)	(0.25, 0.60, 0.80)	(0.95, 0.05, 0.05)
A_8	(0.25, 0.60, 0.80)	(0.25, 0.60, 0.80)	(0.95, 0.05, 0.05)	(0.25, 0.60, 0.80)	(0.10, 0.75, 0.85)	(0.25, 0.60, 0.80)	(0.10, 0.75, 0.85)
A_9	(0.25, 0.60, 0.80)	(0.50, 0.40, 0.60)	(0.50, 0.40, 0.60)	(0.10, 0.75, 0.85)	(0.25, 0.60, 0.80)	(0.95, 0.05, 0.05)	(0.25, 0.60, 0.80)
A_{10}	(0.95, 0.05, 0.05)	(0.65, 0.30, 0.45)	(0.65, 0.30, 0.45)	(0.25, 0.60, 0.80)	(0.10, 0.75, 0.85)	(0.25, 0.60, 0.80)	(0.10, 0.75, 0.85)

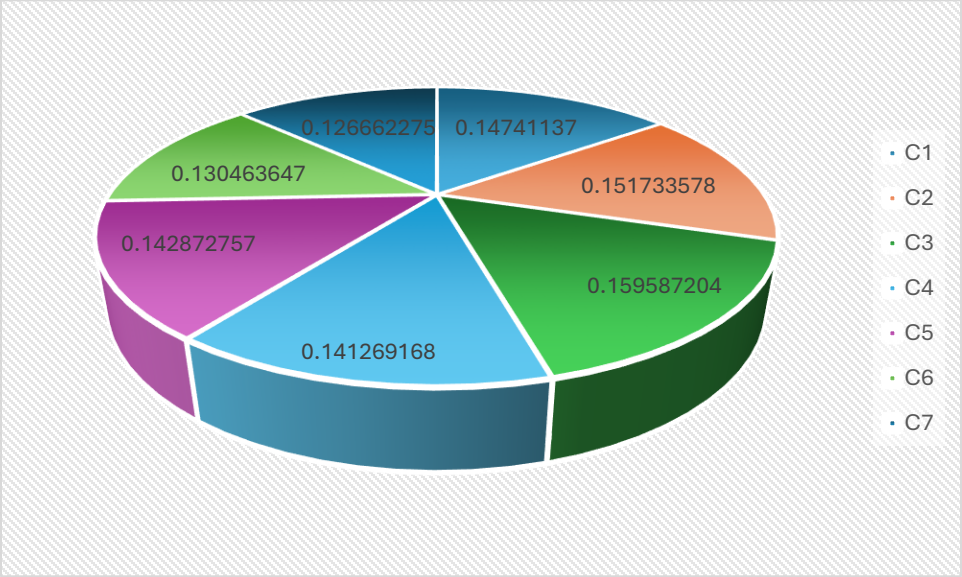

Figure 3. The criteria weights.

Then obtain the product multiplication of decision matrix by the criteria weights as in Table 3. Then we obtain the sum of each value in product multiplication. Then we obtain the rank of the alternatives as in Figure 4.

Table 3. Product multiplication of decision matrix

	C_1	C_2	C_3	C_4	C_5	C_6	C_7
A_1	0.110804	0.101661	0.085379	0.093897	0.149266	0.141292	0.055832
A_2	0.087697	0.069545	0.145417	0.09465	0.10475	0.069798	0.107336
A_3	0.129759	0.138261	0.140477	0.075579	0.130187	0.035638	0.149366
A_4	0.138075	0.176144	0.135237	0.128726	0.065483	0.11888	0.111494
A_5	0.129759	0.104766	0.085379	0.096899	0.170132	0.122201	0.11864
A_6	0.117118	0.093708	0.135237	0.150811	0.146165	0.046739	0.075353
A_7	0.110804	0.176144	0.153044	0.10969	0.179174	0.110557	0.111494

A_8	0.076384	0.142124	0.185261	0.132322	0.054042	0.131866	0.058054
A_9	0.148996	0.155274	0.181012	0.135477	0.0862	0.122201	0.058054
A_{10}	0.184866	0.150077	0.200136	0.115499	0.074032	0.067602	0.065632

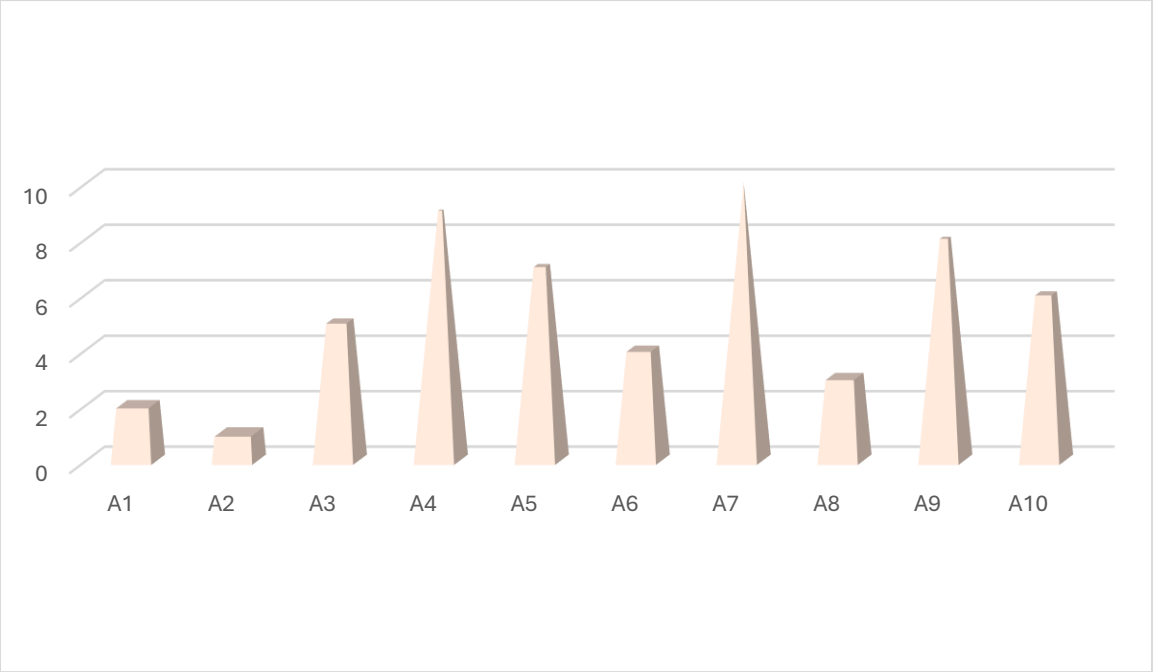

Figure 4. The rank of alternatives.

4. Comparative Analysis of the Proposed Methodology with Other MCDM Methods

This part compares the proposed methodology with other MCDM methods. Figure 5 shows this comparison. We compared our model with different MCDM methods to show different rank of alternatives from different MCDM methods. We show all methods show that the alternative 7 is the best and alternative 2 is the worst.

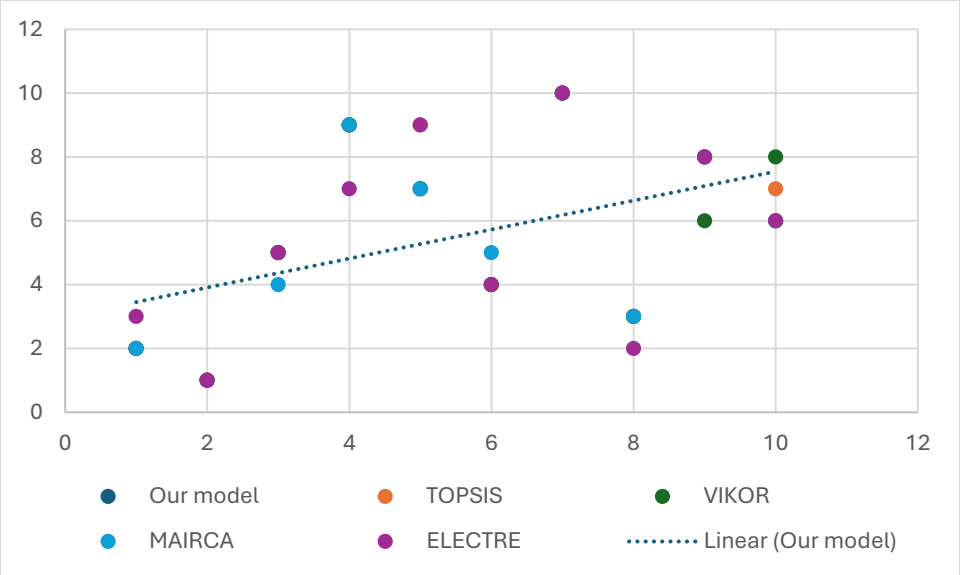

Figure 5. The correlation between our model and other MCDM methods.

5. Conclusions

For more than ten years, a major teaching management initiative in Chinese colleges and universities has been employed to research and put into place quality monitoring methods for teaching art and music. One of the most important tasks in college and university management is to accurately and truthfully assess the teaching skills of professors of art and music. This paper mandates that each university provides direction on the quantitative assessment methodology of music education art for music graduates in accordance with the national guidance on music graduates. This study developed a MCDM methodology with the WPM method to rank the university music programs. This method is integrated with plithogenic sets to deal with uncertainty and vague information. Three experts have evaluated the criteria and alternatives. This study used seven criteria and ten alternatives to be ranked by the WPM method. We compute the criteria weights and rank the alternatives. The results show alternative 7 is the best and alternative 2 is the worst.

5.1 Limitations of the Proposed Methodology

Although the proposed methodology offers notable strengths in managing uncertainties and effectively ranking alternatives, it is not without its limitations. These challenges must be acknowledged to provide a balanced perspective on its applicability and potential areas for improvement.

One key limitation lies in the dependence on expert input. The methodology relies heavily on the expertise and judgment of the evaluators who provide assessments for the

criteria and alternatives. This reliance introduces a level of subjectivity, as any bias, inconsistency, or lack of expertise on the part of the evaluators can directly influence the outcomes. Consequently, ensuring the selection of qualified and unbiased experts becomes a crucial factor in the reliability of the methodology.

Another challenge is the complexity involved in calculations. The use of Plithogenic sets introduces intricate mathematical computations that require a solid understanding of the underlying principles. For individuals or organizations lacking adequate training or technical resources, implementing the methodology can be daunting. This complexity may limit its adoption and practical use, especially in settings where simpler tools or models are preferred.

Additionally, the methodology faces scalability issues. While it demonstrates efficiency and reliability when applied to a moderate number of criteria and alternatives, its performance tends to decline as the dataset grows larger. Managing large-scale decision-making problems with an extensive number of criteria and alternatives may require significant computational resources and further optimization of the model.

Lastly, the study is constrained by limited validation data. The analysis was based on the inputs of three experts and ten alternatives, which, while sufficient for demonstrating the methodology's viability, may not fully capture its robustness and generalizability. Broader datasets with a more diverse range of experts and alternatives are necessary to validate the methodology's performance across varied contexts and scenarios.

5.2 Recommendations for Future Research

Building on the findings of this study and addressing its limitations, several recommendations can be made to enhance the methodology's applicability and effectiveness in broader contexts.

One of the most significant recommendations is the development of automated tools. The complexity of implementing Plithogenic-based Multi-Criteria Decision-Making (MCDM) methods can pose a challenge, particularly for users with limited technical expertise. By creating software solutions or user-friendly platforms, the computational aspects of the methodology can be simplified, making it accessible to a wider audience. Such tools would streamline data input, facilitate calculations, and provide intuitive visualizations of results, encouraging broader adoption.

Another promising avenue for improvement is the incorporation of machine learning algorithms. By integrating advanced algorithms, the methodology could reduce its

reliance on expert evaluations, which are prone to subjectivity and bias. Machine learning models could analyze historical data to identify patterns, refine criteria weights, and generate more consistent and objective evaluations. This would enhance both the reliability and scalability of the framework.

To further enrich the methodology, there is a need for expansion of criteria. While the current study focuses on specific teaching quality metrics, future research should consider additional factors such as student satisfaction, graduate success rates, and interdisciplinary collaboration opportunities. Incorporating these dimensions would provide a more holistic evaluation of teaching quality and ensure that the methodology aligns with evolving educational priorities.

Lastly, cross-cultural validation is essential to test the methodology's universal applicability. Applying the framework to evaluate music programs in different cultural and educational settings would help identify any context-specific challenges and validate their robustness across diverse environments. This would also allow researchers to explore how variations in cultural values and educational practices influence the relative importance of criteria.

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Hypersoft Semi-Connected Spaces

Florentin Smarandache[1], V.Inthumathi[2] and M.Amsaveni[3,*]

[1] "Department of Mathematics, University of New Mexico, Gallup, NM 87301, USA;
smarand@unm.edu

[2] Department of Mathematics, Nallamuthu Gounder Mahalingam College,
Tamil Nadu, India;
inthumathi65@gmail.com

[3] Department of Mathematics, Nallamuthu Gounder Mahalingam College,
Tamil Nadu, India;
amsavenim2000@gmail.com

*Correspondence: amsavenim2000@gmail.com"

Abstract. Neutrosophic sets are an important field of topology that deals with inconsistent, indeterminate, and uncertain circumstances in real life. Hypersoft sets have grown in popularity due to their ability to represent data as a collection of trait-valued disjoint sets that combine various attributes. Motivated by this concept, we introduce a new hyper-approach known as hypersoft semi-separated, which employs hypersoft semi-open sets [HSO]. This article presents several local properties of HSO sets, such as hypersoft semi-neighborhoods of hypersoft points and hypersoft semi-first countable spaces. Additionally, definitions of hypersoft semi-connected [HSCO] spaces, hypersoft locally semi-connected [HLSCO] spaces, and associated properties are provided. The comparison between hypersoft-connected [HCO] spaces, HSCO, and HLSCO spaces is presented along with their examples.

Keywords: HSO sets, hypersoft semi-first countable spaces, hypersoft semi-separation, HSCO, HLSCO.

1. Introduction

Molodtsov [18] developed the mathematical method of soft set theory to deal with uncertainty. The study of soft set theory and its applications has been explored by numerous researchers [6, 7, 17, 33]. Shabir and Naz [36] established the concept of soft topological spaces by building soft sets over an initial universe set with a certain set of parameters.

Sasikala and Sivaraj [35] discovered the soft semi-open and soft semi-closed sets in soft topological spaces [STS]. Bin Chen [4] described the local properties of soft semi-open sets. Krishnaveni and Sekar [16] defined the characteristics of soft semi-connected and soft locally

semi-connected spaces.

Smarandache [37] devised a clever strategy for dealing with ambiguity. By switching to a multiple decision function, the soft sets were extended into hypersoft sets. Based on the hypersoft sets and their expansion, several researchers have developed a wide range of operators, features, and applications [1, 2, 10, 12]. Hypersoft topology, hypersoft closure, hypersoft interior, hypersoft connected spaces, and hypersoft separation axioms were investigated by Musa and Asaad [21–23]. Inthumathi and Amsaveni [14] introduced hypersoft subspace topology, hypersoft basis, hypersoft limit point, and hypersoft Hausdorff spaces. Inthumathi et al. [13] presented HSO sets, hypersoft semi-closed [HSC] sets, hypersoft semi-interior, and hypersoft semi-closure.

As an extension of hypersoft sets, the N-Hypersoft set was introduced by Musa et al. In [32], the N-Hypersoft set was intended to manage analyses using binary and non-binary data while maintaining an intuitive sense of structural symmetry. The superhypersoft set [41] is an extension of the hypersoft set that was first introduced by Smarandache. In addition, he introduced superhyperstructure, superhyperfunction [39], and superhypertopology [40]. An extension of the hypersoft set, from determinate to indeterminate data, is the indetermhypersoft set [8, 38]. The concept of hypersoft sets is expanded to Pythagorean neutrosophic hypersoft sets, and double-frame soft sets are extended to double-frame hypersoft sets by Muhammad Saeed et al. [19,31]. With the help of this extension, we may manage intricate decision-making situations, including several qualities and their interactions, in a strong framework. Novel approaches have been demonstrated by recent studies that have advanced a variety of hypersoft sectors [3, 11, 20, 24–29, 34].

In recent days, Neutrosophic sets play a dominant role in addressing uncertainty in decision-making difficulties. In [9,30,42,43], Smarandache expanded his ideas through neutrosophic sets in various domains. In future, we plan to concentrate on strengthening our concept for the aforementioned extended neutrosophic domains and their applications.

The current work is structured into the subsequent sections: In Section 2, some fundamental concepts regarding hypersoft sets and hypersoft topological spaces (HTS) are summarized. Section 3 defines the hypersoft semi-neighborhood system, hypersoft semi-first countable spaces, and associated properties. New ideas like hypersoft semi-separation and hypersoft semi-connectedness via hypersoft semi-open sets in an HTS are presented in Section 4. Section 5 explains the concept of HLSCO spaces and the connection between HCO, HSCO, and HLSCO spaces. Section 6 narrates the conclusion.

2. Preliminaries

This section aims to provide a clear overview of the planned study by going over some fundamental hypersoft set attributes. For further information and fundamental terminology, see [5, 15]

The following table describes the various abbreviations and definitions used throughout the paper.

Abbreviation	Definition
STS	Soft topological spaces
HTS	Hypersoft topological spaces
HSO	Hypersoft semi-open
HSC	Hypersoft semi-closed
HCO	Hypersoft connected
HSCO	Hypersoft semi-connected
HLSCO	Hypersoft locally semi-connected
HS.S.C	Hypersoft semi component

Definition 2.1. [18] "Let U be an initial universe and E be a set of parameters. Let P(U) denote the power set of U. The pair (F, E) or simply F_E, is called a soft set over U, where F is a mapping given by $F : E \to P(U)$.

Definition 2.2. [37] Let U be a universe of discourse, P(U) the power set of U and $E_1, E_2, \cdots, E_n$ the pairwise disjoint sets of parameters. Let A_i be the nonempty subset of E_i for each $i = 1, 2, ..., n$. A hypersoft set can be identified by the pair $(\mathcal{F}, A_1 * A_2 * \cdots * A_n)$, where $\mathcal{F} : A_1 * A_2 * \cdots * A_n \to P(U)$. For sake of simplicity, we write the symbols $\mathbf{E}$ for $E_1 * E_2 * \cdots * E_n$, $\mathbf{A}$ for $A_1 * A_2 * \cdots * A_n$.

Definition 2.3. [21] Let τ be a collection of hypersoft sets over U, then τ is called a hypersoft topology over U if

(1) $(\emptyset, \mathbf{A})$ and$(\mathcal{F}, \mathbf{A})$ belongs to τ,
(2) The intersection of any two hypersoft sets in τ belongs to τ,
(3) The union of any number of a hypersoft sets in τ belongs to τ.

Then $((\mathcal{F}, \mathbf{A}), \tau)$ is called a HTS over U.

Definition 2.4. [21] Let $((\mathcal{F}, \mathbf{A}), \tau)$ be a HTS and let $(\mathcal{F}, \mathbf{B})$ be a hypersoft set then

(1) The hypersoft interior of $(\mathcal{F}, \mathbf{B})$ is the hypersoft set
$h\text{-}int(\mathcal{F}, \mathbf{B}) = \bigcup\{(\mathcal{F}, \mathbf{C}) : (\mathcal{F}, \mathbf{C})$ is hypersoft open and $(\mathcal{F}, \mathbf{C}) \subseteq (\mathcal{F}, \mathbf{B})\}$.
(2) The hypersoft closure of $(\mathcal{F}, \mathbf{B})$ is the hypersoft set
$h\text{-}cl(\mathcal{F}, \mathbf{B}) = \bigcap\{(\mathcal{F}, \mathbf{C}) : (\mathcal{F}, \mathbf{C})$ is hypersoft closed and $(\mathcal{F}, \mathbf{B}) \subseteq (\mathcal{F}, \mathbf{C})\}$.

Definition 2.5. [13] Let $(\mathcal{F}, \mathbf{B})$ be a hypersoft set of a HTS $((\mathcal{F}, \mathbf{A}), \tau)$. $(\mathcal{F}, \mathbf{B})$ is called a HSO set if $(\mathcal{F}, \mathbf{B}) \subseteq h\text{-}cl(h\text{-}int(\mathcal{F}, \mathbf{B}))$.

Definition 2.6. [13] A hypersoft set $(\mathcal{F}, \mathbf{B})$ in a HTS $((\mathcal{F}, \mathbf{A}), \tau)$ is called a HSC set if its relative complement is a HSO set.

Definition 2.7. [13] Let $((\mathcal{F}, \mathbf{A}), \tau)$ be a HTS and let $(\mathcal{F}, \mathbf{B})$ be a hypersoft set in $(\mathcal{F}, \mathbf{A})$.

(1) The hypersoft semi-interior of $(\mathcal{F}, \mathbf{B})$ is the hypersoft set $\bigcup\{(\mathcal{F}, \mathbf{C}) : (\mathcal{F}, \mathbf{C})$ is HSO and $(\mathcal{F}, \mathbf{C}) \subseteq (\mathcal{F}, \mathbf{B})\}$ and it is denoted by $h\text{-}sint(\mathcal{F}, \mathbf{B})$.

(2) The hypersoft semi-closure of $(\mathcal{F}, \mathbf{B})$ is the hypersoft set $\bigcap\{(\mathcal{F}, \mathbf{C}) : (\mathcal{F}, \mathbf{C})$ is HSC and $(\mathcal{F}, \mathbf{B}) \subseteq (\mathcal{F}, \mathbf{C})\}$ and it is denoted by $h\text{-}scl(\mathcal{F}, \mathbf{B})$. Clearly, $h\text{-}scl(\mathcal{F}, \mathbf{B})$ is the smallest HSC set containing $(\mathcal{F}, \mathbf{B})$ and $h\text{-}sint(\mathcal{F}, \mathbf{B})$ is the largest HSO set contained in $(\mathcal{F}, \mathbf{B})$.

Definition 2.8. [22] Let $((\mathcal{F}, \mathbf{A}), \tau)$ be a HTS. Two non-null hypersoft sets $(\mathcal{F}, \mathbf{A})$ and $(\mathcal{F}, \mathbf{B})$ are said to be separated hypersoft sets iff $(\mathcal{F}, \mathbf{A}) \bigcap h{-}cl(\mathcal{F}, \mathbf{B}) = \Phi$ and $h{-}cl(\mathcal{F}, \mathbf{A}) \bigcap (\mathcal{F}, \mathbf{B}) = \Phi$.

Note that any two separated hypersoft sets are disjoint hypersoft sets.

Definition 2.9. [22] A HTS $((\mathcal{F}, \mathbf{A}), \tau)$ is called hypersoft disconnected iff $(\mathcal{F}, \mathbf{A})$ can be expressed as the union of two non-null separated hypersoft sets. Otherwise, $((\mathcal{F}, \mathbf{A}), \tau)$ is called hypersoft connected."

3. Hypersoft Semi-Neighborhoods

Definition 3.1. A hypersoft set $(\Omega, \mathfrak{A})$ in a HTS $((\Omega, \mathcal{P}), \tau)$ is called a hypersoft semi-neighborhood of the hypersoft point $(x, \Omega_{\mathfrak{A}}(x))$ if there is a HSO set $(\Omega, \mathfrak{B}) \ni (x, \Omega_{\mathfrak{A}}(x)) \in (\Omega, \mathfrak{B}) \subseteq (\Omega, \mathfrak{A})$.

The semi-neighborhood system of a hypersoft point $(x, \Omega_{\mathfrak{A}}(x))$ which is denoted by $\mathcal{H}^{\mathcal{N}}{}_{(x, \Omega_{\mathfrak{A}}(x))}$ is the set of all its semi-neighborhoods.

Example 3.2. Let $Y = \{y_1, y_2, y_3\}$, $P_1 = \{p_1, p_2\}$, $P_2 = \{p_3\}$, $P_3 = \{p_4\}$ and let Ω is a mapping from $\mathcal{P} \longrightarrow P(Y)$.

Then the hypersoft set is defined by

$$(\Omega, \mathcal{P}) = \left\{ \begin{array}{c} (p_1, p_3, p_4), \{y_1, y_2\} \\ ((p_2, p_3, p_4), \{y_2, y_3\}) \end{array} \right\}$$

Consider all the hypersoft subsets of $(\Omega, \mathcal{P})$ as follows:

$(\Omega, \mathcal{P})_1 = \{(p_1, p_3, p_4), \{y_1\}\}$

$(\Omega, \mathcal{P})_2 = \{(p_1, p_3, p_4), \{y_2\}\}$

$(\Omega, \mathcal{P})_3 = \{(p_1, p_3, p_4), \{y_1, y_2\}\}$

$(\Omega, \mathcal{P})_4 = \{(p_2, p_3, p_4), \{y_2\}\}$

$(\Omega,\mathcal{P})_5 = \{(p_2,p_3,p_4),\{y_3\}\}$
$(\Omega,\mathcal{P})_6 = \{(p_2,p_3,p_4),\{y_2,y_3\}\}$
$(\Omega,\mathcal{P})_7 = \{((p_1,p_3,p_4),\{y_1\}),((p_2,p_3,p_4),\{y_2\})\}$
$(\Omega,\mathcal{P})_8 = \{((p_1,p_3,p_4),\{y_1\}),((p_2,p_3,p_4),\{y_3\})\}$
$(\Omega,\mathcal{P})_9 = \{((p_1,p_3,p_4),\{y_1\}),((p_2,p_3,p_4),\{y_2,y_3\})\}$
$(\Omega,\mathcal{P})_{10} = \{((p_1,p_3,p_4),\{y_2\}),((p_2,p_3,p_4),\{y_2\})\}$
$(\Omega,\mathcal{P})_{11} = \{((p_1,p_3,p_4),\{y_2\}),((p_2,p_3,p_4),\{y_3\})\}$
$(\Omega,\mathcal{P})_{12} = \{((p_1,p_3,p_4),\{y_2\}),((p_2,p_3,p_4),\{y_2,y_3\})\}$
$(\Omega,\mathcal{P})_{13} = \{((p_1,p_3,p_4),\{y_1,y_2\}),((p_2,p_3,p_4),\{y_2\})\}$
$(\Omega,\mathcal{P})_{14} = \{((p_1,p_3,p_4),\{y_1,y_2\}),((p_2,p_3,p_4),\{y_3\})\}$
$(\Omega,\mathcal{P})_{15} = (\Omega,\mathcal{P})$
$(\Omega,\mathcal{P})_{16} = (\Omega,\mathcal{P})_\Phi$.
$\tau = \{(\Omega,\mathcal{P}),(\Omega,\mathcal{P})_\Phi,(\Omega,\mathcal{P})_2,(\Omega,\mathcal{P})_{11},(\Omega,\mathcal{P})_{13}\}$
Then $((\Omega,\mathcal{P}),\tau)$ is a HTS. The family of all hypersoft open sets is
$\{(\Omega,\mathcal{P}),(\Omega,\mathcal{P})_\Phi,(\Omega,\mathcal{P})_2,(\Omega,\mathcal{P})_{11},(\Omega,\mathcal{P})_{13}\}$.
The family of all hypersoft closed sets is
$\{(\Omega,\mathcal{P})_\Phi,(\Omega,\mathcal{P}),(\Omega,\mathcal{P})_9,(\Omega,\mathcal{P})_7,(\Omega,\mathcal{P})_5\}$.
The family of all HSO sets is
$\{(\Omega,\mathcal{P})_\Phi,(\Omega,\mathcal{P}),(\Omega,\mathcal{P})_2,(\Omega,\mathcal{P})_3,(\Omega,\mathcal{P})_{10},(\Omega,\mathcal{P})_{11},(\Omega,\mathcal{P})_{12},(\Omega,\mathcal{P})_{13},(\Omega,\mathcal{P})_{14}\}$.
The family of all HSC sets is
$\{(\Omega,\mathcal{P})_\Phi,(\Omega,\mathcal{P}),(\Omega,\mathcal{P})_9,(\Omega,\mathcal{P})_6,(\Omega,\mathcal{P})_8,(\Omega,\mathcal{P})_7,(\Omega,\mathcal{P})_1,(\Omega,\mathcal{P})_5,(\Omega,\mathcal{P})_4\}$.
Consider the hypersoft point $\{(p_1,p_3,p_4),\{y_2\}\}$.
Then the family of hypersoft semi-neighborhood of $\{(p_1,p_3,p_4),\{y_2\}\}$ is
$\mathcal{H}^{\mathcal{N}}{}_{\{(p_1,p_3,p_4),\{y_2\}\}} = \{(\Omega,\mathcal{P})_{10},(\Omega,\mathcal{P})_{11},(\Omega,\mathcal{P})_{12},(\Omega,\mathcal{P})_{13},(\Omega,\mathcal{P})_{14},(\Omega,\mathcal{P})_{15}\}$

Proposition 3.3. *The semi-neighborhood system $\mathcal{H}^{\mathcal{N}}{}_{(x,\Omega_{\mathfrak{A}}(x))}$ at a hypersoft point $(x,\Omega_{\mathfrak{A}}(x))$ in the HTS $((\Omega,\mathcal{P}),\tau)$ has the following results.*

(1) *If $(\Omega,\mathfrak{A}) \in \mathcal{H}^{\mathcal{N}}{}_{(x,\Omega_{\mathfrak{A}}(x))}$, then $(x,\Omega_{\mathfrak{A}}(x)) \in (\Omega,\mathfrak{A})$*
(2) *If $(\Omega,\mathfrak{A}) \in \mathcal{H}^{\mathcal{N}}{}_{(x,\Omega_{\mathfrak{A}}(x))}$ and $(\Omega,\mathfrak{A}) \subseteq (\Omega,\mathfrak{B})$, then $(\Omega,\mathfrak{B}) \in \mathcal{H}^{\mathcal{N}}{}_{(x,\Omega_{\mathfrak{A}}(x))}$.*
(3) *If $(\Omega,\mathfrak{A}) \in \mathcal{H}^{\mathcal{N}}{}_{(x,\Omega_{\mathfrak{A}}(x))}$, then there exists a hypersoft set $(\Omega,\mathfrak{B}) \in \mathcal{H}^{\mathcal{N}}{}_{(x,\Omega_{\mathfrak{A}}(x))} \ni (\Omega,\mathfrak{A}) \in \mathcal{H}^{\mathcal{N}}{}_{(x_1,\Omega_{\mathfrak{A}}(x_1))}$ for each $(x_1,\Omega_{\mathfrak{A}}(x_1)) \in (\Omega,\mathfrak{B})$.*

proof:

(1) If $(\Omega,\mathfrak{A}) \in \mathcal{H}^{\mathcal{N}}{}_{(x,\Omega_{\mathfrak{A}}(x))}$, then we have a HSO set $(\Omega,\mathfrak{B}) \ni (x,\Omega_{\mathfrak{A}}(x)) \in (\Omega,\mathfrak{B}) \subseteq (\Omega,\mathfrak{A})$. So we have $(x,\Omega_{\mathfrak{A}}(x)) \in (\Omega,\mathfrak{A})$.

(2) If $(\Omega,\mathfrak{A}) \in \mathcal{H}^{\mathcal{N}}{}_{(x,\Omega_{\mathfrak{A}}(x))}$ and $(\Omega,\mathfrak{A}) \subseteq (\Omega,\mathfrak{B})$. Because $(\Omega,\mathfrak{A}) \in \mathcal{H}^{\mathcal{N}}{}_{(x,\Omega_{\mathfrak{A}}(x))}$, there exists a HSO set $(\Omega,\mathfrak{F}) \ni (x,\Omega_{\mathfrak{A}}(x)) \in (\Omega,\mathfrak{F}) \subseteq (\Omega,\mathfrak{A})$. So we have $(x,\Omega_{\mathfrak{A}}(x)) \in (\Omega,\mathfrak{F}) \subseteq (\Omega,\mathfrak{A}) \subseteq (\Omega,\mathfrak{B})$ and we have $(\Omega,\mathfrak{B}) \in \mathcal{H}^{\mathcal{N}}{}_{(x,\Omega_{\mathfrak{A}}(x))}$.

(3) If $(\Omega,\mathfrak{A}) \in \mathcal{H}^{\mathcal{N}}{}_{(x,\Omega_{\mathfrak{A}}(x))}$, then there is a HSO set $(\Omega,\mathfrak{F}) \ni (x,\Omega_{\mathfrak{A}}(x)) \in (\Omega,\mathfrak{F}) \subseteq (\Omega,\mathfrak{A})$. Let $(\Omega,\mathfrak{B}) = (\Omega,\mathfrak{F})$ and for each $(x_1,\Omega_{\mathfrak{A}}(x_1)) \in (\Omega,\mathfrak{B})$,
$(x_1,\Omega_{\mathfrak{A}}(x_1)) \in (\Omega,\mathfrak{B}) \subseteq (\Omega,\mathfrak{F}) \subseteq (\Omega,\mathfrak{A})$.
This shows that $(\Omega,\mathfrak{A}) \in \mathcal{H}^{\mathcal{N}}{}_{(x_1,\Omega_{\mathfrak{A}}(x_1))}$.

Definition 3.4. Let $((\Omega,\mathcal{P}),\tau)$ be a HTS and $\mathcal{H}^{\mathcal{N}}{}_{(x,\Omega_{\mathfrak{A}}(x))}$ be a hypersoft semi-neighborhood of a hypersoft point $(x,\Omega_{\mathfrak{A}}(x))$. If for every hypersoft semi-neighborhood $(\Omega,\mathfrak{A})$ of $(x,\Omega_{\mathfrak{A}}(x))$, there is a $(\Omega,\mathfrak{F}) \in \mathcal{H}^{\mathcal{B}}{}_{(x,\Omega_{\mathfrak{A}}(x))} \subseteq \mathcal{H}^{\mathcal{N}}{}_{(x,\Omega_{\mathfrak{A}}(x))} \ni (x,\Omega_{\mathfrak{A}}(x)) \in (\Omega,\mathfrak{F}) \subseteq (\Omega,\mathfrak{A})$, then $\mathcal{H}^{\mathcal{B}}{}_{(x,\Omega_{\mathfrak{A}}(x))}$ is called a hypersoft semi-neighborhoods base of $\mathcal{H}^{\mathcal{N}}{}_{(x,\Omega_{\mathfrak{A}}(x))}$ at $(x,\Omega_{\mathfrak{A}}(x))$.

Example 3.5. From example 3.2, a hypersoft semi-neighborhood base of $\mathcal{H}^{\mathcal{N}}{}_{\{(p_1,p_3,p_4),\{y_2\}\}}$ at $\{(p_1,p_3,p_4),\{y_2\}\}$ is $\mathcal{H}^{\mathcal{B}}{}_{\{(p_1,p_3,p_4),\{y_2\}\}} = (\Omega,\mathcal{P})_2$

Proposition 3.6. *Let $((\Omega,\mathcal{P}),\tau)$ be a HTS and $\mathcal{H}^{\mathcal{N}}{}_{(x,\Omega_{\mathfrak{A}}(x))}$ be a hypersoft semi-neighborhood of a hypersoft point $(x,\Omega_{\mathfrak{A}}(x))$. $\mathcal{H}^{\mathcal{B}}{}_{(x,\Omega_{\mathfrak{A}}(x))}$ is the hypersoft semi-neighborhoods base of $\mathcal{H}^{\mathcal{N}}{}_{(x,\Omega_{\mathfrak{A}}(x))}$ at $(x,\Omega_{\mathfrak{A}}(x))$. Then one has the following:*

(1) *If $(\Omega,\mathfrak{A}) \in \mathcal{H}^{\mathcal{B}}{}_{(x,\Omega_{\mathfrak{A}}(x))}$, then $(x,\Omega_{\mathfrak{A}}(x)) \in (\Omega,\mathfrak{A})$.*

(2) *If $(\Omega,\mathfrak{A}) \in \mathcal{H}^{\mathcal{B}}{}_{(x,\Omega_{\mathfrak{A}}(x))}$ and $(\Omega,\mathfrak{A}) \subseteq (\Omega,\mathfrak{B})$, then $(\Omega,\mathfrak{B}) \in \mathcal{H}^{\mathcal{B}}{}_{(x,\Omega_{\mathfrak{A}}(x))}$.*

(3) *If $(\Omega,\mathfrak{A}) \in \mathcal{H}^{\mathcal{B}}{}_{(x,\Omega_{\mathfrak{A}}(x))}$, then there exists a hypersoft set $(\Omega,\mathfrak{B}) \in \mathcal{H}^{\mathcal{B}}{}_{(x,\Omega_{\mathfrak{A}}(x))} \ni (\Omega,\mathfrak{A}) \in \mathcal{H}^{\mathcal{B}}{}_{(x_1,\Omega_{\mathfrak{A}}(x_1))}$ for each $(x_1,\Omega_{\mathfrak{A}}(x_1)) \in (\Omega,\mathfrak{B})$.*

proof: By looking into the relevant properties of hypersoft semi-neighborhoods in proposition 3.2, one can readily verify these properties.

Definition 3.7. Let $((\Omega,\mathcal{P}),\tau)$ be a HTS and $(x,\Omega_{\mathfrak{A}}(x))$ be a hypersoft point in $((\Omega,\mathcal{P}),\tau)$. If $(x,\Omega_{\mathfrak{A}}(x))$ has a countable hypersoft semi-neighborhoods base, then we say that $((\Omega,\mathcal{P}),\tau)$ is hypersoft semi-first-countable at $(x,\Omega_{\mathfrak{A}}(x))$. If each hypersoft point in $((\Omega,\mathcal{P}),\tau)$ is hypersoft semi-first-countable, then we say that $((\Omega,\mathcal{P}),\tau)$ is a hypersoft semi-first-countable space.

Example 3.8. From example 3.5, the hypersoft point $\{(p_1,p_3,p_4),\{y_2\}\}$ has a countable hypersoft semi-neighborhood base $(\Omega,\mathcal{P})_2$. Therefore a HTS $((\Omega,\mathcal{P}),\tau)$ is hypersoft semi-first-countable at $\{(p_1,p_3,p_4),\{y_2\}\}$.

Proposition 3.9. *Let $((\Omega,\mathcal{P}),\tau)$ be a HTS and $(x,\Omega_{\mathfrak{A}}(x))$ be a hypersoft point in $((\Omega,\mathcal{P}),\tau)$. Then $((\Omega,\mathcal{P}),\tau)$ is hypersoft semi-first countable at $(x,\Omega_{\mathfrak{A}}(x))$ iff there is a countable hypersoft semi-neighborhoods base $\{(\Omega_n,\mathfrak{A}), n \in N\}$ at $(x,\Omega_{\mathfrak{A}}(x)) \ni (\Omega_{n+1},\mathfrak{A}) \subseteq (\Omega_n,\mathfrak{A})$ for each $n \in N$.*

proof: ⇐Obvious.
⇒ Let $\{(\sigma_n, \mathfrak{A}), n \in N\}$ be a countable hypersoft semi-neighborhoods base at $(x, \Omega_{\mathfrak{A}}(x))$. For each $n \in N$ put $(\Omega_n, \mathfrak{A}) = \bigcap_{i=1}^{n} (\sigma_i, \mathfrak{A})$. Then it is easy to see that $\{(\Omega_n, \mathfrak{A}), n \in N\}$ is a hypersoft semi-neighborhoods base at $(x, \Omega_{\mathfrak{A}}(x))$ and $(\Omega_{n+1}, \mathfrak{A}) \subseteq (\Omega_n, \mathfrak{A})$ for each $n \in N$.

Definition 3.10. A hypersoft set $(\Omega, \mathfrak{A})$ in a HTS $((\Omega, \mathcal{P}), \tau)$ is called a hypersoft semi-neighborhood of the hypersoft set $(\Omega, \mathfrak{F})$ if there is a HSO set $(\Omega, \mathfrak{B}) \ni (\Omega, \mathfrak{F}) \subseteq (\Omega, \mathfrak{B}) \subseteq (\Omega, \mathfrak{A})$.

Example 3.11. From example 3.2, consider the hypersoft set $(\Omega, \mathcal{P})_1$. The family of all hypersoft semi-neighborhoods of the hypersoft set $(\Omega, \mathcal{P})_1 = \{(\Omega, \mathcal{P})_{13}, (\Omega, \mathcal{P})_{14}, (\Omega, \mathcal{P})_{15}\}$.

Proposition 3.12. *Let $((\Omega, \mathcal{P}), \tau)$ be a HTS, $(\Omega, \mathfrak{B})$ be a hypersoft set. Then $(\Omega, \mathfrak{B})$ is HSO iff for each hypersoft set $(\Omega, \mathfrak{A})$ contained in $(\Omega, \mathfrak{B})$, $(\Omega, \mathfrak{B})$ is a hypersoft semi-neighborhood of $(\Omega, \mathfrak{A})$.*

proof: ⇒Obvious
⇐ Because $(\Omega, \mathfrak{B}) \subseteq (\Omega, \mathfrak{B})$, then we have a HSO set $(\Omega, \mathfrak{F}) \ni (\Omega, \mathfrak{B}) \subseteq (\Omega, \mathfrak{F}) \subseteq (\Omega, \mathfrak{B})$. So we have $(\Omega, \mathfrak{B}) = (\Omega, \mathfrak{F})$ and $(\Omega, \mathfrak{B})$ is HSO.

Definition 3.13. Let the sequence $\{(\Omega_n, \mathfrak{A}), n \in N\}$ be a hypersoft sequence in a HTS $((\Omega, \mathcal{P}), \tau)$. Then $\{(\Omega_n, \mathfrak{A}), n \in N\}$ is eventually contained in a hypersoft set $(\Omega, \mathfrak{B})$ iff there is an integer $m \ni$ if $n \geq m$, then $(\Omega_n, \mathfrak{A}) \subseteq (\Omega, \mathfrak{B})$.
The sequence is frequently contained in $(\Omega, \mathfrak{B})$ iff for each integer m, there is an integer $n \ni n \geq m$ and $(\Omega_n, \mathfrak{A}) \subseteq (\Omega, \mathfrak{B})$.

Definition 3.14. Let the sequence $\{(\Omega_n, \mathfrak{A}), n \in N\}$ be a hypersoft sequence in a HTS $((\Omega, \mathcal{P}), \tau)$, then one says $\{(\Omega_n, \mathfrak{A}), n \in N\}$ semi converges to a hypersoft point $(x, \Omega_{\mathfrak{A}}(x))$ if it is eventually contained in each semi-neighborhood of $(x, \Omega_{\mathfrak{A}}(x))$.
The hypersoft point $(x, \Omega_{\mathfrak{A}}(x))$ is called a semi-cluster hypersoft point of $\{(\Omega_n, \mathfrak{A}), n \in N\}$ if the sequence is frequently contained in every semi-neighborhood of $(x, \Omega_{\mathfrak{A}}(x))$.

Proposition 3.15. *Let $((\Omega, \mathcal{P}), \tau)$ be a hypersoft semi-first countable, then one has the following:*

(1) *$(\Omega, \mathfrak{B})$ is HSO iff for every hypersoft sequence $\{(\Omega_n, \mathfrak{A}), n \in N\}$ which semi-converges to $(x, \Omega_{\mathfrak{A}}(x))$ in $(\Omega, \mathfrak{B})$ is eventually contained in $(\Omega, \mathfrak{B})$.*
(2) *If $(x, \Omega_{\mathfrak{A}}(x))$ is a semi-cluster hypersoft point of the hypersoft sequence $\{(\Omega_n, \mathfrak{A}), n \in N\}$, then one has a subsequence of $\{(\Omega_n, \mathfrak{A}), n \in N\}$ which semi converges to $(x, \Omega_{\mathfrak{A}}(x))$.*

Proof :

(1) Because $(\Omega, \mathfrak{B})$ is HSO, $(\Omega, \mathfrak{B})$ is a semi-neighborhood of $(x, \Omega_{\mathfrak{A}}(x))$, and $\{(\Omega_n, \mathfrak{A}), n \in N\}$ semi-converges to $(x, \Omega_{\mathfrak{A}}(x))$. Then we have $\{(\Omega_n, \mathfrak{A}), n \in N\}$ is eventually contained in $(\Omega, \mathfrak{B})$.

For each $(x, \Omega_{\mathfrak{A}}(x))$ contained in $(\Omega, \mathfrak{B})$, let $\{(\Omega_n, \mathfrak{A}), n \in N\}$ be the semi-neighborhood systems $\ni (\Omega_{n+1}, \mathfrak{A}) \subseteq (\Omega_n, \mathfrak{A})$ for each $n \in N$ by proposition 3.6.

Then $\{(\Omega_n, \mathfrak{A}), n \in N\}$ is eventually contained in each semi-neighborhood of $(x, \Omega_{\mathfrak{A}}(x))$. ie) $\{(\Omega_n, \mathfrak{A}), n \in N\}$ semi-converges to $(x, \Omega_{\mathfrak{A}}(x))$. So we have an integer $m \ni$ if $n \geq m$, $(\Omega_n, \mathfrak{A}) \subseteq (\Omega, \mathfrak{B})$. Then $(\Omega, \mathfrak{B})$ is semi-neighborhood of $(x, \Omega_{\mathfrak{A}}(x))$ and by proposition 3.8, $(\Omega, \mathfrak{B})$ is HSO.

(2) Let $\{(\Omega_n, \mathfrak{A}), n \in N\}$ be the semi-neighborhoods $\ni (\Omega_{n+1}, \mathfrak{A}) \subseteq (\Omega_n, \mathfrak{A})$ for each $n \in N$ by proposition 3.6. For every non-negative integer i, find n_i satisfies $n_i \geq i$ and $(\Omega_{n_i}, \mathfrak{A}) \subseteq (\Omega_i, \mathfrak{A})$. Then $\{(\Omega_{n_i}, \mathfrak{A}), i \in N\}$ is a subsequence of the sequence $\{(\Omega_n, \mathfrak{A}), n \in N\}$. Obviously this subsequence semi-converges to $(x, \Omega_{\mathfrak{A}}(x))$.

4. HSCO Spaces

Definition 4.1. Let $((\Omega, \mathcal{P}), \tau)$ be a HTS. A hypersoft semi separation on $(\Omega, \mathcal{P})$ is a pair $(\Omega, \mathfrak{A})$ and $(\Omega, \mathfrak{B})$ of non- null HSO sets $\ni (\Omega, \mathcal{P}) = (\Omega, \mathfrak{A}) \cup (\Omega, \mathfrak{B})$,$(\Omega, \mathfrak{A}) \cap (\Omega, \mathfrak{B}) = \Phi$. In otherwords, The two non-null hypersoft subsets $(\Omega, \mathfrak{A}), (\Omega, \mathfrak{B})$ of a HTS $((\Omega, \mathcal{P}), \tau)$ are said to be hypersoft semi-separated iff $h - scl(\Omega, \mathfrak{A}) \cap (\Omega, \mathfrak{B}) = \Phi$ and $(\Omega, \mathfrak{A}) \cap h - scl(\Omega, \mathfrak{B}) = \Phi$.

Definition 4.2. A HTS $((\Omega, \mathcal{P}), \tau)$ is called HSCO space if there is no hypersoft semi-separations on $(\Omega, \mathcal{P})$. If $((\Omega, \mathcal{P}), \tau)$ has hypersoft semi-separations then $((\Omega, \mathcal{P}), \tau)$ is called hypersoft semi disconnected space.

Example 4.3. From example 3.2, we see that $(\Omega, \mathcal{P})$ cannot be expressed as the union of two hypersoft semi-separated sets and hence $((\Omega, \mathcal{P}), \tau)$ is HSCO.

Theorem 4.4. *Let $((\Omega, \mathcal{P}), \tau)$ be a HTS. $(\Omega, \mathfrak{A})$ and $(\Omega, \mathfrak{B})$ are hypersoft semi-separations on $(\Omega, \mathcal{P})$. If $(\Omega, \mathfrak{F})$ is a HSCO subspace of $((\Omega, \mathcal{P}), \tau)$, then one has $(\Omega, \mathfrak{F}) \subseteq (\Omega, \mathfrak{A})$ or $(\Omega, \mathfrak{F}) \subseteq (\Omega, \mathfrak{B})$.*

proof: Since $(\Omega, \mathfrak{A})$ and $(\Omega, \mathfrak{B})$ are HSO sets, then we have $(\Omega, \mathfrak{A}) \cap (\Omega, \mathfrak{F})$ and $(\Omega, \mathfrak{B}) \cap (\Omega, \mathfrak{F})$ are also HSO sets. Hence $(\Omega, \mathfrak{A}) \cap (\Omega, \mathfrak{F})$ and $(\Omega, \mathfrak{B}) \cap (\Omega, \mathfrak{F})$ are hypersoft semi-separations of $(\Omega, \mathfrak{F})$ and this is contradiction. So one of $(\Omega, \mathfrak{A}) \cap (\Omega, \mathfrak{F})$ or $(\Omega, \mathfrak{B}) \cap (\Omega, \mathfrak{F})$ is Φ and thus $(\Omega, \mathfrak{F}) \subseteq (\Omega, \mathfrak{A})$ or $(\Omega, \mathfrak{F}) \subseteq (\Omega, \mathfrak{B})$.

Theorem 4.5. *Let $((\Omega, \mathcal{P}), \tau)$ be a HTS and $(\Omega, \mathfrak{F})$ is a HSCO subspace of $((\Omega, \mathcal{P}), \tau)$. If $(\Omega, \mathfrak{F}) \subseteq (\Omega, \mathfrak{C}) \subseteq \overline{(\Omega, \mathfrak{F})}$, then $(\Omega, \mathfrak{C})$ is HSCO.*

proof: Suppose $(\Omega, \mathfrak{C})$ is not HSCO, then there exist non-null HSO sets $(\Omega, \mathfrak{A})$ and $(\Omega, \mathfrak{B})$ which form a hypersoft semi-separation of $(\Omega, \mathfrak{C})$. Then by theorem 4.4, we have $(\Omega, \mathfrak{F}) \subseteq (\Omega, \mathfrak{A})$ or $(\Omega, \mathfrak{F}) \subseteq (\Omega, \mathfrak{B})$. Suppose $(\Omega, \mathfrak{F}) \subseteq (\Omega, \mathfrak{A})$, then we have $(\Omega, \mathfrak{C}) \subseteq \overline{(\Omega, \mathfrak{F})} \subseteq \overline{(\Omega, \mathfrak{A})}$. So $(\Omega, \mathfrak{C}) \cap (\Omega, \mathfrak{B}) \subseteq \overline{(\Omega, \mathfrak{A})} \cap (\Omega, \mathfrak{B}) = (\Omega, \mathfrak{A}) \cap (\Omega, \mathfrak{B}) = \Phi$. Hence $(\Omega, \mathfrak{B}) = (\Omega, \mathfrak{B}) \cap (\Omega, \mathfrak{C}) = \Phi$ a contradiction. Thus, we have $(\Omega, \mathfrak{C})$ is HSCO.

Theorem 4.6. *A HTS $((\Omega, \mathcal{P}), \tau)$ is HSCO iff the both HSO and HSC sets are only Φ and $(\Omega, \mathcal{P})$.*

proof: $\Rightarrow$ Let the HTS $((\Omega, \mathcal{P}), \tau)$ be HSCO if $(\Omega, \mathfrak{A})$ is both HSO and HSC in $((\Omega, \mathcal{P}), \tau)$ which is different from Φ and $(\Omega, \mathcal{P})$. So $(\Omega, \mathfrak{A})^c$ is HSO set in $((\Omega, \mathcal{P}), \tau)$ which is different from Φ and $(\Omega, \mathcal{P})$. Then $(\Omega, \mathfrak{A})$ and $(\Omega, \mathfrak{A})^c$ is a hypersoft semi-separation of $(\Omega, \mathcal{P})$. This is a contradiction. So the both HSO and HSC hypersoft sets are only Φ and $(\Omega, \mathcal{P})$.

$\Leftarrow$ Let $(\Omega, \mathfrak{A})$ and $(\Omega, \mathfrak{B})$ be a hypersoft semi-separation of $((\Omega, \mathcal{P}), \tau)$. Let $(\Omega, \mathfrak{A}) \neq (\Omega, \mathcal{P})$ and by definition $(\Omega, \mathfrak{A}) = (\Omega, \mathfrak{B})^c$. This proves that $(\Omega, \mathfrak{A})$ is both HSO and HSC in $((\Omega, \mathcal{P}), \tau)$ which is different from Φ and $(\Omega, \mathcal{P})$. This is a contradiction. So $((\Omega, \mathcal{P}), \tau)$ is HSCO.

Proposition 4.7. *A HTS $((\Omega, \mathcal{P}), \tau)$ is hypersoft semi-disconnected iff there exists non-null, non-whole hypersoft set which is both HSO and HSC.*

Proof: Let $(\Omega, \mathfrak{A})$ be a non-null, non-whole hypersoft set which is both HSO and HSC. We have to show that $((\Omega, \mathcal{P}), \tau)$ is hypersoft semi-disconnected. Let $(\Omega, \mathfrak{B}) = (\Omega, \mathfrak{A})^c$. Then, $(\Omega, \mathfrak{B})$ is non-null since $(\Omega, \mathfrak{A})$ is non-whole hypersoft set. Moreover, $(\Omega, \mathfrak{A}) \cup (\Omega, \mathfrak{B}) = (\Omega, \mathcal{P})$ and $(\Omega, \mathfrak{A}) \cap (\Omega, \mathfrak{B}) = \Phi$. Since $(\Omega, \mathfrak{A})$ is both HSO and HSC, then $(\Omega, \mathfrak{B})$ is also HSO and HSC. Hence $h - scl(\Omega, \mathfrak{A}) = (\Omega, \mathfrak{A})$, $h - scl(\Omega, \mathfrak{B}) = (\Omega, \mathfrak{B})$. It follows that $(\Omega, \mathfrak{A}) \cap h - scl(\Omega, \mathfrak{B}) = \Phi$ and $h - scl(\Omega, \mathfrak{A}) \cap (\Omega, \mathfrak{B}) = \Phi$. Thus $(\Omega, \mathcal{P})$ has been expressed as the union of two hypersoft semi-separated sets and so $((\Omega, \mathcal{P}), \tau)$ is a hypersoft disconnected.

Conversely, let $((\Omega, \mathcal{P}), \tau)$ be a hypersoft semi-disconnected. Then there exists non-null hypersoft sets $(\Omega, \mathfrak{A})$ and $(\Omega, \mathfrak{B})$ suth that $(\Omega, \mathfrak{A}) \cap h - scl(\Omega, \mathfrak{B}) = \Phi$ and $h - scl(\Omega, \mathfrak{A}) \cap (\Omega, \mathfrak{B}) = \Phi$ and $(\Omega, \mathcal{P}) = (\Omega, \mathfrak{A}) \cup (\Omega, \mathfrak{B})$. Since $(\Omega, \mathfrak{A}) \subseteq h - scl(\Omega, \mathfrak{A})$, $h - scl(\Omega, \mathfrak{A}) \cap (\Omega, \mathfrak{B}) = \Phi$ then $(\Omega, \mathfrak{A}) \cap (\Omega, \mathfrak{B}) = \Phi$. Hence $(\Omega, \mathfrak{A}) = (\Omega, \mathfrak{B})^c$. Since $(\Omega, \mathfrak{B})$ is non-null and $(\Omega, \mathfrak{B}) \cup (\Omega, \mathfrak{B})^c = (\Omega, \mathcal{P})$, it follows that $(\Omega, \mathfrak{B}) = (\Omega, \mathfrak{A})^c$ is a non-whole hypersoft set. Now, $(\Omega, \mathfrak{A}) \cup h - scl(\Omega, \mathfrak{B}) = (\Omega, \mathcal{P})$. Also $(\Omega, \mathfrak{A}) \cap h - scl(\Omega, \mathfrak{B}) = \Phi$ then $(\Omega, \mathfrak{A}) = (h - scl(\Omega, \mathfrak{B}))^c$ and similarly $(\Omega, \mathfrak{B}) = (h - scl(\Omega, \mathfrak{A}))^c$. Since $h - scl(\Omega, \mathfrak{A})$ and $h - scl(\Omega, \mathfrak{B})$ are HSC sets, it follows that $(\Omega, \mathfrak{A})$ and $(\Omega, \mathfrak{B})$ are HSO sets. Since $(\Omega, \mathfrak{A}) = (\Omega, \mathfrak{B})^c$, $(\Omega, \mathfrak{A})$ is also HSC set. Thus, $(\Omega, \mathfrak{A})$ is non-null, non-whole hypersoft set which is both HSO and HSC. We have shown incidentally that $(\Omega, \mathfrak{B})$ is also a non-null, non-whole hypersoft set which is both HSO and HSC.

Theorem 4.8. *The union of a collection of HSCO subspaces of* $((\Omega,\mathcal{P}),\tau)$ *that have non-null intersection is HSCO.*

proof: Let $((\Omega,\mathfrak{A}_\alpha),\tau_{\mathfrak{A}_\alpha})_{\alpha\in J}$ be an arbitrary collection of hypersoft semi connected subspaces of $((\Omega,\mathcal{P}),\tau)$. Suppose that $(\Omega,\mathfrak{A})=\bigcup_{\alpha\in J}(\Omega,\mathfrak{A}_\alpha)=(\Omega,\mathfrak{B})\cup(\Omega,\mathfrak{C})$. Where $(\Omega,\mathfrak{B})$ and $(\Omega,\mathfrak{C})$ form a hypersoft semi-separation of $(\Omega,\mathfrak{A})$. By hypothesis, we may choose a hypersoft point $(x,\Omega_{\mathfrak{A}_\alpha}(x))\ni(x,\Omega_{\mathfrak{A}_\alpha}(x))\in\bigcap_{\alpha\in J}(\Omega,\mathfrak{A}_\alpha)$ and it must belong to either a hypersoft subset$(\Omega,\mathfrak{B})$ or a hypersoft subset $(\Omega,\mathfrak{C})$. Since $(\Omega,\mathfrak{B}),(\Omega,\mathfrak{C})$ are disjoint, we must have $(\Omega,\mathfrak{A}_\alpha)\subset(\Omega,\mathfrak{B})$ for all $\alpha\in J$, and so $(\Omega,\mathfrak{A})\subset(\Omega,\mathfrak{B})$. From this we obtain that $(\Omega,\mathfrak{C})=\Phi$, which is a contradiction. This proves the theorem.

Proposition 4.9. *If* $(\Omega,\mathfrak{A})$ *and* $(\Omega,\mathfrak{B})$ *are hypersoft semi-separated sets and* $(\Pi,\mathfrak{A})\subseteq(\Omega,\mathfrak{A})$ *and* $(\Pi,\mathfrak{B})\subseteq(\Omega,\mathfrak{B})$*, then* $(\Pi,\mathfrak{A})$ *and* $(\Pi,\mathfrak{B})$ *are also hypersoft semi-separated sets.*

Proof: We are given that $(\Omega,\mathfrak{A})\cap h-scl(\Omega,\mathfrak{B})=\Phi$ and $(\Omega,\mathfrak{B})\cap h-scl(\Omega,\mathfrak{A})=\Phi$. Also $(\Pi,\mathfrak{A})\subseteq(\Omega,\mathfrak{A})$ implies $h-scl(\Pi,\mathfrak{A})\subseteq h-scl(\Omega,\mathfrak{A})$ and $(\Pi,\mathfrak{B})\subseteq(\Omega,\mathfrak{B})$ implies $h-scl(\Pi,\mathfrak{B})\subseteq h-scl(\Omega,\mathfrak{B})$. It follows that $(\Pi,\mathfrak{A})\cap h-scl(\Pi,\mathfrak{B})=\Phi$ and $(\Pi,\mathfrak{A})\cap h-scl(\Pi,\mathfrak{B})=\Phi$. Hence $(\Pi,\mathfrak{A})$ and $(\Pi,\mathfrak{B})$ are hypersoft semi-separated sets.

Proposition 4.10. *Let* $(\Omega,\mathfrak{A})$ *and* $(\Omega,\mathfrak{B})$ *are hypersoft semi-separated sets of a HTS* $((\Omega,\mathcal{P}),\tau)$*. If* $(\Omega,\mathfrak{A})\cup(\Omega,\mathfrak{B})$ *is a HSC, then* $(\Omega,\mathfrak{A})$ *and* $(\Omega,\mathfrak{B})$ *are HSC.*

Proof: Suppose that $(\Omega,\mathfrak{A})\cup(\Omega,\mathfrak{B})$ is a HSC so that $h-scl((\Omega,\mathfrak{A})\cup(\Omega,\mathfrak{B}))=(\Omega,\mathfrak{A})\cup(\Omega,\mathfrak{B})$. To prove that $(\Omega,\mathfrak{A})$ and $(\Omega,\mathfrak{B})$ are HSC, we have to prove that $h-scl(\Omega,\mathfrak{A})=(\Omega,\mathfrak{A})$ and $h-scl(\Omega,\mathfrak{B})=(\Omega,\mathfrak{B})$. Since we have $h-scl((\Omega,\mathfrak{A})\cup(\Omega,\mathfrak{B}))=h-scl(\Omega,\mathfrak{A})\cup h-scl(\Omega,\mathfrak{B})$, then $h-scl(\Omega,\mathfrak{A})\cup h-scl(\Omega,\mathfrak{B})=(\Omega,\mathfrak{A})\cup(\Omega,\mathfrak{B})$.

Evidently, $h-scl(\Omega,\mathfrak{A})=h-scl(\Omega,\mathfrak{A})\cap(h-scl(\Omega,\mathfrak{A})\cup h-scl(\Omega,\mathfrak{B}))=h-scl(\Omega,\mathfrak{A})\cap((\Omega,\mathfrak{A})\cup(\Omega,\mathfrak{B}))=(h-scl((\Omega,\mathfrak{A})\cap(\Omega,\mathfrak{A}))\cup(h-scl((\Omega,\mathfrak{A})\cap(\Omega,\mathfrak{B}))=(\Omega,\mathfrak{A})\cup\Phi=(\Omega,\mathfrak{A})$. Thus $h-scl(\Omega,\mathfrak{A})=(\Omega,\mathfrak{A})$. Similarly, we can prove that $h-scl(\Omega,\mathfrak{B})=(\Omega,\mathfrak{B})$.

Proposition 4.11. *Let* $((\Omega,\mathcal{P}),\tau_1)$ *and* $((\Omega,\mathcal{P}),\tau_2)$ *be two HSCO spaces, then* $((\Omega,\mathcal{P}),\tau_1\cap\tau_2)$ *is a HSCO space.*

Proof: By contradiction, we assume that $((\Omega,\mathcal{P}),\tau_1\cap\tau_2)$ is not a HSCO space. Then there exist two non-empty disjoint HSO sets $(\Omega,\mathfrak{A}),(\Omega,\mathfrak{B})\in(\tau_1\cap\tau_2)\ni$ their union is $(\Omega,\mathcal{P})$ in $((\Omega,\mathcal{P}),\tau_1\cap\tau_2)$. This implies that $(\Omega,\mathfrak{A})$, $(\Omega,\mathfrak{B})$ are two non-null disjoint hypersoft sets $\ni$ their union is $(\Omega,\mathcal{P})$ in $((\Omega,\mathcal{P}),\tau_1)$ and $((\Omega,\mathcal{P}),\tau_2)$. Which is contradiction to given hypothesis. Thus $((\Omega,\mathcal{P}),\tau_1\cap\tau_2)$ is a HSCO space.

Proposition 4.12. *Let* $((\Omega,\mathcal{P}),\tau_1)$ *and* $((\Omega,\mathcal{P}),\tau_2)$ *be two hypersoft semi-disconnected space, then* $((\Omega,\mathcal{P}),\tau_1\cup\tau_2)$ *is a hypersoft semi-disconnected space.*

proof: This is straight forward.

Proposition 4.13. *Let* $((\Omega,\mathcal{P}),\tau_1)$ *and* $((\Omega,\mathcal{P}),\tau_2)$ *be two HTS. If* $((\Omega,\mathcal{P}),\tau_1)$ *is hypersoft semi-disconnected and* $\tau_1 \subseteq \tau_2$*, then* $((\Omega,\mathcal{P}),\tau_2)$ *is hypersoft semi-disconnected.*

Proof: Since $((\Omega,\mathcal{P}),\tau_1)$ is hypersoft semi-disconnected, then there exists non-null, non-whole hypersoft set $(\Omega,\mathfrak{A})$ which is both HSO and HSC in $((\Omega,\mathcal{P}),\tau_1)$. Since τ_2 is finer than τ_1, then $(\Omega,\mathfrak{A})$ is a HSO set belonging to $((\Omega,\mathcal{P}),\tau_2)$. Again, since $(\Omega,\mathfrak{A})$ is a HSC in $((\Omega,\mathcal{P}),\tau_1)$, then $(\Omega,\mathfrak{A})^c$ is a HSO set. Since τ_2 is finer than τ_1, then $(\Omega,\mathfrak{A})^c$ is a HSO set belonging to $((\Omega,\mathcal{P}),\tau_2)$ and consequently $(\Omega,\mathfrak{A})$ is a HSC set in $((\Omega,\mathcal{P}),\tau_2)$. Thus $(\Omega,\mathfrak{A})$ is a non-null, non-whole hypersoft set which is both HSO and HSC in $((\Omega,\mathcal{P}),\tau_2)$. If follows that $((\Omega,\mathcal{P}),\tau_2)$ is hypersoft semi-disconnected.

Corollary 4.14. *Let* $((\Omega,\mathcal{P}),\tau_1)$ *and* $((\Omega,\mathcal{P}),\tau_2)$ *be two HTS. If* $((\Omega,\mathcal{P}),\tau_1)$ *is HSCO and* $\tau_2 \subseteq \tau_1$*, then* $((\Omega,\mathcal{P}),\tau_2)$ *is HSCO.*

Definition 4.15. Let $((\Omega,\mathcal{P}),\tau)$ be a HTS and $(x,\Omega_{\mathfrak{A}}(x)) \in (\Omega,\mathfrak{A})$. The hypersoft semi component of $(x,\Omega_{\mathfrak{A}}(x))$ is the union of all hypersoft semi connected subsets of $(\Omega,\mathfrak{A})$ containing $(x,\Omega_{\mathfrak{A}}(x))$ and it is denoted by $HS.S.C(x,\Omega_{\mathfrak{A}}(x))$
From the above theorem, we see that the hypersoft semi component of $(x,\Omega_{\mathfrak{A}}(x))$ is HSCO.

Example 4.16. Consider a HTS $((\Omega,\mathcal{P}),\tau)$ in example 3.2, let $\{(p_1,p_3,p_4),\{y_2\}\} \in (\Omega,\mathcal{P})_{12}$. The hypersoft semi-connected subsets of $(\Omega,\mathcal{P})_{12}$ containing $\{(p_1,p_3,p_4),\{y_2\}\}$ are $(\Omega,\mathcal{P})_2,(\Omega,\mathcal{P})_{10},(\Omega,\mathcal{P})_{11}$.
The hypersoft semi-component of $\{(p_1,p_3,p_4),\{y_2\}\} = (\Omega,\mathcal{P})_2 \cup (\Omega,\mathcal{P})_{10} \cup (\Omega,\mathcal{P})_{11} = (\Omega,\mathcal{P})_{12}$.

Theorem 4.17. *In a HTS* $((\Omega,\mathcal{P}),\tau)$*,*

(1) *each hypersoft semi component of* $(x,\Omega_{\mathfrak{A}}(x))$ *is a maximal HSCO set in* $(\Omega,\mathcal{P})$*.*
(2) *the set of all distinct hypersoft semi components of hypersoft points of* $(\Omega,\mathcal{P})$ *form a partition of* $(\Omega,\mathcal{P})$*.*
(3) *each hypersoft semi-component of* $(x,\Omega_{\mathfrak{A}}(x))$ *is HSC in* $(\Omega,\mathcal{P})$*.*

proof:

(1) From the definition, the proof is obvious.
(2) Let $HS.S.C(x,\Omega_{\mathfrak{A}}(x))$ and $HS.S.C(x_1,\Omega_{\mathfrak{A}}(x_1))$ be two hypersoft semi components of distinct hypersoft points $(x,\Omega_{\mathfrak{A}}(x))$ and $(x_1,\Omega_{\mathfrak{A}}(x_1))$ in $(\Omega,\mathcal{P})$. If $HS.S.C(x,\Omega_{\mathfrak{A}}(x)) \cap HS.S.C(x_1,\Omega_{\mathfrak{A}}(x_1)) \neq \Phi$, then by theorem 4.8, $HS.S.C(x,\Omega_{\mathfrak{A}}(x)) \cup HS.S.C(x_1,\Omega_{\mathfrak{A}}(x_1))$ is HSCO. But $HS.S.C(x,\Omega_{\mathfrak{A}}(x)) \subset HS.S.C(x,\Omega_{\mathfrak{A}}(x)) \cup HS.S.C(x_1,\Omega_{\mathfrak{A}}(x_1))$, which contradicts the maximality of $HS.S.C(x,\Omega_{\mathfrak{A}}(x))$. Now for any hypersoft point $(x,\Omega_{\mathfrak{A}}(x)) \in (\Omega,\mathcal{P}),(x,\Omega_{\mathfrak{A}}(x)) \in$

$HS.S.C(x, \Omega_{\mathfrak{A}}(x))$

and $\bigcup_{(x,\Omega_{\mathfrak{A}}(x))\in(\Omega,\mathcal{P})} (x, \Omega_{\mathfrak{A}}(x)) \subset \bigcup_{(x,\Omega_{\mathfrak{A}}(x))\in(\Omega,\mathcal{P})} HS.S.C(x, \Omega_{\mathfrak{A}}(x))$.

This implies that $(x, \Omega_{\mathfrak{A}}(x)) \in \bigcup_{(x,\Omega_{\mathfrak{A}}(x))\in(\Omega,\mathcal{P})} HS.S.C(x, \Omega_{\mathfrak{A}}(x)) \subseteq (\Omega, \mathcal{P})$.

(3) Let $(x, \Omega_{\mathfrak{A}}(x))$ be any hypersoft point in $(\Omega, \mathcal{P})$. Then $h - scl(HS.S.C(x, \Omega_{\mathfrak{A}}(x)))$is a HSCO set containing $(x, \Omega_{\mathfrak{A}}(x))$. But $HS.S.C(x, \Omega_{\mathfrak{A}}(x))$ is the maximal HSCO set containing $(x, \Omega_{\mathfrak{A}}(x))$. so $h - scl(HS.S.C(x, \Omega_{\mathfrak{A}}(x))) \subset HS.S.C(x, \Omega_{\mathfrak{A}}(x))$. Hence $HS.S.C(x, \Omega_{\mathfrak{A}}(x))$ is HSC.

5. **HLSCO Spaces**

Definition 5.1. A HTS $((\Omega, \mathcal{P}), \tau)$ is called HLSCO at $(x, \Omega_{\mathfrak{A}}(x)) \in (\Omega, \mathcal{P})$ iff for every HSO set (Ω, U) containing $(x, \Omega_{\mathfrak{A}}(x))$ there exists a HSCO open set $(\Omega, \mathfrak{C}) \ni$
$(x, \Omega_{\mathfrak{A}}(x)) \in (\Omega, \mathfrak{C}) \subset (\Omega, U)$.
$((\Omega, \mathcal{P}), \tau)$ is called HLSCO iff it is a HLSCO at every hypersoft point of $(\Omega, \mathcal{P})$.

Example 5.2. Let $((\Omega, \mathcal{P}), \tau)$ be the HTS, where $(\Omega, \mathcal{P})$ and its hypersoft subsets are consider as in example 3.2. Consider $\tau = \{(\Omega, \mathcal{P}), (\Omega, \mathcal{P})_\Phi, (\Omega, \mathcal{P})_2, (\Omega, \mathcal{P})_3, (\Omega, \mathcal{P})_{11}, (\Omega, \mathcal{P})_{12}, (\Omega, \mathcal{P})_{14}\}$
Then $((\Omega, \mathcal{P}), \tau)$ is a HTS.
The family of all HSO sets is
$\{(\Omega, \mathcal{P})_\Phi, (\Omega, \mathcal{P}), (\Omega, \mathcal{P})_2, (\Omega, \mathcal{P})_3, (\Omega, \mathcal{P})_{10}, (\Omega, \mathcal{P})_{11}, (\Omega, \mathcal{P})_{12}, (\Omega, \mathcal{P})_{13}, (\Omega, \mathcal{P})_{14}\}$.
The family of all HSC sets is
$\{(\Omega, \mathcal{P})_\Phi, (\Omega, \mathcal{P}), (\Omega, \mathcal{P})_9, (\Omega, \mathcal{P})_6, (\Omega, \mathcal{P})_8, (\Omega, \mathcal{P})_7, (\Omega, \mathcal{P})_1, (\Omega, \mathcal{P})_5, (\Omega, \mathcal{P})_4\}$.
The HSO sets containing $((p_1, p_3, p_4), \{y_1, y_2\})$ are $(\Omega, \mathcal{P})_{13}, (\Omega, \mathcal{P})_{14}$ and $(\Omega, \mathcal{P})$. Clearly the hypersoft set $(\Omega, \mathcal{P})_3$ is HSCO and open. Therefore $(\Omega, \mathcal{P})$ is HLSCO at $((p_1, p_3, p_4), \{y_1, y_2\})$. The HSO sets containing $((p_2, p_3, p_4), \{y_2, y_3\})$ are $(\Omega, \mathcal{P})_{12}$ and $(\Omega, \mathcal{P})$. We can easily show that $(\Omega, \mathcal{P})_{12}$ is HSCO. Since $(\Omega, \mathcal{P})_{12}$ is HSCO and hypersoft open therefore $(\Omega, \mathcal{P})$ is HLSCO at $((p_2, p_3, p_4), \{y_2, y_3\})$. Therefore $(\Omega, \mathcal{P})$ is HLSCO.

Theorem 5.3. *A HTS $((\Omega, \mathcal{P}), \tau)$ is HLSCO iff the hypersoft semi-components of HSO set are hypersoft open sets.*

proof: Suppose that $((\Omega, \mathcal{P}), \tau)$ is HLSCO. Let $(\Omega, \mathfrak{A}) \subset (\Omega, \mathcal{P})$ be HSO and $(\Omega, \mathfrak{B})$ be a hypersoft semi-component of $(\Omega, \mathfrak{A})$. If $(x, \Omega_{\mathfrak{A}}(x)) \in (\Omega, \mathfrak{B})$, then because $(x, \Omega_{\mathfrak{A}}(x)) \in (\Omega, \mathfrak{A})$, there is a HSCO open set $(\Omega, U) \ni (x, \Omega_{\mathfrak{A}}(x)) \in (\Omega, U) \subset (\Omega, \mathfrak{A})$. Since $(\Omega, \mathfrak{B})$ is the hypersoft semi component of $(x, \Omega_{\mathfrak{A}}(x))$ and (Ω, U) is HSCO, we have $(x, \Omega_{\mathfrak{A}}(x)) \in (\Omega, U) \subset (\Omega, \mathfrak{B})$. This shows that $(\Omega, \mathfrak{B})$ is hypersoft open.
Conversely, let $(x, \Omega_{\mathfrak{A}}(x)) \in (\Omega, \mathcal{P})$ be arbitrary and let $(\Omega, \mathfrak{A})$ be a HSO set containing

$(x, \Omega_{\mathfrak{A}}(x))$. Let $(\Omega, \mathfrak{B})$ be the hypersoft semi component of $(\Omega, \mathfrak{A}) \ni (x, \Omega_{\mathfrak{A}}(x)) \in (\Omega, \mathfrak{B})$. Now $(\Omega, \mathfrak{B})$ is HSCO open set $\ni (x, \Omega_{\mathfrak{A}}(x)) \in (\Omega, \mathfrak{B}) \subset (\Omega, \mathfrak{A})$. This proves the theorem.

Theorem 5.4. *A HTS $((\Omega, \mathcal{P}), \tau)$ is HLSCO iff given any hypersoft point $(x, \Omega_{\mathfrak{A}}(x)) \in (\Omega, \mathfrak{A})$ and a HSO set (Ω, U) containing $(x, \Omega_{\mathfrak{A}}(x))$, there is a hypersoft open set $(\Omega, \mathfrak{B})$ containing $(x, \Omega_{\mathfrak{A}}(x)) \ni (\Omega, \mathfrak{B})$ is contained in a single hypersoft semi component of (Ω, U).*

proof: Let $(\Omega, \mathcal{P})$ be HLSCO, $(x, \Omega_{\mathfrak{A}}(x)) \in (\Omega, \mathcal{P})$ and (Ω, U) be a HSO set containing $(x, \Omega_{\mathfrak{A}}(x))$. Let (Ω, V) be a hypersoft semi component of (Ω, U) that contains $(x, \Omega_{\mathfrak{A}}(x))$. Since $(\Omega, \mathcal{P})$ is HLSCO and (Ω, U) is HSO, there is a HSCO open set $(\Omega, \mathfrak{B}) \ni (x, \Omega_{\mathfrak{A}}(x))$ and so $(x, \Omega_{\mathfrak{A}}(x)) \in (\Omega, \mathfrak{B}) \subset (\Omega, V) \subset (\Omega, U)$. Since hypersoft semi components are disjoint sets, it follows that $(\Omega, \mathfrak{B})$ is not contained in any other hypersoft semi component of (Ω, U).

Conversely, we suppose that given any hypersoft point $(x, \Omega_{\mathfrak{A}}(x)) \in (\Omega, \mathcal{P})$ and any HSO set (Ω, U) containing $(x, \Omega_{\mathfrak{A}}(x))$, there is a hypersoft open set $(\Omega, \mathfrak{B})$ containing $(x, \Omega_{\mathfrak{A}}(x))$ which is contained in a single hypersoft semi component $(\Omega, \mathfrak{F})$ of (Ω, U). Then $(x, \Omega_{\mathfrak{A}}(x)) \in (\Omega, \mathfrak{B}) \subset (\Omega, \mathfrak{F}) \subset (\Omega, U)$. Let $(x_1, \Omega_{\mathfrak{A}}(x_1)) \in (\Omega, \mathfrak{F})$, then $(x_1, \Omega_{\mathfrak{A}}(x_1)) \in (\Omega, U)$. Thus there is a hypersoft open set $(\Omega, \mathfrak{Q}) \ni (x_1, \Omega_{\mathfrak{A}}(x_1)) \in (\Omega, \mathfrak{Q})$ and $(\Omega, \mathfrak{Q})$ is contained in a single hypersoft semi component of (Ω, U). As the hypersoft semi components are disjoint hypersoft sets and $(x_1, \Omega_{\mathfrak{A}}(x_1)) \in (\Omega, \mathfrak{F})$, $(x_1, \Omega_{\mathfrak{A}}(x_1)) \in (\Omega, \mathfrak{Q}) \subset (\Omega, \mathfrak{F})$. Thus $(\Omega, \mathfrak{F})$ is hypersoft open. Thus for every $(x, \Omega_{\mathfrak{A}}(x)) \in (\Omega, \mathfrak{F})$ and for every HSO set (Ω, U) containing $(x, \Omega_{\mathfrak{A}}(x))$, there is a HSCO open set $(\Omega, \mathfrak{F}) \ni (x, \Omega_{\mathfrak{A}}(x)) \in (\Omega, \mathfrak{F}) \subset (\Omega, U)$. Thus $((\Omega, \mathcal{P}), \tau)$ is HLSCO at $(x, \Omega_{\mathfrak{A}}(x))$. Since $(x, \Omega_{\mathfrak{A}}(x)) \in (\Omega, \mathcal{P})$ is arbitrary $((\Omega, \mathcal{P}), \tau)$ is HLSCO. This proves the theorem.

Remark 5.5. HLSCO does not imply HSCO as shown by the following example.

Example 5.6. From example 5.2, $(\Omega, \mathcal{P})$ is HLSCO. But $(\Omega, \mathcal{P})$ can be expressed as the union of two hypersoft semi separated sets $(\Omega, \mathcal{P})_4$ and $(\Omega, \mathcal{P})_9$. Therefore $(\Omega, \mathcal{P})$ is not HSCO.

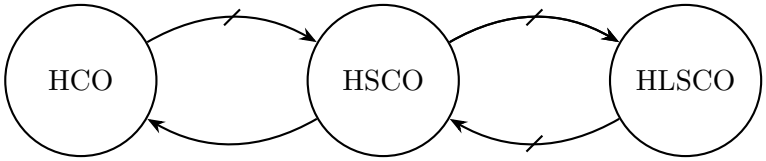
Remark 5.7. HSCO does not imply HLSCO as shown by the following example.

Example 5.8. From example 4.3, $((\Omega, \mathcal{P}), \tau)$ is HSCO. We show that it is HLSCO. Here $(\Omega, \mathcal{P})_3$ is a hypersoft semi-open set containing $((p_1, p_3, p_4), \{y_1, y_2\})$. But there is no hypersoft open subset of $(\Omega, \mathcal{P})_3$ containing $((p_1, p_3, p_4), \{y_1, y_2\})$ and so $((\Omega, \mathcal{P}), \tau)$ is not HLSCO at $((p_1, p_3, p_4), \{y_1, y_2\})$. Therefore $((\Omega, \mathcal{P}), \tau)$ is not HLSCO.

Remark 5.9. Since every HSCO is HCO. But the converse is not true as shown by the following example.

Example 5.10. It is evident from example 5.2 that $((\Omega, \mathcal{P}), \tau)$ is HCO but it is not HSCO.

Remark 5.11. Below is an outline of our research results from the earlier examples:



6. Conclusion

In this study, the concepts of hypersoft semi-neighbourhood of hypersoft point, hypersoft semi-first countable spaces, semi-converges, and semi-cluster hypersoft point were introduced. After that, using HSO sets, we have defined hypersoft semi-separation, HSCO, and HLSCO for strengthening the field of hypersoft set theory. In this work, we also demonstrated the relationship between the HCO, HSCO, and HLSCO spaces through examples. Future research may investigate the concepts of hypersoft pre-separation, hypersoft pre-neighborhood, hypersoft pre-connectedness, and hypersoft locally pre-connectedness. The resulting data will be immensely beneficial for future researchers to improve this work on the Neutrosophic hypersoft set, Plithogenic hypersoft set, Indetermhypersoft set, and Superhypersoft set.

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Conflict of interest :

The authors declare that there is no conflict of interest in the research.

Ethical approval :

This article does not contain any studies with human participants or animals performed by any of the authors.

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A Tree Soft Set Framework for Evaluating Teaching Quality in University Physics Programs: Enhancing Precision and Decision-Making

Hongzhi Zhao[1*], Fengqiang Zhang[2]

[1]Physics and Electronic Engineering Department, Xinxiang University, Xinxiang, 453003, Henan, China

[2]Xinxiang NO.1 Middle School, Xinxiang, 453003, Henan, China

*Corresponding author E-mail: zhhzh007@xxu.edu.cn; wuli2009@126.com

Abstract: Education in physics is at a crossroads. Numerous nations have middling or worse levels of scientific literacy, according to international research, and their students are viewed as being ill-equipped to handle the challenges going forward. The governmental level has acknowledged the necessity of high-quality development. The article focuses on evaluating physics education is taught and learned through experiments and real-world experiences. We propose a multi-criteria decision making (MCDM) approach to deal with various factors in evaluation of teaching quality in physics programs. We integrate the MCDM method with the Tree Soft Set (TSS) to show the relationship between the different nodes. The root node is the main objective in this study, the first level the main factors, and the second level is the sub factors. The MCDM is used with the single valued neutrosophic sets (SVNSs) to deal with vague data. We gathered five main factors and 15 sub factors in this equation. We compute the factors weights using the AHP method to build the pairwise comparison matrix to evaluate them.

Keywords: AHP Method; MCDM; University Physics Programs; Neutrosophic Sets; Teaching Quality.

1. Introduction and literature Review

Science, technology, and culture are changing dramatically as we approach the next millennium. Rapid advances in a variety of fields are leading to scientific and technical breakthroughs that have a profound impact on our daily lives and change our society. There is no immediate end in sight to the significant changes brought about by innovations in numerous industries, such as information and communication technology. A shift towards what is referred to as the "knowledge society" is now taking place in post-

industrial society. Our society' most valuable resource and a major determinant of political decision-making is knowledge[1-2].

As an academic discipline, physics has a long and successful history of generating new knowledge that informs technological advancement and relates to a wide variety of human experience sizes. A new worldview that questions both naïve ideas and long-held philosophical convictions was produced by the physics program[3-4].

New information and communication technologies that have fundamentally altered our lives in recent decades are based on physics. Working at the forefront of emerging technology, physicists are likely to bring about even more significant developments. One important area for anticipated advancements in both physics and biosciences is the fascinating field of nanotechnology. Building models with a physical inspiration has helped people in various professions gain a better grasp of intricate processes[5-6].

Like most universalists, physicists excel in a wide range of professional domains, including the economy, the geo- and environmental sciences, engineering, computer science, medicine, and life sciences, to name a few[7-8]. Physics training's professional openness and universal orientation encompass a wide range of knowledge and skills that are highly beneficial not only in science fields closely related to physics but also in a wide range of subject areas, including those with a very "unphysical" appeal, such as risk assessment in insurance companies or stock market data analysis[9-10].

The multi-criteria decision making (MCDM) process must be understood before talking about AHP. MCDM techniques are useful for making significant judgments that are difficult to decide on directly. These days, a variety of MCDM techniques are accessible for choice. Understanding the broad features of many approaches is essential to choose the best one for this investigation[11-12].

It is evident that AHP performs very well when handling interdependent criteria and local difficulties that involve both quantitative and qualitative concerns. Since urban revitalization is frequently seen as a social issue, it is anticipated that AHP will be an appropriate methodology for this study [13-14]. AHP is one of the MCDM approaches and the essential premise of MCDM is that these judgments have to be made by means of sets of criteria. AHP, which represents a hierarchical decision issue framework with many tiers of criteria with unidirectional linkages, was established by Saaty by using this idea. The hierarchy used by AHP can incorporate both objective (physical) and subjective (intangible) elements[15-16].

Smarandache's neutrosophic sets expand on Atanassov's intuitionistic fuzzy sets (IFSs) notion, offering a fresh perspective on ambiguity, imprecision, inconsistency, and uncertainty. Smarandache defined a neutrosophic set with three components: truth membership, indeterminacy membership, and falsity membership [18-17]. He also established the degree of indeterminacy/neutrality as a new and independent component of fuzzy sets. Better outcomes can be obtained when neutrosophic sets are used in decision-making because the indeterminacy parameter aids in a more thorough description of membership functions. However, using a neutrosophic set in actual scientific and technical domains is more difficult[17-19].

In logic, neutrophilic logic is highly helpful in differentiating between absolute truth and relative truth, absolute membership and relative membership, absolute non-membership and relative non-membership, and absolute truth and relative truth. A decision-maker does not have to be convinced that the total of the components in a membership function for a particular event should equal one when neutrosophic sets are favored. The total might rise to three if those components are independent[20-21].

1.1 Gaps in Literature

Despite advancements in decision-making techniques, the following gaps remain:

1. Limited use of TSS and SVNSs in teaching quality evaluations.
2. Insufficient focus on physics-specific challenges in existing frameworks.
3. Lack of comparative studies integrating multiple MCDM methods.

1.2 Novelty of the Proposed Methodology

The proposed methodology integrates Analytical Hierarchy Process (AHP), Single-Valued Neutrosophic Sets (SVNSs), and Tree Soft Sets (TSS) for evaluating teaching quality in university physics programs. While MCDM techniques have been applied extensively in education quality assessment, this work advances the field by addressing several critical gaps:

a. Unlike conventional MCDM methods, the Tree Soft Set (TSS) approach effectively handles hierarchical relationships between main factors and sub-factors, offering a clear visualization of dependencies.
b. The incorporation of SVNSs allows for precise modeling of vagueness and uncertainty in expert evaluations, providing a more nuanced representation than fuzzy sets.

c. This study represents one of the few instances where AHP, SVNSs, and TSS are combined, leveraging their individual strengths to create a robust and adaptable evaluation framework.

2. Tree Soft Set Approach (TSS)

The method of the TSS is developed by Smarandache who developed the neutrosophic uncertainty framework to deal with vague data in evaluation process. The main goal of TSS is to determine the relationship between the factors and sub factors in the MCDM methodology[22-23].

Let the × be a universe of discourse and y a non-empty subset of x, with the powerset of y p(y).

Suppose x be a set of factors for main nodes as $x = \{x_1, x_2, \ldots., x_n\}$ where $n \geq 1$ and considering factors of x in the first level.

The sub factors are in the second level as $\{x_{11}, x_{1-2}, \ldots., x_{1-n}\}$

The root node is at zero level and then level 1 and level 2 up to level n.

2.1 Applications of TSS in Decision-Making

The strengths of TSS have enabled its application across various fields, showcasing its adaptability and effectiveness in diverse decision-making scenarios.

2.1.1 Educational Evaluations

TSS has been widely applied in the educational sector to evaluate teaching quality, curriculum design, and student performance. For instance, it has been used to assess university-level teaching programs, incorporating hierarchical structures such as main factors (e.g., student engagement) and sub-factors (e.g., communication and feedback mechanisms). By capturing the nuanced relationships between these criteria, TSS ensures a comprehensive evaluation framework that aligns with institutional goals

2.1.2. Healthcare Prioritization

In the healthcare domain, TSS has been employed to prioritize medical interventions and allocate resources effectively. For example, a study used TSS to rank healthcare policies based on criteria such as cost-effectiveness, patient satisfaction, and accessibility. The hierarchical structure of TSS allowed policymakers to weigh these factors at different levels, leading to informed and balanced decisions

2.1.3. Industrial Decision-Making

The flexibility of TSS has also made it a valuable tool in industrial settings. It has been applied to evaluate suppliers, optimize production processes, and select project alternatives. For example, TSS was used to rank suppliers based on criteria like quality, delivery time, and cost. By integrating subjective assessments with measurable data, TSS provided a holistic decision-making framework that addressed both operational and strategic objectives

2.2. Justification for TSS Integration

The Tree Soft Set framework is particularly well-suited for problems involving hierarchical relationships and multi-level dependencies. Its integration with AHP in this study is justified by the following advantages:

a. TSS organizes evaluation criteria into levels (e.g., root node, main factors, sub-factors), simplifying the complexity of decision-making problems.
b. This hierarchical organization improves interpretability for decision-makers.
c. TSS handles imprecise and vague data efficiently, especially when combined with SVNSs.

2.2 Comparative Strengths

Compared to other MCDM techniques (e.g., fuzzy sets or TOPSIS), TSS excels in handling multi-layered problems, ensuring a comprehensive evaluation process. Table 1 illustrates a comparative highlights the unique capabilities of TSS, particularly in the context of hierarchical and uncertain decision-making (see Table 1).

Table 1: Advantages of TSS over Other MCDM Methods

Method	Handles Hierarchical Structure	Handles Vagueness	Simplifies Dependencies	Ease of Visualization
TSS	✓	✓	✓	✓
Fuzzy Sets	✗	✓	✗	✗
TOPSIS	✗	✗	✗	✗
AHP Alone	✓	✗	✗	✓

2.3 Role of SVNSs in Handling Vagueness

Single-Valued Neutrosophic Sets (SVNSs) extend fuzzy sets by incorporating three independent components:

Truth Membership (T): Degree to which a statement is true.

Indeterminacy Membership (I): Degree of uncertainty or ambiguity.

Falsity Membership (F): Degree to which a statement is false.

These components allow SVNSs to model uncertainty more comprehensively.
SVNSs capture vagueness and contradictions in expert evaluations, enabling a more realistic representation of subjective judgments. For example, a physics program's experimental learning component might be rated with T = 0.7, I = 0.2, F = 0.1, indicating strong agreement with minor ambiguity.

2.3.1. Advantages of SVNSs

1. Captures ambiguity in expert inputs.
2. Provides a flexible framework for multi-criteria evaluations.
3. Integrates seamlessly with TSS for hierarchical modeling.

2.4. Criteria Selection and Relevance

The study evaluates teaching quality based on five main factors and 15 sub-factors. Their selection is grounded in literature review and expert consultations, ensuring relevance to physics education:

2.5 Main Factors and Sub-Factors

To effectively evaluate teaching quality in university physics programs, it is essential to structure the evaluation criteria in a hierarchical manner. This approach allows for a comprehensive assessment that encompasses both high-level educational dimensions and detailed measurable attributes. In this section, the **m**ain factors and their corresponding sub-factors are presented. These factors represent critical aspects of teaching quality, while the sub-factors delve deeper into specific, actionable elements under each main factor.
The hierarchical structure of these criteria is designed to reflect the multifaceted nature of teaching quality. By breaking down each main factor into detailed sub-factors, the framework ensures precision, clarity, and relevance to the study's objectives. This structure also enables the identification of strengths and weaknesses across different aspects of teaching quality, providing insights for targeted improvements. Figure 1 below illustrates the relationship between the main factors and their associated sub-factors, structured in alignment with the study's objectives. The figure highlights how the criteria are interconnected and organized to provide a systematic approach to evaluation.

Figure 1 demonstrates the hierarchical framework used for evaluating teaching quality in university physics programs, showing the relationships between the overall objective, the main factors, and their associated sub-factors. At the top of the hierarchy is the root node, which represents the primary goal: Teaching Quality. This goal is supported by five

main factors, each representing a critical dimension of teaching quality. Each main factor is further subdivided into sub-factors, making a total of 15 sub-factors. These sub-factors provide measurable and actionable criteria for evaluation.

The main factors and their associated sub-factors are as follows:

1. *Conceptual Understanding*
    1. **PS**: Problem-Solving – Assessing students' ability to approach and solve complex problems systematically.
    2. **TK**: Theoretical Knowledge – Evaluating the depth and breadth of students' understanding of core physics concepts.
2. *Experimental Learning*
    1. **LBS**: Lab-Based Skills – Measuring proficiency in conducting experiments and handling laboratory equipment.
    2. **HOE**: Hands-On Experiments – Emphasizing practical applications and experimental techniques.
3. *Use of Technology*
    1. **SIM**: Simulations – Utilizing computer-based simulations to model and understand physical phenomena.
    2. **VL**: Virtual Labs – Integrating virtual laboratory environments to enhance practical learning experiences.
4. *Curriculum Design*
    1. **REL**: Relevance – Ensuring the curriculum aligns with current scientific and industrial advancements.
    2. **ID**: Interdisciplinarity – Incorporating cross-disciplinary elements to provide a holistic educational experience.
5. *Teacher-Student Engagement*
    1. **COM**: Communication – Assessing the effectiveness of interactions between teachers and students.
    2. **MEN**: Mentoring – Evaluating the quality of guidance and support provided by instructors.
    3. **SM**: Student Motivation – Measuring the ability to inspire and sustain students' interest in physics.
    4. **FBM**: Feedback Mechanisms – Ensuring timely and constructive feedback to improve student learning.
    5. **CS**: Collaboration Skills – Encouraging teamwork and cooperative learning among students.
    6. **CT**: Critical Thinking – Promoting analytical thinking and problem-solving capabilities.
    7. **IP**: Innovative Practices – Assessing the use of creative teaching methods and approaches.

This hierarchical organization ensures that all critical aspects of teaching quality are systematically addressed. The root node, main factors, and sub-factors collectively provide a comprehensive framework, enabling precise evaluation and targeted improvements. By connecting the overarching objective to detailed sub-factors, Figure 1 visualizes how the evaluation process captures the multi-faceted nature of teaching quality in physics programs.

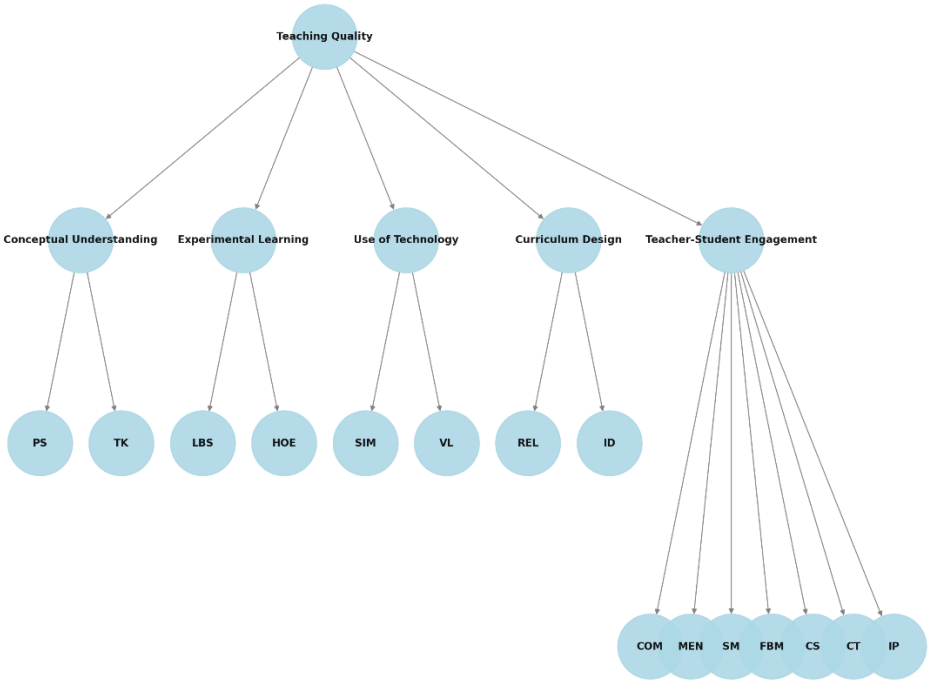

Figure 1: Hierarchical Representation of Main Factors and Sub-Factors

3. TSS-SVN-AHP approach

The TSS-SVN-AHP methodology integrates Tree Soft Sets (TSS), Single-Valued Neutrosophic Numbers (SVNNs), and the Analytical Hierarchy Process (AHP) to compute the criteria weights in a structured and hierarchical manner. This approach is particularly suited for addressing complex decision-making scenarios involving multiple criteria and sub-criteria, as it efficiently handles vagueness, uncertainty, and hierarchical relationships.

In this methodology, a set of experts or decision-makers (DMs) evaluate the factors and sub-factors relevant to the problem being addressed. These factors are organized into hierarchical levels, with the main factors acting as the initial nodes and the sub-factors branching out beneath them. This layered organization ensures a clear and logical representation of the criteria. Figure 2 illustrates the main steps of the proposed methodology.

Step 1, where decision-makers evaluate the criteria using a pairwise comparison matrix. The evaluations are expressed in terms of Single-Valued Neutrosophic Numbers (SVNNs), which capture three dimensions: truth membership (T), indeterminacy membership (I), and falsity membership (F). Each SVNN represents the degree of truth, uncertainty, and falsity in the decision-maker's judgment. To simplify these evaluations, the score function:

$S(X) = \frac{2+T-I-F}{3}$ is applied to obtain a single crisp value for each comparison. This step allows for the aggregation of subjective judgments into a standardized format.

In **Step 2**, the pairwise comparison matrix $Z = \begin{bmatrix} 1 & \cdots & x_{1n} \\ \vdots & \ddots & \vdots \\ x_{n1} & \cdots & 1 \end{bmatrix}$ is constructed, with each element represented by the aggregated SVNN values. This matrix quantifies the relative importance of one criterion compared to another. The diagonal elements of the matrix are always 1, reflecting the equal importance of a criterion itself. The matrix serves as the foundation for calculating weights in subsequent steps.

Step 3 involves normalizing the matrix. For each element $y_{ij} = \frac{x_{ij}}{\sum_{j=1}^{n} x_{ij}}$, the value is obtained by dividing the element in a given column by the sum of all elements in that column. This normalization ensures that the relative importance of each criterion is expressed proportionally and consistently. The normalized values are then used to compute the criteria weights.

In **Step 4**, the weights for each criterion are calculated using the average method. For each row in the normalized matrix, the average of the values is computed, resulting in the weight for the corresponding criterion. These weights reflect the relative significance of each criterion in the overall evaluation framework.

To ensure the reliability and consistency of the judgments, the methodology incorporates additional calculations in Step 5 through Step 7.

Step 5 calculates the eigenvalue $\lambda_{max}$of the pairwise comparison matrix by multiplying the matrix (Z) with the weight vector (W). The eigenvalue represents the degree of consistency in the judgments. $Zw^T = \lambda_{max} W^T$

Step 6 computes the Consistency Index (CI) using the formula:

$CI = \frac{\lambda_{max}-n}{n-1}$

Where n is the number of criteria. The CI measures the level of inconsistency in pairwise comparisons. Ideally, the CI should be close to zero, indicating highly consistent judgments.

In **Step 7**, the Consistency Ratio (CR) is calculated to validate the consistency of the evaluations. The CR is obtained using the formula:

$$CR = \frac{CI}{RI}$$

Where RI represents the Random Index, a pre-determined value based on the size of the matrix. A CR value less than 0.1 indicates that the judgments are consistent and reliable. If the CR exceeds this threshold, the decision-makers need to revise their evaluations to reduce inconsistencies.

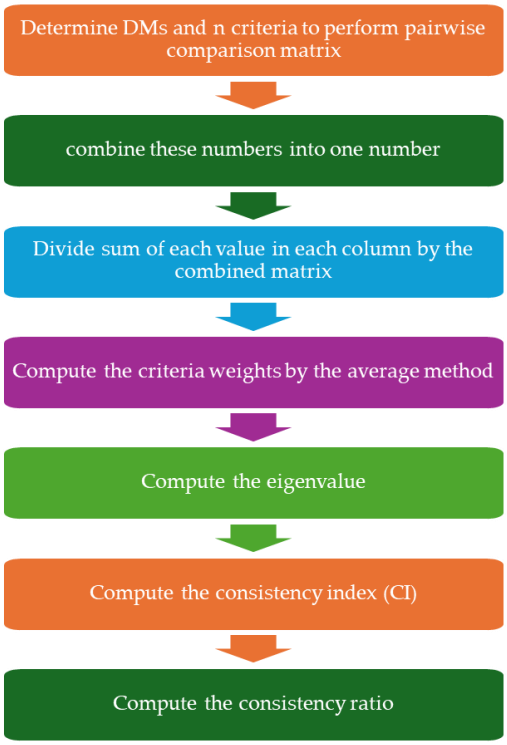
Determine DMs and n criteria to perform pairwise comparison matrix
combine these numbers into one number
Divide sum of each value in each column by the combined matrix
Compute the criteria weights by the average method
Compute the eigenvalue
Compute the consistency index (CI)
Compute the consistency ratio

Figure 2. Steps of the proposed methodology.

4. Application: Evaluating Teaching Quality in University Physics Programs

In this section, we implement the proposed methodology. We gathered five main factors in this study. Figure 2 shows the factors and sub factors. Three experts and DMs are evaluated the factors and sub factors. The root node is the objective of this study. Then the first level is the main factors and sub factors are in second level.

$$Factors = N_1 \times N_2 \times N_3 \times N_4 \times N_5$$

$$Sub\ Factors\ 1 = N_{1-1} \times N_{1-2} \times N_{1-3}$$

$$Sub\ Factors\ 2 = N_{2-1} \times N_{2-2} \times N_{2-3}$$

$$Sub\ Factors\ 3 = N_{3-1} \times N_{3-2} \times N_{3-3}$$

$$Sub\ Factors\ 4 = N_{4-1} \times N_{4-2} \times N_{4-3}$$

$$Sub\ Factors\ 5 = N_{5-1} \times N_{5-2} \times N_{5-3}$$

Three experts were involved in evaluating the factors and sub-factors using SVNSs. These terms allow the experts to express their judgments in a structured way, accounting for the degree of truth, indeterminacy, and falsity associated with each evaluation. Their assessments were then used to construct a pairwise comparison matrix, as shown in Table 2, which compares the relative importance of each factor against others.

Each expert's opinion was replaced by its corresponding SVNS values, ensuring that subjective judgments were systematically incorporated into the matrix. To simplify these neutrosophic values for further analysis, a score function was applied. This function converts the neutrosophic terms into crisp numerical values, retaining the essence of the experts' evaluations while making the data easier to process.

After obtaining the crisp values for all comparisons, these values were combined into a single unified matrix, consolidating the judgments of all three experts. This combined matrix provides a clear representation of the relative importance of each factor based on the collective input of the experts. Next, to normalize the data, the sum of each column in the matrix was calculated. Each value in a column was then divided by the total sum of that column, as shown in Table 3. This step ensures that the data is standardized and proportional, forming a consistent basis for calculating the criteria weights in subsequent steps.

Table 2. pairwise comparison matrix.

	C_1	C_2	C_3	C_4	C_5
C_1	1	(0.9 ,0.1 ,0.2)	(0.8 ,0.2 ,0.3)	(0.7 ,0.3 ,0.4)	(0.6 ,0.4 ,0.5)
C_2	1/(0 .9,0 .1,0 .2)	1	(0.2 ,0.8 ,0.9)	(0.3 ,0.7 ,0.8)	(0.4 ,0.6 ,0.7)
C_3	1/(0 .8,0 .2,0 .3)	1/(0 .2,0 .8,0 .9)	1	(0.9 ,0.1 ,0.2)	(0.8 ,0.2 ,0.3)
C_4	1/(0 .7,0 .3,0 .4)	1/(0 .3,0 .7,0 .8)	1/(0 .9,0 .1,0 .2)	1	(0.8 ,0.2 ,0.3)
C_5	1/(0 .6,0 .4,0 .5)	1/(0 .4,0 .6,0 .7)	1/(0 .8,0 .2,0 .3)	1/(0 .8,0 .2,0 .3)	1

	C_1	C_2	C_3	C_4	C_5
C_1	1	(0.9,0.1,0.2)	(0.8,0.2,0.3)	(0.7,0.3,0.4)	(0.4,0.6,0.7)
C_2	1/(0.9,0.1,0.2)	1	(0.2,0.8,0.9)	(0.3,0.7,0.8)	(0.3,0.7,0.8)
C_3	1/(0.8,0.2,0.3)	1/(0.2,0.8,0.9)	1	(0.9,0.1,0.2)	(0.8,0.2,0.3)
C_4	1/(0.7,0.3,0.4)	1/(0.3,0.7,0.8)	1/(0.9,0.1,0.2)	1	(0.8,0.2,0.3)
C_5	1/(0.4,0.6,0.7)	1/(0.3,0.7,0.8)	1/(0.8,0.2,0.3)	1/(0.8,0.2,0.3)	1
	C_1	C_2	C_3	C_4	C_5
C_1	1	(0.9,0.1,0.2)	(0.8,0.2,0.3)	(0.7,0.3,0.4)	(0.6,0.4,0.5)
C_2	1/(0.9,0.1,0.2)	1	(0.2,0.8,0.9)	(0.3,0.7,0.8)	(0.1,0.9,0.9)
C_3	1/(0.8,0.2,0.3)	1/(0.2,0.8,0.9)	1	(0.9,0.1,0.2)	(0.9,0.1,0.2)
C_4	1/(0.7,0.3,0.4)	1/(0.3,0.7,0.8)	1/(0.9,0.1,0.2)	1	(0.3,0.7,0.8)
C_5	1/(0.6,0.4,0.5)	1/(0.1,0.9,0.9)	1/(0.9,0.1,0.2)	1/(0.3,0.7,0.8)	1

Table 3. The normalization pairwise matrix.

	C_1	C_2	C_3	C_4	C_5
C_1	0.14197	0.06908	0.17659	0.13553	0.15901
C_2	0.16381	0.07973	0.03839	0.05420	0.07773
C_3	0.18517	0.34951	0.23034	0.17617	0.25441
C_4	0.21295	0.22994	0.26578	0.20327	0.19085
C_5	0.31610	0.26082	0.28890	0.43083	0.31800

C_5	0.296087	0.271749	0.288891	0.430844	0.318021

4.1 Validation of Findings

The validation of the proposed framework for evaluating teaching quality in university physics programs was conducted through three primary methods: comparison with real-world data, sensitivity analysis, and expert feedback.

4.1.1 Comparison with Real-World Data

The criteria weights derived from the proposed methodology were compared with insights from existing literature on effective teaching practices in physics education. The following weights were assigned to the five main factors:

Criteria	Weight
Conceptual Understanding	28%
Experimental Learning	22%
Use of Technology	18%
Curriculum Design	16%
Teacher-Student Engagement	16%

The analysis confirmed that Conceptual Understanding and Experimental Learning are the most significant criteria for teaching quality. These findings align with studies emphasizing the importance of theoretical knowledge and hands-on learning experiences in fostering student success in physics.

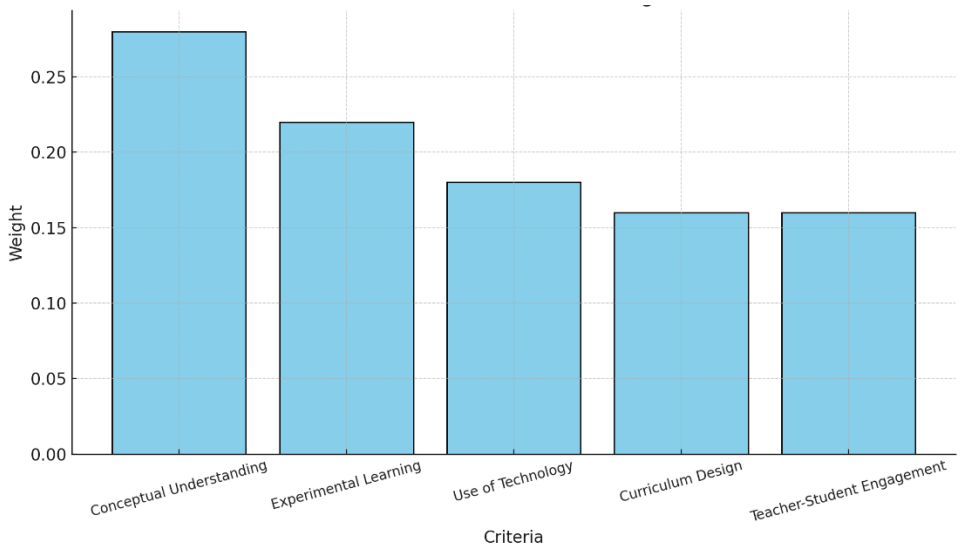

Figure 3 Vision of Criteria Weights

Figure 3 illustrates the weights assigned to the five main criteria used in the evaluation of teaching quality. Each criterion represents a fundamental aspect of effective physics education and contributes uniquely to the overall framework. At the top of the hierarchy is Conceptual Understanding, which carries the highest weight at 28%. This criterion emphasizes the importance of building a strong foundation in theoretical knowledge and problem-solving skills. It reflects the core of physics education, ensuring that students develop a deep and comprehensive understanding of essential concepts and can apply them effectively.

The second most significant criterion, Experimental Learning, is weighted at 22%. This factor highlights the critical role of hands-on activities and practical applications in physics education. Through laboratory experiments and real-world engagement, students can reinforce their theoretical knowledge, develop practical skills, and gain a deeper appreciation for the subject's relevance.

The importance of Use of Technology is also evident, with a weight of 18%. This criterion reflects the growing reliance on tools such as virtual labs and simulations to enhance the learning experience. These technologies make abstract and complex physics concepts more accessible, interactive, and engaging, bridging the gap between theory and practice in innovative ways.

Curriculum Design, with a weight of 16%, focuses on the relevance and interdisciplinarity of teaching content. A well-designed curriculum ensures alignment with modern scientific advancements and prepares students for diverse academic and professional opportunities by connecting physics with other fields of study.

Finally, Teacher-Student Engagement, also weighted at 16%, emphasizes the importance of strong, interactive relationships between educators and students. This criterion assesses aspects such as mentoring, effective communication, and collaborative learning environments. A positive and supportive relationship fosters student motivation and creates a more inclusive and productive educational experience.

Figure 3 provides a clear understanding of the relative importance of each criterion in the framework. It highlights the need to prioritize conceptual understanding and experimental learning while integrating technology and maintaining strong teacher-student engagement. Together, these elements form a balanced and comprehensive approach to improving teaching quality in university physics programs.

4.1.2. Sensitivity Analysis

To test the robustness of the proposed framework, a sensitivity analysis was performed. This analysis examined the impact of increasing or decreasing the weight of each criterion on the overall evaluation results. The results are summarized in Table 4.

Table 4 Original and Adjusted Weights for Sensitivity Analysis

Criteria	Original Weights	Increased Impact Weights	Decreased Impact Weights	Increased Impact (%)	Decreased Impact (%)
Conceptual Understanding	0.28	0.33	0.24	+5%	-4%
Experimental Learning	0.22	0.29	0.16	+7%	-6%
Use of Technology	0.18	0.22	0.15	+4%	-3%
Curriculum Design	0.16	0.22	0.11	+6%	-5%
Teacher-Student Engagement	0.16	0.19	0.14	+3%	-2%

Table 4 illustrates how weights change under two scenarios: increasing and decreasing the relative importance of each criterion. For instance, increasing the weight of Conceptual Understanding by 5% raises its adjusted weight to 0.33, while a 4% decrease lowers it to 0.24. Similarly, Experimental Learning shows the most significant variation, with a 7% increase pushing its weight to 0.29 and a 6% decrease reducing it to 0.16. Figure 4 illustrates the original, increased, and decreased weights for all five criteria, showing how their importance shifts under different impact scenarios.

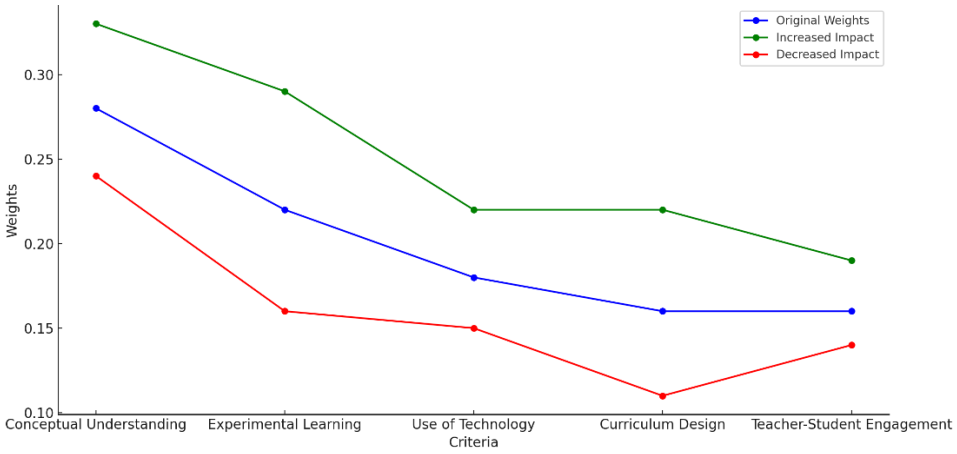

Figure 4. Sensitivity Analysis: Impact on Criteria Weights

The sensitivity analysis reveals how adjustments in weights affect the importance of each criterion in the evaluation process. Conceptual Understanding, as shown in Figure 5, responds evenly to increases (+5%) and decreases (-4%), maintaining its position as a critical factor due to its balanced sensitivity. On the other hand, Experimental Learning, illustrated in Figure 6, is the most sensitive criterion. A 7% increase in its weight significantly boosts its impact, highlighting the crucial role of hands-on activities in

enhancing teaching quality. In contrast, Figure 7 demonstrates that Use of Technology experiences moderate changes with adjusted weights, reflecting its steady but less dominant role compared to experimental learning and conceptual understanding. The analysis of Curriculum Design, as depicted in Figure 8, shows moderate sensitivity, suggesting its importance in structuring effective educational programs. Lastly, Teacher-Student Engagement, presented in Figure 9, exhibits minimal sensitivity, with only slight changes observed under adjusted weights, emphasizing its stable yet less variable influence in the overall evaluation.

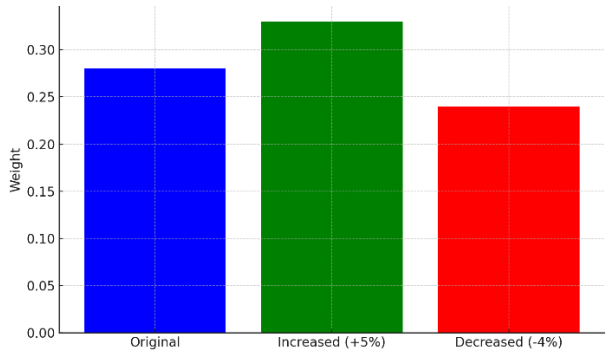

Figure 5. Sensitivity analysis for Conceptual Understanding

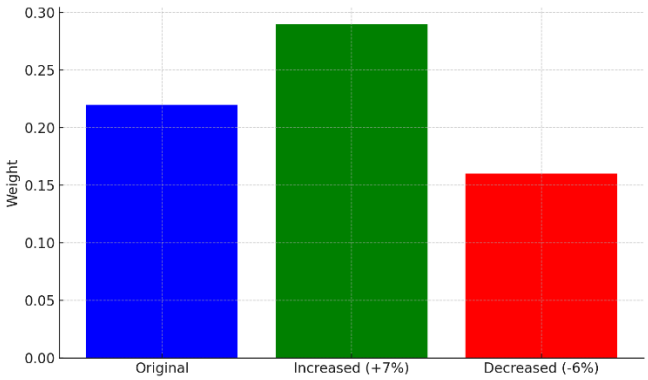

Figure 6. Sensitivity Analysis for Experimental Learning

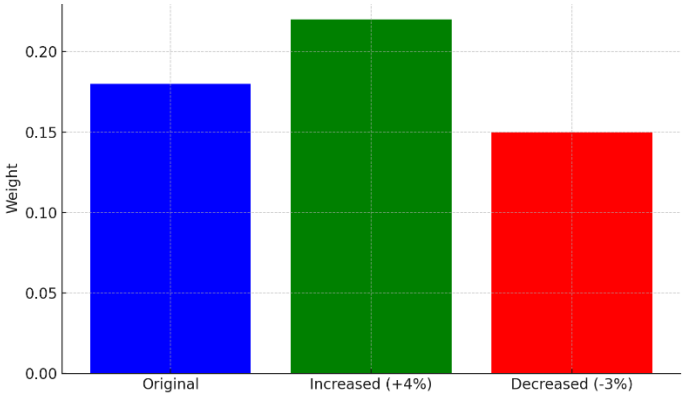

Figure 7. Sensitivity Analysis for Use of Technology

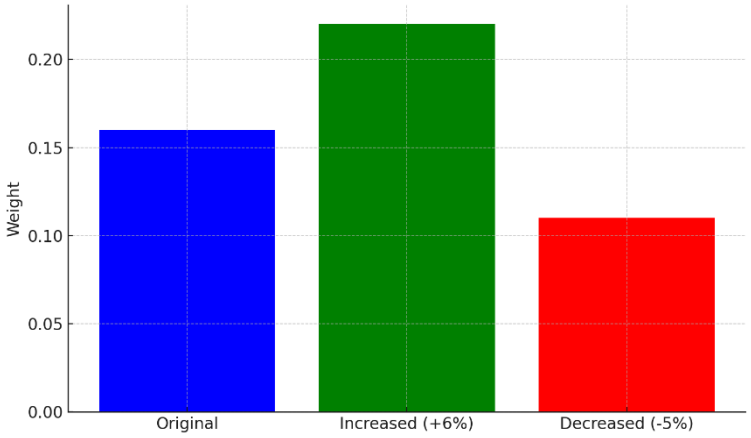

Figure 8. Sensitivity Analysis for Curriculum Design

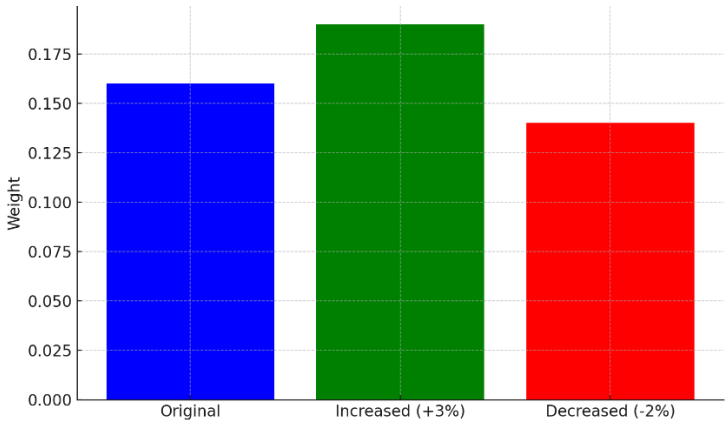

Figure 9. Sensitivity Analysis for Teacher-Student Engagement

The sensitivity analysis underscores the importance of prioritizing Experimental Learning and Curriculum Design for significant improvements in teaching quality. These criteria demonstrate higher sensitivity, suggesting that targeted investments in these areas can yield noticeable changes in rankings. Meanwhile, Conceptual Understanding remains a foundational element with balanced sensitivity, highlighting its critical role in building a strong theoretical base. The steady influence of Use of Technology and Teacher-Student Engagement indicates their consistent contributions to overall teaching quality.

4.1.3 Expert Feedback

Feedback from experienced physics educators played a vital role in validating the selection of criteria and their assigned weights in the evaluation framework. This input not only reinforced the robustness of the proposed methodology but also highlighted the

practical relevance of the criteria in real-world teaching environments. Below is a detailed explanation of the key observations made by the experts:

1. Experts agreed on the prominence of Conceptual Understanding and Experimental Learning, emphasizing their foundational role in physics education.
2. Educators highlighted the growing importance of virtual labs and simulations, confirming the relevance of the Use of Technology criterion.
3. While assigned a slightly lower weight, experts stressed the necessity of effective communication, mentoring, and feedback in creating a positive learning environment.

The feedback provided by the experts not only validates the framework but also offers insights into its practical application. The high emphasis on Conceptual Understanding and Experimental Learning underscores their critical role in physics education, aligning with global best practices. Meanwhile, the recognition of technology integration reflects the evolving nature of teaching methodologies, ensuring that the framework remains relevant in the context of modern educational challenges. Teacher-Student Engagement, although weighted lower, remains an essential component, ensuring that the human element of teaching is not overshadowed by technical advancements.

4.1.4 Practical Implications

The findings of this study offer several practical implications for improving teaching quality in university physics programs. By addressing critical dimensions of teaching and learning, the proposed framework provides actionable insights that can guide educators, institutions, and policymakers toward more effective practices and strategies.

For educators, the results emphasize the importance of enhancing hands-on experimental learning and fostering conceptual clarity. Physics educators can leverage these insights to design lesson plans that integrate practical, real-world applications with theoretical concepts. By focusing on improving students' problem-solving abilities and providing opportunities for engaging in laboratory-based experiments, educators can create a more dynamic and impactful learning environment.

For institutions, the study highlights the need for investment in advanced educational technologies, such as virtual labs and simulations. These tools not only make complex concepts more accessible to students but also allow for interactive and remote learning opportunities, which are increasingly relevant in modern education. Additionally, institutions can benefit from designing interdisciplinary curricula that bridge physics with other scientific and technological fields, fostering a broader skill set among students and preparing them for diverse career paths.

For policymakers, the results underline the importance of creating frameworks that assess and reward innovative teaching practices. Policies that encourage the adoption of cutting-edge teaching methods, collaborative learning, and critical thinking can drive improvements in educational quality. Recognizing and incentivizing educators who implement these practices can also promote a culture of continuous improvement and innovation in teaching. By translating these findings into actionable steps, this framework has the potential to elevate the overall quality of physics education, ensuring that students are better equipped with the knowledge, skills, and experiences needed to succeed in a rapidly evolving world.

5. Conclusions and Future Work

The goal of physics education is to equip students to handle a world that is becoming more complicated by the day. Both the emotional and the cognitive channels must be addressed to solve the issues with physics education. We must come up with more suitable methods to give the students a sense of purpose in physics. This entails integrating physics more widely into a variety of settings and establishing it in society. Meaningful settings can range from a deeper comprehension of the natural and technological world to metacognitive and epistemic concerns that address the role of our own mental activity in creating models of the world and reflect on our place in the cosmos, depending on the target audience. This study used the AHP method to evaluate the teaching quality in physics programs. The AHP method is integrated with the SVNSs to deal with vague data. This methodology is integrated with the TSS to show the relationships between the factors and sub factors. The results show the Conceptual Understanding and Theoretical Knowledge of Physics has the highest weights.

5.1 Future Work

The proposed framework for evaluating teaching quality in university physics programs provides a strong foundation, but there is significant scope for future enhancements. One potential direction is the incorporation of machine learning techniques to further refine the evaluation process. By analyzing large datasets and historical performance trends, machine learning algorithms could assist in reducing reliance on expert evaluations while identifying patterns and correlations that may not be immediately apparent. This approach could also enable predictive modeling to anticipate the impact of curriculum changes or teaching strategies on student outcomes.

Additionally, expanding the framework to include cross-disciplinary and cultural perspectives could enhance its applicability and relevance. Future studies could apply

the methodology to other disciplines or educational settings to test its robustness across diverse learning environments. Further validation with larger datasets and broader participant groups, including international educators and institutions, would also provide deeper insights and increase the generalizability of the findings. By addressing these directions, the framework could evolve into a more comprehensive tool for improving teaching quality in higher education globally.

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Neutrosophic Mean Estimators Using Extreme Indeterminate Observations in Sample Surveys

Vinay Kumar Yadav[1], Deepak Majhi[2], Alia A. Alkhathami[3] and Shakti Prasad[4,*]

[1]Department of Data Sciences and Analytics, School of Social Sciences, M S Ramaiah University of Applied Sciences, 560054, Bengaluru, India; vkyadavbhu@gmail.com

[2]PG Department of Mathematics, Veer Kunwar Singh University Ara Bihar-802301, India; deepak.ismd@gmail.com

[3]Department of Basic Science,College of Science and Theoretical Studies, Saudi Electronic University, Riyadh 11673, Kingdom of Saudi Arabia; A.alkhathami@seu.edu.sa

[4]Department of Mathematics, National Institute of Technology Jamshedpur, 831014, Jamshedpur, Jharkhand, India; shakti.pd@gmail.com

*Correspondence: shakti.pd@gmail.com;

Abstract. In classical statistics, research typically relies on precise data to estimate the population mean, especially when auxiliary information is available. However, in the presence of outliers, conventional statistical approaches that depend on accurate data and auxiliary information encounter challenges. The primary objective is to attain the most accurate population mean estimates while minimizing the mean square error. Neutrosophic statistics, a more attractive framework than classical statistics, deals with data characterized by imprecision and uncertainty. In this current article, we adapt Särndal's strategy and introduce neutrosophic mean estimators, applying them to meteorological data, specifically stratified dew point data. In these proposed estimators, the incorporation of auxiliary information and the application of robust techniques address issues that arise due to outliers and imprecise observations. These factors can otherwise undermine the effectiveness of neutrosophic estimation methods. The article also suggests combining auxiliary information with extremely indeterminate neutrosophic observations, utilizing robust regression methods (Huber-M, Hampel-M, and Tukey-M), as well as the quantile regression technique. These approaches enhance the neutrosophic mean estimation process. The outcomes, which include the utilization of dew point data, showcase the superior performance of the proposed estimators compared to adapted estimators in a neutrosophic context. Ultimately, this study provides valuable insights by taking an initial step in defining and utilizing the concept of neutrosophic indeterminate extreme observations.

Keywords: Neutrosophic Mean estimation, Särndal Approach, Extreme Values, Neutrosophic Robust Regression, Neutrosophic Robust Quantile Regression.

1. Introduction

The term 'dew point' holds significant meaning in the field of meteorology, as it represents the crucial temperature at which moisture begins to condense, leading to the formation of visible dew, mist, or clouds. While traditionally viewed as a single value, the concept of dew point gains a fascinating complexity when examined through the lens of neutrosophic logic—a theory that accounts for uncertainty and indeterminacy. Traditionally, the dew point is considered the temperature at which the air can no longer absorb moisture, leading to visible condensation. However, neutrosophic logic challenges this binary perspective. In neutrosophy, the dew point is seen as a triadic truth, consisting of the region of true saturation (where condensation is inevitable), the region of indeterminate saturation (where the distinction between saturation and unsaturation is unclear), and the region of false saturation (indicating a lack of saturation). This trinity captures the intricate interplay between temperature, moisture, and atmospheric dynamics.

In the neutrosophic context, the relationship between the dew point and moisture undergoes a transformative shift. Humidity, as a measure of atmospheric moisture, transcends a simple binary concept. Neutrosophic humidity encompasses the presence of moisture, a hazy realm of uncertainty where the state of moisture's existence remains unclear, as well as the absence of moisture. This aligns with the complex nature of humidity levels, where distinguishing between moist and dry conditions isn't always straightforward. Neutrosophic dew point introduces an element of uncertainty into predictions. High dew point often indicate favorable conditions for precipitation, while low dew point typically signal drier circumstances. Neutrosophic logic incorporates this uncertainty. High dew point not only suggest the possibility of precipitation but also acknowledge the inherent uncertainty in meteorological forecasts. Forecasts become increasingly unpredictable when the value indeterminate comes into play. Even low dew point, commonly viewed as clear indicators of dryness, find representation within the neutrosophic spectrum. This underscores the idea that outcomes may not always align with expectations based on dryness.

Viewing human comfort through the neutrosophic lens adds a layer of depth, especially in relation to humidity. High humidity typically leads to discomfort because evaporative cooling becomes less effective. Neutrosophy accommodates varying degrees of ambiguity in this context. The indeterminate element acknowledges that while high humidity often coincides with discomfort as a true condition, comfort isn't solely determined by humidity levels. Similar to the false aspect, which represents lower humidity, it doesn't guarantee everyone's comfort. Amidst fluctuating humidity, neutrosophic reflection allows us to explore the intricate range of human sensations. The dew point serves as the conductor of the neutrosophic atmosphere, illustrating atmospheric events. Within this realm of uncertainty, weather patterns are shaped

by warm currents and colder masses. In this symphony, the neutrosophic triad is expressed. The true value signifies air saturation at specific times, while the indeterminate section records transitions from one definitive condition to another. The false aspect pertains to situations where saturation is still absent. This triadic dance mirrors the complex choreography of atmospheric components.

In statistical inference, estimators that incorporate auxiliary data have proven to be essential. Researchers have developed various estimation approaches, including ratio estimation methods, to compute population metrics like the mean. These techniques have received extensive research attention and have shown promise in reducing sampling error, particularly when a strong connection exists between the study variable and auxiliary variable. By considering both the auxiliary variable and the primary research variable, these estimators can enhance precision and potentially reduce the required sample size. Cochran [9] emphasized the advantages of ratio estimation techniques in obtaining more accurate estimates.

Fuzzy logic has emerged as an effective tool for data analysis, particularly when dealing with data that is fuzzy, ambiguous, uncertain, or imprecise— in contrast to classical statistics, which assumes precise data. Such data can be effectively handled and processed using fuzzy statistics. However, fuzzy statistics do not explicitly consider the level of data uncertainty. Neutrosophic logic, an extension of fuzzy logic, offers a more comprehensive approach by incorporating both the determinate and indeterminate aspects of observations. Smarandache [30] highlights that neutrosophic logic enables the measurement and interpretation of data with hazy or unclear observations. Fuzzy logic techniques, involving complex fuzzy sets and their broadening to complex neutrosophic sets, have discovered practical applications in decision-making contexts. Researchers have also investigated the use of different kinds of auxiliary information to improve the performance of ratio-type estimators, including incorporating the coefficient of variation, as noted by Robson [17] applications.

The growing demand for statistical approaches capable of handling and analyzing data in a more robust and flexible manner is evident in the application of neutrosophic numbers in decision-making scenarios. Neutrosophic statistics open up new avenues for research, enabling a deeper understanding and analysis of data with varying degrees of indeterminacy. Motivated by these insights, we propose Särndal-type neutrosophic robust mean estimators that utilize extreme indeterminate observations. We assess the performance of these estimators through an application to a dew point data set.

The article is organized into several sections to provide a structured exploration of the proposed concepts and methodologies. In Section 2, we delve into the notion of Extreme Observations and robust mean estimation. Section 3 delves into the realm of Neutrosophic

Statistics and Adapted Estimators. Section 4 introduces our proposal for Neutrosophic Estimators, encompassing Neutrosophic Ratio Estimators, Neutrosophic Product Estimators (both in combined and separate versions), and the Särndal method. Further, Section 4.1 presents our proposed Neutrosophic OLS Regression and its Proposed Estimators, while Section 4.2 covers the intricacies of the proposed Neutrosophic Robust Regression and its Proposed Estimators. Subsection 4.3 provides insights into Neutrosophic Robust Quantile Regression along with its Proposed Estimators. Sections 5 and 6 are dedicated to showcasing practical applications and discussing the outcomes of our proposed methodologies. Finally, in Section 7, we draw the article to a close by outlining the potential utility of neutrosophic estimators in addressing the complexities and ambiguities inherent in real-world data analysis.

2. **Extreme Observations and Neutrosophic Robust Estimation of Mean**

The characteristics of neutrosophic observations high, low include ambiguity, indeterminacy, and paraconsistency. As a result, it is impossible to determine with certainty whether a neutrosophic observation is high, low, or neither. When there is a lot of uncertainty or when the data is insufficient, this kind of observation can be helpful. A neutrosophic observation of high and low, for instance, may be used to characterise the stock market, which is renowned for being erratic. Additionally, it might be used to the possibility of a natural disaster like a tsunami or hurricane. A neutrosophic number, which is a number with three elements including truth, indeterminacy, and falsehood, may be used to describe neutrosophic observations such as high and low. The degree to which an observation is true is represented by its truth component, while its degree of uncertainty is represented by its indeterminacy component and its degree of falsehood by its falsity component. The observation "the stock market will go up tomorrow" might be represented by a neutrosophic number such as (0.6, 0.2, 0.2). In other words, there is a 60% chance the stock market will rise tomorrow, a 20% chance it won't rise or fall, and a 20% chance it will fall.

The inclusion of unusual observations can significantly impact mean estimators' outcomes, which is not uncommon in sample survey data. These estimators, designed to calculate a variable's average value, are particularly sensitive to extremely large or small values when present in the sample. The presence of extreme values can introduce bias into estimations, leading researchers to often consider excluding them from the dataset entirely. However, an alternative approach is to retain known extreme values in the data and use them as supplementary information to enhance estimation accuracy. By incorporating these high values as additional data points, estimators can leverage a broader range of information, potentially leading to more accurate and reliable results. This approach underscores the importance of considering auxiliary variables to improve estimation precision alongside the utilization of extreme values.

When available, auxiliary variables provide additional data that can enhance the accuracy of the estimation process. It becomes clear that these estimators are often based on single-point estimate when looking for estimators that make use of the maximum and lowest values of X. They concentrate to include the extreme values of the variable to offer insights into the data distribution or to draw conclusions about the characteristics of the population. Such as Ali et al. [1], Zaman et al. [41] [43], Bulut and Zaman [7] have done numerious contributions in this field.

Reserchers such as Tahir et al. [31], Kumar et al. [14], Singh et al. [29], Singh et al. [27], Yadav and Smarandache [33], Alomair and Shahzad [4], Yadav and Prasad ([34], [35]), Kumari et al. [15], Singh and Kumari [28], Alqudah et al. [3] and Ullah et al. [32] present a variety of different neutrosophic mean estimators and discuss their applications.

It is important to note that there aren't many research especially looking at how to use extreme observations in neutrosophic mean estimators. The current article focuses on this particular point.

3. **Neutrosophic Statistics and Adapted Estimators**

Neutrosophic statistical approaches have grown in popularity for analysing datasets with variable degrees of uncertainty, often known as neutrosophic data. This statistical technique recognises the difficulty of precisely establishing sample sizes, as noted in Smarandache's study [30]. Smarandache's [30] work has proved the use of neutrosophic statistics in investigating systems with inherent uncertainty and ambiguity. In addition to more general uses, neutrosophic statistics are useful in the discipline of rock engineering. Researchers have utilised neuosophic numbers to investigate issues like the scale effect and the anisotropy of joint roughness coefficients. According to Chen et al. [8], these efforts have resulted in the development of effective methods for addressing information loss and creating well-fitting functions.

A fresh method for analysing neutrosophic data has also been made available through neutrosophic analysis of variance. This method broadens the scope of data analysis in the context of ambiguous or missing data by allowing researchers to draw important conclusions from datasets with varied degrees of ambiguity. Neutosophic interval value statistics is a particular method within Neutosophic Statistics that expresses data using Neutosophic Intervals in the context of sample surveys. A generalisation of conventional intervals, neutrosophic intervals allow for indeterminacy and partial truth but lack sharp boundary definitions. To handle such ambiguous interval data, neutrosophic interval value statistics expands on conventional statistical techniques.

To demonstrate neutrosophic interval value statistics in a representative survey, let's look at the following example:

Let's say you were to survey a population to find out how tall each person was. The conventional method would entail gathering exact height measurements, such as 165 cm, 172 cm, and so on. The data that is currently accessible, nevertheless, could occasionally be ambiguous or inaccurate. For instance, during the survey, individuals may give estimations of their height that are general or ambiguous, such as "around 170 cm" or "between 160 cm and 180 cm." These replies muddy the data and leave it up to interpretation. These ambiguous height measurements would be represented by neutrosophic intervals in neutrosophic interval value statistics. For instance, the response "around 170 cm" may be represented as a neutrosophic interval [165 cm, 175 cm, 0.6], where the bounds of the interval denote the range of feasible heights and the truth-membership value 0.6 denotes the degree of trust that the height falls within that range. The response "between 160 cm and 180 cm" might be expressed similarly as a neutrosophic interval [155 cm, 185 cm, 0.8], suggesting a broader range with a greater level of conviction. After gathering a collection of neutrosophic interval data, you may analyse the data using statistical methods designed specifically for neutrosophic statistics. This might entail doing hypothesis testing using neutrosophic intervals or computing other statistical measures, such as the mean, median, mode, and variance. By using neutrosophic interval value statistics, it is possible to include ambiguous and imprecise data in the analysis and gain a deeper knowledge of the survey data.

Consider a population of the size N is divided into L mutually exclusive strata of sizes $N_h(h = 1, 2, 3, ..., L)$ such that $\sum_{h=1}^{L} N_h = N$. We consider a neutrosophic variable, denoted by $Y_{(\delta N)} \in [Y_{(\delta N)_L}, Y_{(\delta N)_U}]$, on a finite population $U_{(\delta N)}$ consisting of N identifiable units. To estimate the unknown mean of neutosophic study variable $Y_{(\delta N)}$, denoted by $\bar{Y}_{(\delta N)} \in [\bar{Y}_{(\delta N)_L}, \bar{Y}_{(\delta N)_U}]$, a neutrosophic auxiliary variable $X_{(\delta N)} \in [X_{(\delta N)_L}, X_{(\delta N)_U}]$ is used. Note that additional information may be available for the entire population through knowledge of $X_{(\delta N)}$ or $\bar{X}_{(\delta N)} \in [\bar{X}_{(\delta N)_L}, \bar{X}_{(\delta N)_U}]$, or on a sample from $U_{(\delta N)}$ in the absence of population information related to study variable.

In stratified random sampling $Y_{hi_{(\delta N)}}$ and $X_{hi_{(\delta N)}}$ be the values of the neutrosophic study variable and the neutrosophic auxiliary variable at i^{th} unit $(i = 1, 2, 3, ..., N_h)$ in the h^{th} $(h = 1, 2, 3, ..., L)$ stratum respectively. Let a sample of size n_h $(h = 1, 2, 3, ..., L)$ is drawn from each stratum independently by simple random sampling without replacement (SRSWOR) such that $\sum_{h=1}^{L} n_h = n$ where n is total number of units in a sample. Let $y_{hi_{(\delta N)}}$ and $x_{hi_{(\delta N)}}$ be the values of the neutrosophic study variable and the neutrosophic auxiliary variable of the i^{th} unit $(i = 1, 2, 3, ..., N_h)$ in a sample.

Define:

$\bar{Y}_{(\delta N)} = \frac{\sum_{i=1}^{N} Y_{i_{(\delta N)}}}{N}$, $\bar{X}_{(\delta N)} = \frac{\sum_{i=1}^{N} X_{i_{(\delta N)}}}{N}$ = Population means of the neutrosophic study and the neutrosophic auxiliary variables respectively,

$\bar{Y}_{h_{(\delta N)}} = \frac{\sum_{i=1}^{N} Y_{hi_{(\delta N)}}}{N}$, $\bar{X}_{h_{(\delta N)}} = \frac{\sum_{i=1}^{N} X_{hi_{(\delta N)}}}{N}$ = Population means of the neutrosophic study and the neutrosophic auxiliary variables in the h^{th} statum,

$\bar{y}_{h_{(\delta N)}} = \frac{\sum_{i=1}^{N} y_{hi_{(\delta N)}}}{N}$, $\bar{x}_{h_{(\delta N)}} = \frac{\sum_{i=1}^{N} x_{hi_{(\delta N)}}}{N}$ = Sample means of the neutrosophic study and the neutrosophic auxiliary variables in the h^{th} statum,

$S^2_{hx_{(\delta N)}}$ = Neutrosophic auxiliary variable Population variance in the h^{th} stratum,

$S^2_{hy_{(\delta N)}}$ = Neutrosophic Population study variable variance in the h^{th} stratum,

$S_{hyx_{(\delta N)}}$ = Neutrosophic Population covariance of $Y_{(\delta N)}$ and $X_{(\delta N)}$ in the h^{th} stratum,

$\beta_{h_{(\delta N)}}$ = Neutrosophic Population regression coefficient for the h^{th} stratum,

$\beta_{c_{(\delta N)}}$ = Neutrosophic Population regression coefficient across the strata,

W_h = Neutrosophic Stratum weight in the h^{th} stratum,

f_h = Neutrosophic Sampling fraction in the h^{th} stratum,

$R_{(\delta N)}$ = Neutrosophic Population ratio,

$R_{h_{(\delta N)}}$ = Neutrosophic Population ratio in the h^{th} stratum.

when the study variable's and the auxiliary variables' regression line does not pass through the origin, it indicates that the regression model has an intercept term. The intercept term causes a change in the regression line, expressing the intrinsic link between the study variable and the auxiliary variables that goes beyond a simple proportionate relationship. In these circumstances, the regression estimate outperforms the ratio and product estimators.

Hansen and Hurwitz [12] suggested two distinct strategies for building ratio estimators in stratified random sampling. Stratified random sampling entails segmenting the population into various groups or strata and then sampling from each stratum separately. Ratio estimators estimate population parameters by calculating the ratio of auxiliary variables to the study variable within each stratum. The methods suggested in have introduced innovative methodologies or adjustments to current ratio estimating systems, therefore improving the estimates' accuracy.

In this paper, we consider the use of maximum and minimum values in neutrosophic statistics using both combined and separate estimators. To deal with uncertainty and inconsistencies, neutrosophic statistics mixes fuzzy logic, intuitionistic logic, and paraconsistent logic. We can investigate the upper and lower boundaries of neutrosophic statistics by employing maximum and minimum values.

The Neutrosopic mean per unit estimator in stratified random sampling are given by:

$$\bar{y}_{st_{(\delta N)}} = \sum_{h=1}^{L} W_h \bar{y}_{h_{(\delta N)}} \tag{1}$$

$$Var(\bar{y}_{st_{(\delta N)}}) = \sum_{h=1}^{L} W_h^2 \lambda_h S^2_{hy_{(\delta N)}} \tag{2}$$

where $\lambda_h = \frac{1-f_h}{n_h}$

Adapted Neutrosophic Combined Estimators

Neutrosophic Combined ratio estimators in stratified random sampling of a finite population mean ($\bar{Y}_{(\delta N)}$) are given as:

$$\hat{\bar{Y}}_{RC_{0_{(\delta N)}}} = \frac{\bar{y}_{st_{(\delta N)}}}{\bar{x}_{st_{(\delta N)}}} \bar{X}_{(\delta N)} \tag{3}$$

$$MSE(\hat{\bar{Y}}_{RC_{0_{(\delta N)}}}) \cong \sum_{h=1}^{L} W_h^2 \lambda_h \Big(S^2_{hy_{(\delta N)}} + R^2_{(\delta N)} S^2_{hx_{(\delta N)}} - 2R_{(\delta N)} S_{hyx_{(\delta N)}} \Big) \tag{4}$$

Neutrosophic Combined product estimators in stratified random sampling of a finite population mean ($\bar{Y}_{(\delta N)}$) are given as:

$$\hat{\bar{Y}}_{PC_{0_{(\delta N)}}} = \frac{\bar{y}_{st_{(\delta N)}}}{\bar{X}_{(\delta N)}} \bar{x}_{st_{(\delta N)}} \tag{5}$$

$$MSE(\hat{\bar{Y}}_{PC_{0_{(\delta N)}}}) \cong \sum_{h=1}^{L} W_h^2 \lambda_h \Big(S^2_{hy_{(\delta N)}} + R^2_{(\delta N)} S^2_{hx_{(\delta N)}} + 2R_{(\delta N)} S_{hyx_{(\delta N)}} \Big) \tag{6}$$

Neutrosophic Combined regression estimators in stratified random sampling of a finite population mean ($\bar{Y}_{(\delta N)}$) are given as:

$$\hat{\bar{Y}}_{lrC_{0_{(\delta N)}}} = \bar{y}_{st_{(\delta N)}} + b_{c_{(\delta N)}} \Big(\bar{X}_{(\delta N)} - \bar{x}_{st_{(\delta N)}} \Big) \tag{7}$$

where $b_{c_{(\delta N)}}$ is a neutrosophic combined sample regression coefficient.

$$MSE(\hat{\bar{Y}}_{lrC_{0_{(\delta N)}}}) = \sum_{h=1}^{L} W_h^2 \lambda_h S^2_{hy_{(\delta N)}} \Big(1 - p_c^2 \Big) \tag{8}$$

where $p_c = \frac{\sum_{h=1}^{L} W_h^2 \lambda_h S_{hyx_{(\delta N)}}}{\sqrt{\sum_{h=1}^{L} W_h^2 \lambda_h S^2_{hy_{(\delta N)}} \sum_{h=1}^{L} W_h^2 \lambda_h S^2_{hx_{(\delta N)}}}}$

We can also write $MSE(\hat{\bar{Y}}_{lrC_{0_{(\delta N)}}})$ expression as

$$MSE(\hat{\bar{Y}}_{lrC_{0_{(\delta N)}}}) = \Big\{ \sum_{h=1}^{L} W_h^2 \lambda_h S^2_{hy_{(\delta N)}} + \beta^2_{c_{(\delta N)}} \sum_{h=1}^{L} W_h^2 \lambda_h S^2_{hx_{(\delta N)}} - 2\beta_{c_{(\delta N)}} \sum_{h=1}^{L} W_h^2 \lambda_h S^2_{hyx_{(\delta N)}} \Big\} \tag{9}$$

Where $\beta_{c_{(\delta N)}} = \frac{\sum_{h=1}^{L} W_h^2 \lambda_h S^2_{hyx_{(\delta N)}}}{\sum_{h=1}^{L} W_h^2 \lambda_h S^2_{hx_{(\delta N)}}}$

Adapted Neutrosophic Separate Estimators

Neutrosophic Separate ratio estimators in stratified random sampling of a finite population mean ($\bar{Y}_{(\delta N)}$) are given as:

$$\hat{\bar{Y}}_{RS_{0_{(\delta N)}}} = \sum_{h=1}^{L} W_h \bar{y}_{hR_{(\delta N)}}, \ where \ \bar{y}_{hR_{(\delta N)}} = \frac{\bar{y}_{h_{(\delta N)}}}{\bar{x}_{h_{(\delta N)}}} \bar{X}_{h_{(\delta N)}} \tag{10}$$

$$MSE(\hat{\bar{Y}}_{RS_{0_{(\delta N)}}}) \cong \sum_{h=1}^{L} W_h^2 \lambda_h \Big(S^2_{hy_{(\delta N)}} + R^2_{h_{(\delta N)}} S^2_{hx_{(\delta N)}} - 2R_{h_{(\delta N)}} S_{hyx_{(\delta N)}} \Big) \tag{11}$$

Neutrosophic Separate product estimators in stratified random sampling of a finite population mean ($\bar{Y}_{(\delta N)}$) are given as:

$$\hat{\bar{Y}}_{PS_{0_{(\delta N)}}} = \sum_{h=1}^{L} W_h \bar{y}_{hP_{(\delta N)}}, \ where \ \bar{y}_{hP_{(\delta N)}} = \frac{\bar{y}_{h_{(\delta N)}}}{\bar{X}_{h_{(\delta N)}}} \bar{x}_{h_{(\delta N)}} \tag{12}$$

$$MSE(\hat{\bar{Y}}_{PS_{0_{(\delta N)}}}) \cong \sum_{h=1}^{L} W_h^2 \lambda_h \Big(S^2_{hy_{(\delta N)}} + R^2_{h_{(\delta N)}} S^2_{hx_{(\delta N)}} + 2R_{h_{(\delta N)}} S_{hyx_{(\delta N)}} \Big) \tag{13}$$

Neutrosophic Separate regression estimators in stratified random sampling of a finite population mean ($\bar{Y}_{(\delta N)}$) are given as:

$$\hat{\bar{Y}}_{lrS_{0_{(\delta N)}}} = \sum_{h=1}^{L} W_h \bar{y}_{hlr_{(\delta N)}} \tag{14}$$

where $\bar{y}_{hlr_{(\delta N)}} = \bar{y}_{h_{(\delta N)}} + b_{h_{(\delta N)}} (\bar{X}_{h_{(\delta N)}} - \bar{x}_{h_{(\delta N)}})$ and $b_{h_{(\delta N)}} = \frac{S_{hyx_{(\delta N)}}}{S^2_{hx_{(\delta N)}}}$

$$MSE(\hat{\bar{Y}}_{lrS_{0_{(\delta N)}}}) = \sum_{h=1}^{L} W_h^2 \lambda_h S^2_{hy_{(\delta N)}} \Big(1 - p_h^2 \Big) \tag{15}$$

where $p_h = \frac{S_{hyx_{(\delta N)}}}{S_{hy_{(\delta N)}} S_{hx_{(\delta N)}}}$

We can also write $MSE(\hat{\bar{Y}}_{lrS_{0_{(\delta N)}}})$ expression as

$$MSE(\hat{\bar{Y}}_{lrS_{0_{(\delta N)}}}) = \sum_{h=1}^{L} W_h^2 \lambda_h \Big[S^2_{hy_{(\delta N)}} + \beta^2_{h_{(\delta N)}} S^2_{hx_{(\delta N)}} - 2\beta_{h_{(\delta N)}} S^2_{hyx_{(\delta N)}} \Big] \tag{16}$$

Where $\beta_{h_{(\delta N)}} = \frac{S^2_{hyx_{(\delta N)}}}{S^2_{hx_{(\delta N)}}}$

4. **Särndal approach and proposed neutrosophic estimators**

In many cases, real databases contain instances of exceptionally big or tiny numbers. This is also true in the context of neutrosophic statistics, which considers the existence of indeterminacy, ambiguity, and inconsistency in data. Hybrid seed production firms, for example, produce new seed kinds and provide specified productivity ranges per acre that might help farmers. These ranges enable the detection of maximum and lowest values within the collection. Similarly, when calculating the average income of families, it is typically well-known what the income of the richest individual (maximum) in a community is, but determining the income of the lowest person (minimum) is simple. Furthermore, several surveys undertaken at regular intervals give significant information on both the maximum and minimum values.

However, a problem in neutrosophic statistics is that the mean per unit estimate for the mean of a finite population is very sensitive to extreme values. Accordingly, the estimator can yield false findings if any of these unexpected values are chosen for the sample. Särndal [19] provided an unbiased estimator to handle this issue precisely within the context of extreme

observations in the population. The purpose of this estimate is to lessen the influence of the extreme values seen in the dataset. Särndal's method can be used to enhance the precision and dependability of statistical analysis in neutrosophic statistics.

Let $y_{h_{max_{(\delta N)}}}$ ($y_{h_{min_{(\delta N)}}}$) and $x_{h_{max_{(\delta N)}}}$ ($x_{h_{min_{(\delta N)}}}$) be the neutrosophic study's maximum (minimum) and auxiliary variables' respective maximum (minimum) values in stratum h. The suggested estimate for finite population mean is ($\bar{Y}_{(\delta N)}$) under neutrosophic stratified random sampling are as follows:

$$\bar{y}_{st_{(\delta N)}} = \sum_{h=1}^{L} W_h \bar{y}_{h_{c_{(\delta N)}}} \tag{17}$$

$$\bar{y}_{h_{c_{(\delta N)}}} = \begin{cases} \bar{y}_{h_{(\delta N)}} + c_{h_{(\delta N)}} & \text{if a sample from the } h^{th} \text{ stratum contains } y_{h_{min_{(\delta N)}}} \text{ but not } y_{h_{max_{(\delta N)}}}, \\ \bar{y}_{h_{(\delta N)}} - c_{h_{(\delta N)}} & \text{if a sample from the } h^{th} \text{ stratum contains } y_{h_{max_{(\delta N)}}} \text{ but not } y_{h_{min_{(\delta N)}}}, \\ \bar{y}_{h_{(\delta N)}} & \text{in other cases,} \end{cases} \tag{18}$$

where $\bar{y}_{h_{c_{(\delta N)}}}$ $(h = 1, 2, 3, ..., L)$ are arbitrary constants.

The selection of values in a sample is based on the relationship between the neutrosophic study variable $Y_{h_{(\delta N)}}$ and the neutrosophic auxiliary variable $X_{h_{(\delta N)}}$. When the relationship is positive, it is predicted that a big value of $Y_{h_{(\delta N)}}$ will be picked for large values of $X_{h_{(\delta N)}}$ and a small value of $Y_{h_{(\delta N)}}$ for small values of $X_{h_{(\delta N)}}$. The definition of ratio estimators follows from this. In contrast, when $Y_{h_{(\delta N)}}$ and $X_{h_{(\delta N)}}$ have a negative relationship, a big value of $Y_{h_{(\delta N)}}$ is anticipated for small values of $X_{h_{(\delta N)}}$ and a small value of $Y_{h_{(\delta N)}}$ is anticipated for large values of $X_{h_{(\delta N)}}$. Consequently, product estimators are defined. These estimators are based on the assumption that the sample's selected values will be consistent.

(1) : Neutrosophic Ratio Estimators

Neutrosophic Combined ratio estimators in stratified random sampling using extreme values of data.

$$\hat{\bar{Y}}_{RC_{1_{(\delta N)}}} = \frac{\bar{y}_{st.c11_{(\delta N)}}}{\bar{x}_{st.c21_{(\delta N)}}} \bar{X}_{(\delta N)} \tag{19}$$

Neutrosophic Separate ratio estimators in stratified random sampling using extreme values of data.

$$\hat{\bar{Y}}_{RS_{1_{(\delta N)}}} = \sum_{h=1}^{L} W_h \frac{\bar{y}_{h.c11_{(\delta N)}}}{\bar{x}_{h.c21_{(\delta N)}}} \bar{X}_{h_{(\delta N)}} \tag{20}$$

where; $\bar{y}_{st.c11_{(\delta N)}} = \sum_{h=1}^{L} W_h \bar{y}_{h.c11_{(\delta N)}}$ and $\bar{x}_{st.c21_{(\delta N)}} = \sum_{h=1}^{L} W_h \bar{x}_{h.c21_{(\delta N)}}$

and

$$(\bar{y}_{h_{c.11_{(\delta N)}}}, \bar{x}_{h_{c.21_{(\delta N)}}}) = \begin{cases} \Delta_{R1} & \text{if a sample contains } y_{h_{min_{(\delta N)}}} \text{ but not } y_{h_{max_{(\delta N)}}}, \\ \Delta_{R2} & \text{if a sample contains } y_{h_{max_{(\delta N)}}} \text{ but not } y_{h_{min_{(\delta N)}}}, \\ \Delta_{R3} & \text{in other cases,} \end{cases} \tag{21}$$

where $\Delta_{R1} = \left(\bar{y}_{h_{(\delta N)}} + c_{1h_{(\delta N)}}, \bar{x}_{h_{(\delta N)}} + c_{2h_{(\delta N)}}\right)$, $\Delta_{R2} = \left(\bar{y}_{h_{(\delta N)}} - c_{1h_{(\delta N)}}, \bar{x}_{h_{(\delta N)}} - c_{2h_{(\delta N)}}\right)$, $\Delta_{R3} = \left(\bar{y}_{h_{(\delta N)}}, \bar{x}_{h_{(\delta N)}}\right)$, $c_{1h_{(\delta N)}}$ and $c_{2h_{(\delta N)}}$ $(h = 1, 2, 3, ..., L)$ are arbitrary constant.

(2) : Neutrosophic Product Estimators

Neutrosophic combined product estimators in stratified random sampling using extreme values of data.

$$\hat{\bar{Y}}_{PC_{1_{(\delta N)}}} = \frac{\bar{y}_{st.c12_{(\delta N)}}}{\bar{X}_{(\delta N)}} \bar{x}_{st.c22_{(\delta N)}} \tag{22}$$

Neutrosophic Separate product estimators in stratified random sampling using extreme values of data.

$$\hat{\bar{Y}}_{PS_{1_{(\delta N)}}} = \sum_{h=1}^{L} W_h \frac{\bar{y}_{h.c12_{(\delta N)}}}{\bar{X}_{h_{(\delta N)}}} \bar{x}_{h.c22_{(\delta N)}} \tag{23}$$

where; $\bar{y}_{st.c11_{(\delta N)}} = \sum_{h=1}^{L} W_h \bar{y}_{h.c11_{(\delta N)}}$ and $\bar{x}_{st.c21_{(\delta N)}} = \sum_{h=1}^{L} W_h \bar{x}_{h.c21_{(\delta N)}}$
and

$$(\bar{y}_{h_{c.12_{(\delta N)}}}, \bar{x}_{h_{c.22_{(\delta N)}}}) = \begin{cases} \Delta_{P1} & \text{if a sample contains } y_{h_{min_{(\delta N)}}} \text{ but not } y_{h_{max_{(\delta N)}}}, \\ \Delta_{P2} & \text{if a sample contains } y_{h_{max_{(\delta N)}}} \text{ but not } y_{h_{min_{(\delta N)}}}, \\ \Delta_{P3} & \text{in other cases,} \end{cases} \tag{24}$$

where $\Delta_{P1} = \left(\bar{y}_{h_{(\delta N)}} + c_{1h_{(\delta N)}}, \bar{x}_{h_{(\delta N)}} - c_{2h_{(\delta N)}}\right)$, $\Delta_{P2} = \left(\bar{y}_{h_{(\delta N)}} - c_{1h_{(\delta N)}}, \bar{x}_{h_{(\delta N)}} + c_{2h_{(\delta N)}}\right)$, $\Delta_{P3} = \left(\bar{y}_{h_{(\delta N)}}, \bar{x}_{h_{(\delta N)}}\right)$, $c_{1h_{(\delta N)}}$ and $c_{2h_{(\delta N)}}$ $(h = 1, 2, 3, ..., L)$ are arbitrary constant.

Properties of the Neutrosophic estimators

To study the large sample properties of neutrosophic estimators let's consider the following transformations:

$e_{st.c_{1j_{(\delta N)}}} = \frac{\left(\bar{y}_{st.c1j_{(\delta N)}} - \bar{Y}_{(\delta N)}\right)}{\bar{Y}_{(\delta N)}}$, $e_{st.c_{2j_{(\delta N)}}} = \frac{\left(\bar{x}_{st.c2j_{(\delta N)}} - \bar{X}_{(\delta N)}\right)}{\bar{X}_{(\delta N)}}$, $j = 1, 2$

$e_{h.c_{1j_{(\delta N)}}} = \frac{\left(\bar{y}_{h.c1j_{(\delta N)}} - \bar{Y}_{h_{(\delta N)}}\right)}{\bar{Y}_{h_{(\delta N)}}}$, $e_{h.c_{2j_{(\delta N)}}} = \frac{\left(\bar{x}_{h.c2j_{(\delta N)}} - \bar{X}_{h_{(\delta N)}}\right)}{\bar{X}_{h_{(\delta N)}}}$, $j = 1, 2$

Up to the first order of the approximation, we will get

$E(e_{st.c_{1j_{(\delta N)}}}) = E(e_{st.c_{2j_{(\delta N)}}}) = E(e_{h.c_{1j_{(\delta N)}}}) = E(e_{h.c_{2j_{(\delta N)}}}) = 0,$

$E(e^2_{st.c_{1j_{(\delta N)}}}) = \frac{1}{\bar{Y}^2_{(\delta N)}} \sum_{h=1}^{L} W_h^2 \lambda_h \left\{ S^2_{hy_{(\delta N)}} - \frac{2n_h c_{1h_{(\delta N)}}}{N_h - 1} (y_{h_{max_{(\delta N)}}} - y_{h_{min_{(\delta N)}}} - n_h c_{1h_{(\delta N)}}) \right\},$

$E(e^2_{st.c_{2j_{(\delta N)}}}) = \frac{1}{\bar{X}^2_{(\delta N)}} \sum_{h=1}^{L} W_h^2 \lambda_h \left\{ S^2_{hx_{(\delta N)}} - \frac{2n_h c_{2h_{(\delta N)}}}{N_h - 1} (x_{h_{max_{(\delta N)}}} - x_{h_{min_{(\delta N)}}} - n_h c_{2h_{(\delta N)}}) \right\},$

$E(e^2_{h.c_{1j_{(\delta N)}}}) = \frac{1}{\bar{Y}^2_{h_{(\delta N)}}} \lambda_h \left\{ S^2_{hy_{(\delta N)}} - \frac{2n_h c_{1h_{(\delta N)}}}{N_h - 1} (y_{h_{max_{(\delta N)}}} - y_{h_{min_{(\delta N)}}} - n_h c_{1h_{(\delta N)}}) \right\},$

$$E(e^2_{h.c_{2j_{(\delta N)}}}) = \frac{1}{\bar{X}^2_{h_{(\delta N)}}}\lambda_h\left\{S^2_{hx_{(\delta N)}} - \frac{2n_h c_{2h_{(\delta N)}}}{N_h-1}(x_{h_{max_{(\delta N)}}} - x_{h_{min_{(\delta N)}}} - n_h c_{2h_{(\delta N)}})\right\},$$

$$\begin{aligned} E(e_{st.c_{1j_{(\delta N)}}}, e_{st.c_{2j_{(\delta N)}}}) &= \frac{1}{\bar{Y}_{(\delta N)}\bar{X}_{(\delta N)}}\sum_{h=1}^{L} W_h^2\lambda_h\Big[S_{hyx_{(\delta N)}} - \frac{n_h}{N_h-1}\Big\{c_{1h_{(\delta N)}} \\ &\quad (x_{h_{max_{(\delta N)}}} - x_{h_{min_{(\delta N)}}}) + c_{2h_{(\delta N)}}(y_{h_{max_{(\delta N)}}} - y_{h_{min_{(\delta N)}}}) \\ &\quad -2n_h c_{1h_{(\delta N)}} c_{2h_{(\delta N)}}\Big\}\Big] \end{aligned}$$

$$\begin{aligned} E(e_{h.c_{1j_{(\delta N)}}}, e_{h.c_{2j_{(\delta N)}}}) &= \frac{1}{\bar{Y}_{h_{(\delta N)}}\bar{X}_{h_{(\delta N)}}}\lambda_h\Big[S_{hyx_{(\delta N)}} - \frac{n_h}{N_h-1}\Big\{c_{1h_{(\delta N)}} \\ &\quad (x_{h_{max_{(\delta N)}}} - x_{h_{min_{(\delta N)}}}) + c_{2h_{(\delta N)}}(y_{h_{max_{(\delta N)}}} - y_{h_{min_{(\delta N)}}}) \\ &\quad -2n_h c_{1h_{(\delta N)}} c_{2h_{(\delta N)}}\Big\}\Big] \end{aligned}$$

for $j = 1, 2$.

Properties of Neutrosophic Combined Estimators :

Neutrosophic combined ratio estimators $\hat{\bar{Y}}_{RC_{1_{(\delta N)}}}$ in terms of relatives error is given by

$$\hat{\bar{Y}}_{RC_{1_{(\delta N)}}} = \bar{Y}_{(\delta N)}\Big(1 + e_{st.c_{11_{(\delta N)}}}\Big)\Big(1 + e_{st.c_{21_{(\delta N)}}}\Big)^{-1}, \tag{25}$$

Approximation upto first order, we have

$$\begin{aligned} (\hat{\bar{Y}}_{RC_{1_{(\delta N)}}} - \bar{Y}_{(\delta N)}) &\cong \bar{Y}_{(\delta N)}\Big(e_{st.c_{11_{(\delta N)}}} + e_{st.c_{21_{(\delta N)}}} - e_{st.c_{11_{(\delta N)}}} e_{st.c_{21_{(\delta N)}}} \\ &\quad + e^2_{st.c_{21_{(\delta N)}}}\Big), \end{aligned} \tag{26}$$

The Bias of $\hat{\bar{Y}}_{RC_{1_{(\delta N)}}}$ is given by

$$\begin{aligned} Bias(\hat{\bar{Y}}_{RC_{1_{(\delta N)}}}) &\cong \sum_{h=1}^{L} W_h^2\frac{\lambda_h}{\bar{X}_{(\delta N)}}\Big[R_{(\delta N)}\Big\{S^2_{hx_{(\delta N)}} - \frac{2n_h c_{2h_{(\delta N)}}}{N_h-1}(x_{h_{max_{(\delta N)}}} \\ &\quad -x_{h_{min_{(\delta N)}}} - n_h c_{2h_{(\delta N)}})\Big\} - \Big\{S_{hyx_{(\delta N)}} - \frac{n_h}{N_h-1}(c_{1h_{(\delta N)}}(x_{h_{max_{(\delta N)}}} \\ &\quad -x_{h_{min_{(\delta N)}}}) + c_{2h_{(\delta N)}}(y_{h_{max_{(\delta N)}}} - y_{h_{min_{(\delta N)}}}) \\ &\quad -2n_h c_{1h_{(\delta N)}} c_{2h_{(\delta N)}})\Big\}\Big] \end{aligned} \tag{27}$$

The MSE of $\hat{\bar{Y}}_{RC_{1_{(\delta N)}}}$ is given by

$$\begin{aligned} MSE(\hat{\bar{Y}}_{RC_{1_{(\delta N)}}}) &\cong MSE(\hat{\bar{Y}}_{RC_{0_{(\delta N)}}}) - \sum_{h=1}^{L} W_h^2\frac{2\lambda_h n_h}{N_h-1}(c_{1h_{(\delta N)}} - R_{(\delta N)}c_{2h_{(\delta N)}}) \\ &\quad \Big\{(y_{h_{max_{(\delta N)}}} - y_{h_{min_{(\delta N)}}}) - R_{(\delta N)}(x_{h_{max_{(\delta N)}}} - x_{h_{min_{(\delta N)}}}) \\ &\quad -n_h(c_{1h_{(\delta N)}} - Rc_{2h_{(\delta N)}})\Big\} \end{aligned} \tag{28}$$

Differentiating $MSE(\hat{\bar{Y}}_{RC_{1_{(\delta N)}}})$ with respect to $c_{1h_{(\delta N)}}$ and $c_{2h_{(\delta N)}}$ $(h = 1, 2, 3, ..., L)$ respectively and equate to zero, we have

$$\frac{\partial MSE(\hat{\bar{Y}}_{RC_{1_{(\delta N)}}})}{\partial c_{1h_{(\delta N)}}} = 0 \tag{29}$$

$$(y_{h_{max_{(\delta N)}}} - y_{h_{min_{(\delta N)}}}) - R_{(\delta N)}(x_{h_{max_{(\delta N)}}} - x_{h_{min_{(\delta N)}}}) - 2n_h(c_{1h_{(\delta N)}} - Rc_{2h_{(\delta N)}}) = 0 \tag{30}$$

Since there are $2L$ unknowns in these equations, a unique solution for the constants is not feasible. We believe that $c_{1h_{(\delta N)}} = \frac{\left(y_{h_{max_{(\delta N)}}} - y_{h_{min_{(\delta N)}}}\right)}{2n_h}$ and it implies that $c_{2h_{(\delta N)}} = \frac{\left(x_{h_{max_{(\delta N)}}} - x_{h_{min_{(\delta N)}}}\right)}{2n_h}$. For the optimum values of $c_{1h_{(\delta N)}}$ and $c_{2h_{(\delta N)}}$, the MSE of ($\hat{\bar{Y}}_{RC_{1_{(\delta N)}}}$), is given by

$$\begin{aligned} MSE(\hat{\bar{Y}}_{RC_{1_{(\delta N)}}})_{min} &= MSE(\hat{\bar{Y}}_{RC_{0_{(\delta N)}}}) - \sum_{h=1}^{L} W_h^2 \frac{\lambda_h}{2(N_h - 1)} \Big\{ (y_{h_{max_{(\delta N)}}} \\ &\quad - y_{h_{min_{(\delta N)}}}) - R_{(\delta N)}(x_{h_{max_{(\delta N)}}} - x_{h_{min_{(\delta N)}}}) \Big\}^2 \end{aligned} \tag{31}$$

Similarly, the combined product estimator's bias and MSE $\hat{\bar{Y}}_{PC_{1_{(\delta N)}}}$, are given by

$$\begin{aligned} Bias(\hat{\bar{Y}}_{PC_{1_{(\delta N)}}}) &\cong \sum_{h=1}^{L} W_h^2 \frac{\lambda_h}{\bar{X}_{(\delta N)}} \Big[S_{hyx_{(\delta N)}} - \frac{n_h}{N_h - 1} \Big\{ c_{1h_{(\delta N)}} (x_{h_{max_{(\delta N)}}} \\ &\quad - x_{h_{min_{(\delta N)}}}) + c_{2h_{(\delta N)}} (y_{h_{max_{(\delta N)}}} - y_{h_{min_{(\delta N)}}}) \\ &\quad - 2n_h c_{1h_{(\delta N)}} c_{2h_{(\delta N)}} \Big\} \Big] \end{aligned} \tag{32}$$

$$\begin{aligned} MSE(\hat{\bar{Y}}_{PC_{1_{(\delta N)}}}) &\cong MSE(\hat{\bar{Y}}_{PC_{0_{(\delta N)}}}) - \sum_{h=1}^{L} W_h^2 \frac{2\lambda_h n_h}{N_h - 1} (c_{1h_{(\delta N)}} - R_{(\delta N)} c_{2h_{(\delta N)}}) \\ &\quad \Big\{ (y_{h_{max_{(\delta N)}}} - y_{h_{min_{(\delta N)}}}) + R_{(\delta N)} (x_{h_{max_{(\delta N)}}} - x_{h_{min_{(\delta N)}}}) \\ &\quad - n_h (c_{1h_{(\delta N)}} + R_{(\delta N)} c_{2h_{(\delta N)}}) \Big\} \end{aligned} \tag{33}$$

Utlizing values of $c_{1h_{(\delta N)}}$ and $c_{2h_{(\delta N)}}$, MSE of $\hat{\bar{Y}}_{PC_{1_{(\delta N)}}}$, is given as

$$\begin{aligned} MSE(\hat{\bar{Y}}_{PC_{1_{(\delta N)}}})_{min} &= MSE(\hat{\bar{Y}}_{PC_{0_{(\delta N)}}}) - \sum_{h=1}^{L} W_h^2 \frac{\lambda_h}{2(N_h - 1)} \Big\{ \Big(y_{h_{max_{(\delta N)}}} \\ &\quad - y_{h_{min_{(\delta N)}}} \Big) - R_{(\delta N)} \Big(x_{h_{max_{(\delta N)}}} - x_{h_{min_{(\delta N)}}} \Big) \Big\}^2 \end{aligned} \tag{34}$$

Properties of Neutrosophic Separate Estimators :

Neutrosophic separate ratio estimators $\hat{\bar{Y}}_{RS_{1_{(\delta N)}}}$ in terms of relatives error is given by

$$\hat{\bar{Y}}_{RS_{1_{(\delta N)}}} = \sum_{h=1}^{L} W_h \bar{Y}_{h_{(\delta N)}} \Big(1 + e_{h.c_{11_{(\delta N)}}} \Big) \Big(1 + e_{h.c_{21_{(\delta N)}}} \Big)^{-1}, \tag{35}$$

Under the above considered transformations on expanding the expression we have

$$(\hat{\bar{Y}}_{RS_{1_{(\delta N)}}} - \bar{Y}_{(\delta N)}) \cong \sum_{h=1}^{L} W_h \bar{Y}_{h_{(\delta N)}} \Big(e_{h.c_{11_{(\delta N)}}} + e_{h.c_{21_{(\delta N)}}} - e_{h.c_{11_{(\delta N)}}} e_{h.c_{21_{(\delta N)}}} + e^2_{h.c_{21_{(\delta N)}}} \Big), \tag{36}$$

The Bias of $\hat{\bar{Y}}_{RS_{1_{(\delta N)}}}$ is given by

$$\begin{aligned} Bias(\hat{\bar{Y}}_{RS_{1_{(\delta N)}}}) \cong \sum_{h=1}^{L} W_h^2 \frac{\lambda_h}{\bar{X}_{h_{(\delta N)}}} \Big[R_{h_{(\delta N)}} \Big\{ S^2_{hx_{(\delta N)}} - \frac{2n_h c_{h_{(\delta N)}}}{N_h - 1} (x_{h_{max_{(\delta N)}}} \\ - x_{h_{min_{(\delta N)}}} - n_h c_{2h_{(\delta N)}}) \Big\} - \Big\{ S_{hyx_{(\delta N)}} - \frac{n_h}{N_h - 1} (c_{1h_{(\delta N)}} (x_{h_{max_{(\delta N)}}} \\ - x_{h_{min_{(\delta N)}}}) + c_{2h_{(\delta N)}} (y_{h_{max_{(\delta N)}}} - y_{h_{min_{(\delta N)}}}) \\ - 2n_h c_{1h_{(\delta N)}} c_{2h_{(\delta N)}}) \Big\} \Big] \end{aligned} \tag{37}$$

The MSE of $\hat{\bar{Y}}_{RS_{1_{(\delta N)}}}$ is given by

$$\begin{aligned} MSE(\hat{\bar{Y}}_{RS_{1_{(\delta N)}}}) \cong \sum_{h=1}^{L} W_h^2 \lambda_h \Big[S^2_{hy_{(\delta N)}} - \frac{2n_h c_{1h_{(\delta N)}}}{N_h - 1} (y_{h_{max_{(\delta N)}}} - y_{h_{min_{(\delta N)}}} \\ - n_h c_{1h_{(\delta N)}}) + R^2_{h_{(\delta N)}} \{ S^2_{hx_{(\delta N)}} - \frac{2n_h c_{2h_{(\delta N)}}}{N_h - 1} (x_{h_{max_{(\delta N)}}} \\ - x_{h_{min_{(\delta N)}}} - n_h c_{2h_{(\delta N)}}) \} - 2R_{h_{(\delta N)}} \{ S_{hyx_{(\delta N)}} - \frac{n_h}{N_h - 1} (\\ c_{1h_{(\delta N)}} (x_{h_{max_{(\delta N)}}} - x_{h_{min_{(\delta N)}}}) + c_{2h_{(\delta N)}} (y_{h_{max_{(\delta N)}}} - y_{h_{min_{(\delta N)}}}) \\ - 2n_h c_{1h_{(\delta N)}} c_{2h_{(\delta N)}}) \} \Big] \end{aligned} \tag{38}$$

Min MSE of $\hat{\bar{Y}}_{RS_{1_{(\delta N)}}}$, is as follows

$$\begin{aligned} MSE(\hat{\bar{Y}}_{RS_{1_{(\delta N)}}})_{min} = MSE(\hat{\bar{Y}}_{RS_{0_{(\delta N)}}}) - \sum_{h=1}^{L} W_h^2 \frac{\lambda_h}{2(N_h - 1)} \Big\{ \Big(y_{h_{max_{(\delta N)}}} \\ - y_{h_{min_{(\delta N)}}} \Big) - R_{h_{(\delta N)}} \Big(x_{h_{max_{(\delta N)}}} - x_{h_{min_{(\delta N)}}} \Big) \Big\}^2 \end{aligned} \tag{39}$$

Similarly bias and MSE of estimator $\hat{\bar{Y}}_{PS_{1_{(\delta N)}}}$, are as follows

$$\begin{aligned} Bias(\hat{\bar{Y}}_{PS_{1_{(\delta N)}}}) \cong \sum_{h=1}^{L} W_h^2 \frac{\lambda_h}{\bar{X}_{(\delta N)}} \Big[S_{hyx_{(\delta N)}} - \frac{n_h}{N_h - 1} \Big\{ c_{1h_{(\delta N)}} (x_{h_{max_{(\delta N)}}} \\ - x_{h_{min_{(\delta N)}}}) + c_{2h_{(\delta N)}} (y_{h_{max_{(\delta N)}}} - y_{h_{min_{(\delta N)}}}) \\ - 2n_h c_{1h_{(\delta N)}} c_{2h_{(\delta N)}} \Big\} \Big] \end{aligned} \tag{40}$$

$$MSE(\hat{\bar{Y}}_{PS_{1_{(\delta N)}}}) \cong MSE(\hat{\bar{Y}}_{PS_{0_{(\delta N)}}}) - \sum_{h=1}^{L} W_h^2 \frac{2\lambda_h n_h}{N_h - 1}(c_{1h_{(\delta N)}} + R_{h_{(\delta N)}} c_{2h_{(\delta N)}}) \left\{(y_{h_{max_{(\delta N)}}} - y_{h_{min_{(\delta N)}}}) + R_{h_{(\delta N)}}(x_{h_{max_{(\delta N)}}} - x_{h_{min_{(\delta N)}}}) - n_h(c_{1h_{(\delta N)}} + R_{h_{(\delta N)}} c_{2h_{(\delta N)}})\right\} \quad (41)$$

Min MSE of $\hat{\bar{Y}}_{PS_{1_{(\delta N)}}}$, is as follows

$$MSE(\hat{\bar{Y}}_{PS_{1_{(\delta N)}}})_{min} = MSE(\hat{\bar{Y}}_{PS_{0_{(\delta N)}}}) - \sum_{h=1}^{L} W_h^2 \frac{\lambda_h}{2(N_h - 1)} \left\{ \left(y_{h_{max_{(\delta N)}}} - y_{h_{min_{(\delta N)}}} \right) - R_{h_{(\delta N)}} \left(x_{h_{max_{(\delta N)}}} - x_{h_{min_{(\delta N)}}} \right) \right\}^2 \quad (42)$$

4.1. *Neutrosophic OLS Regression and Proposed Estimators*

In the discipline of statistics, parametric regression techniques are widely used, providing useful tools for modeling and analysing correlations between variables. Among these strategies, Ordinary Least Squares (OLS) stands out as one of the most often used. The primary goal of OLS is to estimate model parameters by minimising the sum of squared residuals, which indicate the differences between observed and projected values. We are utilising the Särndal method for addressing extreme values as we pursue thorough statistical analysis.

Neutrosophic statistics addresses uncertainty that is characterised by ambiguity, vagueness, and indeterminacy, posing particular difficulties for data interpretation. Neutosophic statistics incorporates the Särndal technique, and we want to create novel approaches that efficiently manage high values in the presence of uncertain data. This combination strategy has a great deal of promise to enhance the reliability and validity of statistical analysis, facilitating better informed decision-making across a variety of disciplines.

4.1.1. *When Positive Correlated*

(1) : Neutrosophic Combined Regression Estimators

Neutrosophic combined regression estimators in stratified random sampling of a finite population mean ($\bar{Y}_{(\delta N)}$) are given as:

$$\hat{\bar{Y}}_{lrC_{11_{(\delta N)}}} = \bar{y}_{st.c11_{(\delta N)}} + b_{c_{(\delta N)}} \left(\bar{X}_{(\delta N)} - \bar{x}_{st.c21_{(\delta N)}} \right) \quad (43)$$

where $b_{c_{(\delta N)}}$ is a neutrosophic combined sample regression coefficient.

(2) : Neutrosophic Separate Regression Estimators

Neutrosophic separate regression estimators in stratified random sampling of a finite population mean ($\bar{Y}_{(\delta N)}$) are given as:

$$\hat{\bar{Y}}_{lrS_{11_{(\delta N)}}} = \sum_{h=1}^{L} W_h \left[\bar{y}_{h.c11_{(\delta N)}} + b_{h_{(\delta N)}} \left(\bar{X}_{h_{(\delta N)}} - \bar{x}_{h.c21_{(\delta N)}} \right) \right] \tag{44}$$

4.1.2. *When Negative Correlated*

(1) : Neutrosophic Combined Regression Estimators

Neutrosophic combined regression estimators in stratified random sampling of a finite population mean ($\bar{Y}_{(\delta N)}$) are given as:

$$\hat{\bar{Y}}_{lrC_{12_{(\delta N)}}} = \bar{y}_{st.c12_{(\delta N)}} + b_{c_{(\delta N)}} \left(\bar{X}_{(\delta N)} - \bar{x}_{st.c22_{(\delta N)}} \right) \tag{45}$$

(2) : Neutrosophic Separate Regression Estimators

Neutrosophic separate regression estimators in stratified random sampling of a finite population mean ($\bar{Y}_{(\delta N)}$) are given as:

$$\hat{\bar{Y}}_{lrS_{12_{(\delta N)}}} = \sum_{h=1}^{L} W_h \left[\bar{y}_{h.c12_{(\delta N)}} + b_{h_{(\delta N)}} \left(\bar{X}_{h_{(\delta N)}} - \bar{x}_{h.c22_{(\delta N)}} \right) \right] \tag{46}$$

Properties of Neutrosophic Combined Regression Estimators

Bias and MSE of combined neutrosophic regression estimator $\hat{\bar{Y}}_{lrC_{11_{(\delta N)}}}$ in case positive correlation given by

$$Bias(\hat{\bar{Y}}_{lrC_{11_{(\delta N)}}}) = -Cov\left(\bar{x}_{st.c21_{(\delta N)}}, b_{c_{(\delta N)}} \right) \tag{47}$$

$$\begin{aligned} MSE(\hat{\bar{Y}}_{lrC_{11_{(\delta N)}}}) \;\cong\; & MSE(\hat{\bar{Y}}_{lrC_{0_{(\delta N)}}}) - \sum_{h=1}^{L} W_h^2 \frac{2\lambda_h n_h}{N_h - 1} (c_{1h_{(\delta N)}} - \beta_{c_{(\delta N)}} c_{2h_{(\delta N)}}) \\ & \Big\{ (y_{h_{max_{(\delta N)}}} - y_{h_{min_{(\delta N)}}}) - \beta_{c_{(\delta N)}} (x_{h_{max_{(\delta N)}}} - x_{h_{min_{(\delta N)}}}) \\ & - n_h (c_{1h_{(\delta N)}} - \beta_{c_{(\delta N)}} c_{2h_{(\delta N)}}) \Big\} \end{aligned} \tag{48}$$

where $\beta_{c_{(\delta N)}} = \dfrac{Cov\left(\bar{y}_{st_{(\delta N)}}, \bar{x}_{st_{(\delta N)}} \right)}{Var\left(\bar{x}_{st_{(\delta N)}} \right)}$ is population regression coefficient.

Min MSE of $\hat{\bar{Y}}_{lrC_{11_{(\delta N)}}}$, is as follows:

$$\begin{aligned} MSE(\hat{\bar{Y}}_{lrC_{11_{(\delta N)}}})_{min} \;=\; & MSE(\hat{\bar{Y}}_{lrC_{0_{(\delta N)}}}) - \sum_{h=1}^{L} W_h^2 \frac{\lambda_h}{2(N_h - 1)} \\ & \left\{ \left(y_{h_{max_{(\delta N)}}} - y_{h_{min_{(\delta N)}}} \right) - \beta_{c_{(\delta N)}} \left(x_{h_{max_{(\delta N)}}} - x_{h_{min_{(\delta N)}}} \right) \right\}^2 \end{aligned} \tag{49}$$

When there is negative correlation , the MSE of neutrosophic combined regression estimator $\hat{\bar{Y}}_{lrC_{12_{(\delta N)}}}$, is given by

$$MSE(\hat{\bar{Y}}_{lrC_{12_{(\delta N)}}}) \cong MSE(\hat{\bar{Y}}_{lrC_{0_{(\delta N)}}}) - \sum_{h=1}^{L} W_h^2 \frac{2\lambda_h n_h}{N_h - 1}\Big(c_{1h_{(\delta N)}} + \beta_{c_{(\delta N)}} c_{2h_{(\delta N)}}\Big) \Big\{\Big(y_{h_{max_{(\delta N)}}} - y_{h_{min_{(\delta N)}}}\Big) + \beta_{c_{(\delta N)}}\Big(x_{h_{max_{(\delta N)}}} - x_{h_{min_{(\delta N)}}}\Big) - n_h\Big(c_{1h_{(\delta N)}} + \beta_{c_{(\delta N)}} c_{2h_{(\delta N)}}\Big)\Big\} \quad (50)$$

Min MSE of $\hat{\bar{Y}}_{lrC_{12_{(\delta N)}}}$, is as follows:

$$MSE(\hat{\bar{Y}}_{lrC_{12_{(\delta N)}}})_{min} \cong MSE(\hat{\bar{Y}}_{lrC_{0_{(\delta N)}}}) - \sum_{h=1}^{L} W_h^2 \frac{\lambda_h}{2(N_h - 1)} \Big\{\Big(y_{h_{max_{(\delta N)}}} - y_{h_{min_{(\delta N)}}}\Big) + \beta_{c_{(\delta N)}}\Big(x_{h_{max_{(\delta N)}}} - x_{h_{min_{(\delta N)}}}\Big)\Big\}^2 \quad (51)$$

Equations (49) and (51)'s findings for the variances of combined regression estimators allow us to calculate the minimum mean square error of the combined regression estimator as follows.

$$MSE(\hat{\bar{Y}}_{lrC_{1_{(\delta N)}}})_{min} \cong MSE(\hat{\bar{Y}}_{lrC_{0_{(\delta N)}}}) - \sum_{h=1}^{L} W_h^2 \frac{\lambda_h}{2(N_h - 1)} \Big\{\Big(y_{h_{max_{(\delta N)}}} - y_{h_{min_{(\delta N)}}}\Big) - |\beta_{c_{(\delta N)}}|\Big(x_{h_{max_{(\delta N)}}} - x_{h_{min_{(\delta N)}}}\Big)\Big\}^2 \quad (52)$$

Properties of Neutrosophic Separate Regression Estimators

Bias and MSE of separate neutrosophic regression estimator $\hat{\bar{Y}}_{lrS_{11_{(\delta N)}}}$ in case positive correlation given by

$$Bias(\hat{\bar{Y}}_{lrS_{11_{(\delta N)}}}) = -\sum_{h=1}^{L} W_{h_{(\delta N)}} Cov\Big(\bar{x}_{h.c21_{(\delta N)}}, b_{h_{(\delta N)}}\Big) \quad (53)$$

$$MSE(\hat{\bar{Y}}_{lrS_{11_{(\delta N)}}}) \cong MSE(\hat{\bar{Y}}_{lrS_{0_{(\delta N)}}}) - \sum_{h=1}^{L} W_h^2 \frac{2\lambda_h n_h}{N_h - 1}\Big(c_{1h_{(\delta N)}} - \beta_{h_{(\delta N)}} c_{2h_{(\delta N)}}\Big) \Big\{\Big(y_{h_{max_{(\delta N)}}} - y_{h_{min_{(\delta N)}}}\Big) - \beta_{h_{(\delta N)}}\Big(x_{h_{max_{(\delta N)}}} - x_{h_{min_{(\delta N)}}}\Big) - n_h\Big(c_{1h_{(\delta N)}} - \beta_{h_{(\delta N)}} c_{2h_{(\delta N)}}\Big)\Big\} \quad (54)$$

where $\beta_{h_{(\delta N)}} = \frac{S_{hyx_{(\delta N)}}}{S^2_{hx_{(\delta N)}}}$ is population regression coefficient in h^{th} stratum.

Min MSE of $\hat{\bar{Y}}_{lrS_{11_{(\delta N)}}}$, is as follows:

$$MSE(\hat{\bar{Y}}_{lrS_{11_{(\delta N)}}})_{min} = MSE(\hat{\bar{Y}}_{lrS_{0_{(\delta N)}}}) - \sum_{h=1}^{L} W_h^2 \frac{\lambda_h}{2(N_h - 1)} \Big\{\Big(y_{h_{max_{(\delta N)}}} - y_{h_{min_{(\delta N)}}}\Big) - \beta_{h_{(\delta N)}}\Big(x_{h_{max_{(\delta N)}}} - x_{h_{min_{(\delta N)}}}\Big)\Big\}^2 \quad (55)$$

When there is negative correlation , the MSE of neutrosophic Separate regression estimator $\hat{\hat{Y}}_{lrS_{12_{(\delta N)}}}$, is given by

$$\begin{aligned} MSE(\hat{\hat{Y}}_{lrS_{12_{(\delta N)}}}) &\cong MSE(\hat{\hat{Y}}_{lrS_{0_{(\delta N)}}}) - \sum_{h=1}^{L} W_h^2 \frac{2\lambda_h n_h}{N_h - 1}\Big(c_{1h_{(\delta N)}} + \beta_{h_{(\delta N)}} c_{2h_{(\delta N)}}\Big) \\ &\Big\{\Big(y_{h_{max_{(\delta N)}}} - y_{h_{min_{(\delta N)}}}\Big) + \beta_{h_{(\delta N)}}\Big(x_{h_{max_{(\delta N)}}} - x_{h_{min_{(\delta N)}}}\Big) \\ &-n_h\Big(c_{1h_{(\delta N)}} + \beta_{h_{(\delta N)}} c_{2h_{(\delta N)}}\Big)\Big\} \end{aligned} \tag{56}$$

Min MSE of $\hat{\hat{Y}}_{lrS_{12_{(\delta N)}}}$, is as follows:

$$\begin{aligned} MSE(\hat{\hat{Y}}_{lrS_{12_{(\delta N)}}})_{min} &= MSE(\hat{\hat{Y}}_{lrS_{0_{(\delta N)}}}) - \sum_{h=1}^{L} W_h^2 \frac{\lambda_h}{2(N_h - 1)} \\ &\Big\{\Big(y_{h_{max_{(\delta N)}}} - y_{h_{min_{(\delta N)}}}\Big) + \beta_{h_{(\delta N)}}\Big(x_{h_{max_{(\delta N)}}} - x_{h_{min_{(\delta N)}}}\Big)\Big\}^2 \end{aligned} \tag{57}$$

Generally the minimum mean square error of Separate regression estimator can be written as

$$\begin{aligned} MSE(\hat{\hat{Y}}_{lrS_{1_{(\delta N)}}})_{min} &= MSE(\hat{\hat{Y}}_{lrS_{0_{(\delta N)}}}) - \sum_{h=1}^{L} W_h^2 \frac{\lambda_h}{2(N_h - 1)} \\ &\Big\{\Big(y_{h_{max_{(\delta N)}}} - y_{h_{min_{(\delta N)}}}\Big) - |\beta_{h_{(\delta N)}}|\Big(x_{h_{max_{(\delta N)}}} - x_{h_{min_{(\delta N)}}}\Big)\Big\}^2 \end{aligned} \tag{58}$$

4.2. *Neutrosophic Robust Regression and Proposed Estimators*

When working with datasets, the presence of outliers or exceptional data points may affect the accuracy of estimates obtained through the ordinary least squares (OLS) approach. OLS, a commonly employed approach for establishing relationships between variables, becomes susceptible in the face of outliers, as even a single outlier can exert undue influence on the results of investigation. The vulnerability has been demonstrated by researchers Hampel et al. [11] and Rousseeuw and Leroy [18],

Seber and Lee [21] explain the causes of OLS's susceptibility to outliers. First off, greater residuals have a disproportionately strong impact on the total error assessment, dominating the contributions of smaller residuals, when mistakes are quantified using the squared deviations among observed and predicted results (residuals). Furthermore, common measures of central tendency, such as the arithmetic mean, are not designed to deal with outliers. Due to the impact of greater squared values, the existence of outliers causes significant distortions in error estimates, magnifying their effect on regression findings.

Alternative techniques, including robust regression, are used to mitigate the negative impact of outliers on regression results. These methods are made to be less impacted by outliers and to give more accurate predictions even when there are odd data points present. Interested parties can read Yu and Yao's study [37] for an in-depth explanation of these robust regression

techniques. Regression using ordinary least squares (OLS) is a popular method for estimating a linear regression model's parameters. Its primary process entails minimising the sum of squared differences between the dependent variable's expected and actual values. Finding the best line that best explains the relationship between the independent and dependent variables while reducing model errors is the major goal. However, OLS relies on several presumptions that could not hold true in all real-world situations, such as the normal distribution of the residuals (the differences between estimated and observed values) and a constant variance.

OLS's increased sensitivity to outliers—data values that significantly deviate from the norm—is an important attribute. Natural variances or mistakes in the data collection process might lead to outliers. Even though it could be irregular, their influence might be significant, skewing predictions of regression coefficients and producing inaccurate forecasts. Enter robust regression, a strategy that has gained interest to address OLS's sensitivity to outliers. Robust regression offers a more robust analysis since it is made to be less impacted by these unusual instances. This is especially helpful when working with real-world data when OLS's presumptions might not hold true. Robust regression allows for the diversity and complexity that are frequently seen in real-world data, allowing analysts to gain more trustworthy conclusions. The differences between the two approaches highlight the constantly developing approaches in statistics and data analysis, highlighting the demand for flexible tools that can negotiate the complexities of real datasets and produce accurate findings even in the face of anomalies. Within robust statistical techniques, robust regression is a subset that focuses on minimising the sum of absolute residues instead of the sum of squared residuals. This novel strategy avoids the weaknesses to outliers that afflict conventional approaches. Extreme data points have less disproportionate effect on results when the absolute values of residuals are highlighted. In robust regression, outliers are given less weight rather than being given the same weight as other data points. This tactic greatly reduces their influence on the regression's estimated coefficients. Ordinary Least Squares (OLS) analysis frequently falls short of robust regression because it is less adept at navigating the turbulent seas of extreme values, especially in circumstances rife with outliers.

The homoscedasticity assumption, which underlies OLS regression, states that residual variance should be consistent over the range of the independent variable. This principle, however, can be undermined by real-world data, particularly if the distribution of the dependent variable is atypical. The difference may result in inaccurate coefficient estimations and uncertain projections. Another obstacle for OLS is sensitivity to outliers, which can reduce precision. With its weighted least squares technique, robust regression intervenes in this situation as a solution to the problem. With this strategy, observations with smaller variations receive heavier weights while those with bigger variances receive lesser weights. The end result is a

more accurate estimation of the coefficients, which holds up well even when there are outliers or important data points. Regression analyses are therefore kept reliable and perceptive even while dealing with the complexity and uncertainties present in real-world data thanks to this strong technique.

When dealing with data that vary from a normal distribution and hence violate the assumption of normality, robust regression outperforms ordinary least squares (OLS) regression. In order for OLS regression to work, the dependent variable must have a normal distribution, which is a requirement that is frequently not satisfied by real-world data. As a result, predictions may turn out to be inaccurate and the coefficients obtained using OLS may be biased. Contrastingly, robust regression does not rely on the normality assumption and has a better ability to deal with non-normal data. Huber-M, Hampel-M, and Tukey-M are three robust regression approaches that are specially used in this work. For a complete understanding of these strategies, please refer to the in-depth study conducted by Zaman [38], Zaman and Bulut ([39], [40], [42]), Shahzad et al.([24], [26], [25]), Sajjad et al. [20], Ali et al. [2], Mukhtar and Shahzad [16], Bulut and Zaman [6].

4.2.1. *When Positive Correlated*

(1) : Neutrosophic Combined Robust Regression Estimators

Neutrosophic combined regression estimators ($\bar{Y}_{(\delta N)}$) are given as:

$$\hat{\bar{Y}}_{(P_i).C11_{(\delta N)}} = \bar{y}_{st.c11_{(\delta N)}} + \beta_{(rob_i).c_{(\delta N)}}\left(\bar{X}_{(\delta N)} - \bar{x}_{st.c21_{(\delta N)}}\right) \tag{59}$$

(2) : Neutrosophic Separate Robust Regression Estimators

Neutrosophic separate regression estimators ($\bar{Y}_{(\delta N)}$) are given as:

$$\hat{\bar{Y}}_{(P_i).S11_{(\delta N)}} = \sum_{h=1}^{L} W_h\left[\bar{y}_{h.c11_{(\delta N)}} + \beta_{(rob_i).h_{(\delta N)}}\left(\bar{X}_{h_{(\delta N)}} - \bar{x}_{h.c21_{(\delta N)}}\right)\right] \tag{60}$$

4.2.2. *When Negative Correlated*

(1) : Neutrosophic Combined Robust Regression Estimators

Neutrosophic combined regression estimators ($\bar{Y}_{(\delta N)}$) are given as:

$$\hat{\bar{Y}}_{(P_i).C12_{(\delta N)}} = \bar{y}_{st.c12_{(\delta N)}} + \beta_{(rob_i).c_{(\delta N)}}\left(\bar{X}_{(\delta N)} - \bar{x}_{st.c22_{(\delta N)}}\right) \tag{61}$$

(2) : Neutrosophic Separate Robust Regression Estimators

Neutrosophic separate regression estimators ($\bar{Y}_{(\delta N)}$) are given as:

$$\hat{\bar{Y}}_{(P_i).S12_{(\delta N)}} = \sum_{h=1}^{L} W_h\left[\bar{y}_{h.c12_{(\delta N)}} + \beta_{(rob_i).h_{(\delta N)}}\left(\bar{X}_{h_{(\delta N)}} - \bar{x}_{h.c22_{(\delta N)}}\right)\right] \tag{62}$$

where $\beta_{(rob_i).c_{(\delta N)}}$ and $\beta_{(rob_i).h_{(\delta N)}}$ are computed from robust regression tools like Huber-M, Hample-M, and Tukey-M for $i = 1, 2, 3$.

Properties of Neutrosophic combined Robust Regression Estimators

The bias and MSE of combined neutrosophic regression estimator $\hat{\bar{Y}}_{(P_i).C_{11_{(\delta N)}}}$ in case positive correlation given by

$$Bias\left(\hat{\bar{Y}}_{(P_i).C_{11_{(\delta N)}}}\right) = -Cov\left(\bar{x}_{st.c21_{(\delta N)}}, b_{c_{(\delta N)}}\right) \tag{63}$$

$$\begin{aligned} MSE\left(\hat{\bar{Y}}_{(P_i).C_{11_{(\delta N)}}}\right) \cong\ & MSE(\hat{\bar{Y}}_{(P_i).C_{0_{(\delta N)}}}) - \sum_{h=1}^{L} W_h^2 \frac{2\lambda_h n_h}{N_h - 1}\left(c_{1h_{(\delta N)}} - \hat{\beta}_{(rob_i).c_{(\delta N)}} c_{2h_{(\delta N)}}\right) \\ & \left\{\left(y_{h_{max_{(\delta N)}}} - y_{h_{min_{(\delta N)}}}\right) - \hat{\beta}_{(rob_i).c_{(\delta N)}}\left(x_{h_{max_{(\delta N)}}} - x_{h_{min_{(\delta N)}}}\right)\right. \\ & \left. -n_h\left(c_{1h_{(\delta N)}} - \hat{\beta}_{(rob_i).c_{(\delta N)}} c_{2h_{(\delta N)}}\right)\right\} \end{aligned} \tag{64}$$

where $\hat{\beta}_{(rob_i).c_{(\delta N)}}$ is population regression coefficient.

Min MSE of $\hat{\bar{Y}}_{(P_i).C_{11_{(\delta N)}}}$, is as follows:

$$\begin{aligned} MSE(\hat{\bar{Y}}_{(P_i).C_{11_{(\delta N)}}})_{min} =\ & MSE(\hat{\bar{Y}}_{(P_i).C_{0_{(\delta N)}}}) - \sum_{h=1}^{L} W_h^2 \frac{\lambda_h}{2(N_h - 1)} \\ & \left\{\left(y_{h_{max_{(\delta N)}}} - y_{h_{min_{(\delta N)}}}\right) - \hat{\beta}_{(rob_i).c_{(\delta N)}}\left(x_{h_{max_{(\delta N)}}} - x_{h_{min_{(\delta N)}}}\right)\right\}^2 \end{aligned} \tag{65}$$

When there is negative correlation , the MSE of neutrosophic combined robust regression estimator $\hat{\bar{Y}}_{(P_i).C_{12_{(\delta N)}}}$, is as follows

$$\begin{aligned} MSE(\hat{\bar{Y}}_{(P_i).C_{12_{(\delta N)}}}) \cong\ & MSE(\hat{\bar{Y}}_{(P_i).C_{0_{(\delta N)}}}) - \sum_{h=1}^{L} W_h^2 \frac{2\lambda_h n_h}{N_h - 1}\left(c_{1h_{(\delta N)}} + \hat{\beta}_{(rob_i).c_{(\delta N)}} c_{2h_{(\delta N)}}\right) \\ & \left\{(y_{h_{max_{(\delta N)}}} - y_{h_{min_{(\delta N)}}}) + \hat{\beta}_{(rob_i).c_{(\delta N)}}\left(x_{h_{max_{(\delta N)}}} - x_{h_{min_{(\delta N)}}}\right)\right. \\ & \left. -n_h\left(c_{1h_{(\delta N)}} + \hat{\beta}_{(rob_i).c_{(\delta N)}} c_{2h_{(\delta N)}}\right)\right\} \end{aligned} \tag{66}$$

Min MSE of $\hat{\bar{Y}}_{(P_i).C_{12_{(\delta N)}}}$, is given by:

$$\begin{aligned} MSE(\hat{\bar{Y}}_{(P_i).C_{12_{(\delta N)}}})_{min} \cong\ & MSE(\hat{\bar{Y}}_{(P_i).C_{0_{(\delta N)}}}) - \sum_{h=1}^{L} W_h^2 \frac{\lambda_h}{2(N_h - 1)} \\ & \left\{\left(y_{h_{max_{(\delta N)}}} - y_{h_{min_{(\delta N)}}}\right) + \hat{\beta}_{(rob_i).c_{(\delta N)}}\left(x_{h_{max_{(\delta N)}}} - x_{h_{min_{(\delta N)}}}\right)\right\}^2 \end{aligned} \tag{67}$$

Equations (65) and (67)'s findings for the variances of combined regression estimators allow us to calculate the minimum mean square error of the combined regression estimator as follows.

$$\begin{aligned} MSE(\hat{\bar{Y}}_{(P_i).C_{1_{(\delta N)}}})_{min} \cong\ & MSE(\hat{\bar{Y}}_{(P_i).C_{0_{(\delta N)}}}) - \sum_{h=1}^{L} W_h^2 \frac{\lambda_h}{2(N_h - 1)} \\ & \left\{\left(y_{h_{max_{(\delta N)}}} - y_{h_{min_{(\delta N)}}}\right) - |\hat{\beta}_{(rob_i).c_{(\delta N)}}|\left(x_{h_{max_{(\delta N)}}} - x_{h_{min_{(\delta N)}}}\right)\right\}^2 \end{aligned} \tag{68}$$

Properties of Neutrosophic Separate Robust Regression Estimators

Bias and MSE of separate neutrosophic regression estimator $\hat{\bar{Y}}_{(P_i).S_{11_{(\delta N)}}}$ in case positive correlation given by

$$Bias(\hat{\bar{Y}}_{(P_i).S_{11_{(\delta N)}}}) = -\sum_{h=1}^{L} W_{h_{(\delta N)}} Cov\Big(\bar{x}_{h.c21_{(\delta N)}}, b_{h_{(\delta N)}}\Big) \tag{69}$$

$$\begin{aligned} MSE(\hat{\bar{Y}}_{(P_i).S_{11_{(\delta N)}}}) \cong\ & MSE(\hat{\bar{Y}}_{(P_i).S_{0_{(\delta N)}}}) - \sum_{h=1}^{L} W_h^2 \frac{2\lambda_h n_h}{N_h - 1}(c_{1h_{(\delta N)}} - \hat{\beta}_{(rob_i).h_{(\delta N)}} \\ & \Big\{(y_{h_{max_{(\delta N)}}} - y_{h_{min_{(\delta N)}}}) - \hat{\beta}_{(rob_i).h_{(\delta N)}}(x_{h_{max_{(\delta N)}}} - x_{h_{min_{(\delta N)}}}) \\ & -n_h(c_{1h_{(\delta N)}} - \hat{\beta}_{(rob_i).h_{(\delta N)}})\Big\} \end{aligned} \tag{70}$$

where $\hat{\beta}_{(rob_i).h_{(\delta N)}}$ is population regression coefficient in h^{th} stratum.

Min MSE of $\hat{\bar{Y}}_{(P_i).S_{11_{(\delta N)}}}$, is as follows:

$$\begin{aligned} MSE(\hat{\bar{Y}}_{(P_i).S_{11_{(\delta N)}}})_{min} =\ & MSE(\hat{\bar{Y}}_{(P_i).S_{0_{(\delta N)}}}) - \sum_{h=1}^{L} W_h^2 \frac{\lambda_h}{2(N_h - 1)} \\ & \Big\{(y_{h_{max_{(\delta N)}}} - y_{h_{min_{(\delta N)}}}) - \hat{\beta}_{(rob_i).h_{(\delta N)}}(x_{h_{max_{(\delta N)}}} - x_{h_{min_{(\delta N)}}})\Big\}^2 \end{aligned} \tag{71}$$

When there is negative correlation , the MSE of neutrosophic separate robust regression estimator $\hat{\bar{Y}}_{(P_i).S_{12_{(\delta N)}}}$, is given by

$$\begin{aligned} MSE(\hat{\bar{Y}}_{(P_i).S_{12_{(\delta N)}}}) \cong\ & MSE(\hat{\bar{Y}}_{(P_i).S_{0_{(\delta N)}}}) - \sum_{h=1}^{L} W_h^2 \frac{2\lambda_h n_h}{N_h - 1}(c_{1h_{(\delta N)}} + \hat{\beta}_{(rob_i).h_{(\delta N)}} \\ & \Big\{(y_{h_{max_{(\delta N)}}} - y_{h_{min_{(\delta N)}}}) + \hat{\beta}_{(rob_i).h_{(\delta N)}}(x_{h_{max_{(\delta N)}}} - x_{h_{min_{(\delta N)}}}) \\ & -n_h(c_{1h_{(\delta N)}} + \hat{\beta}_{(rob_i).h_{(\delta N)}})\Big\} \end{aligned} \tag{72}$$

Min MSE of $\hat{\bar{Y}}_{(P_i).S_{12_{(\delta N)}}}$, is as follows:

$$\begin{aligned} MSE\Big(\hat{\bar{Y}}_{(P_i).S_{12_{(\delta N)}}}\Big)_{min} =\ & MSE\Big(\hat{\bar{Y}}_{(P_i).S_{0_{(\delta N)}}}\Big) - \sum_{h=1}^{L} W_h^2 \frac{\lambda_h}{2(N_h - 1)} \\ & \Big\{\Big(y_{h_{max_{(\delta N)}}} - y_{h_{min_{(\delta N)}}}\Big) + \hat{\beta}_{(rob_i).h_{(\delta N)}}\Big(x_{h_{max_{(\delta N)}}} - x_{h_{min_{(\delta N)}}}\Big)\Big\}^2 \end{aligned} \tag{73}$$

Generally the Min MSE of Separate Robust neutrosophic regression estimator can be written as

$$\begin{aligned} MSE(\hat{\bar{Y}}_{(P_i).S_{1_{(\delta N)}}})_{min} =\ & MSE(\hat{\bar{Y}}_{(P_i).S_{0_{(\delta N)}}}) - \sum_{h=1}^{L} W_h^2 \frac{\lambda_h}{2(N_h - 1)} \\ & \Big\{\Big(y_{h_{max_{(\delta N)}}} - y_{h_{min_{(\delta N)}}}\Big) - |\hat{\beta}_{(rob_i).h_{(\delta N)}}|\Big(x_{h_{max_{(\delta N)}}} - x_{h_{min_{(\delta N)}}}\Big)\Big\}^2 \end{aligned} \tag{74}$$

4.3. *Neutrosophic Robust Quantile Regression and Proposed Estimators*

In contrast to the conventional linear regression strategy, quantile regression presents an alternative that does not rely on the assumptions of constant variance and a normal distribution in error terms. The adaptability of this alternative method avoids the necessity for certain presumptions to be true. Quantile regression predicts the conditional values within the dependent variable's distribution in a

linear model rather than depending on averages [10]. The quartiles used to calculate the coefficients are frequently 0.25, 0.35, 0.45, 0.55, 0.65, 0.75, and 0.85. Robust quantile coefficients for regression take center stage in place of the typical coefficients generated from Ordinary Least Squares (OLS) regression, in contrast to the traditional emphasis on averages.

In our suggestion, neutrosophic estimators are used, providing improved resilience to the disruptive influence of data outliers. When datasets are impacted by outlier data points, these estimators should be used in estimation approaches due to their exceptional resilience. The robust quantile regression method that is suggested delivers estimates that are more accurate even when outliers are present, therefore it is not necessary to exclude these data points from the study.

Together, these viewpoints provide a more comprehensive understanding of the trans formative properties of quantile regression, its stable coefficients, and the novel incorporation of neutrosophic estimators, all of which support a more flexible and accurate analysis of data that may be affected by outliers researchers such as Anas et al. [5], Yadav and Prasad [36], Shahzad et al. ([22], [23]) and Koç and Koç [13] have made significant contributions in the filed of robust quantile regression.

4.3.1. *When Positive Correlated*

(1) : Neutrosophic Combined Robust Quantile Regression Estimators

Neutrosophic combined robust quantile regression estimators in stratified random sampling of a finite population mean ($\bar{Y}_{(\delta N)}$) are given as:

$$\hat{\bar{Y}}_{(Q_i).C_{11_{(\delta N)}}} = \bar{y}_{st.c11_{(\delta N)}} + \beta_{(q_j).c_{(\delta N)}}\left(\bar{X}_{(\delta N)} - \bar{x}_{st.c21_{(\delta N)}}\right) \tag{75}$$

(2) : Neutrosophic Separate Robust Quantile Regression Estimators

Neutrosophic Separate robust quantile regression estimators in stratified random sampling of a finite population mean ($\bar{Y}_{(\delta N)}$) are given as:

$$\hat{\bar{Y}}_{(Q_i).S_{11_{(\delta N)}}} = \sum_{h=1}^{L} W_h\left[\bar{y}_{h.c11_{(\delta N)}} + \beta_{(q_j).h_{(\delta N)}}\left(\bar{X}_{h_{(\delta N)}} - \bar{x}_{h.c21_{(\delta N)}}\right)\right] \tag{76}$$

4.3.2. *When Negative Correlated*

(1) : Neutrosophic Combined Robust Quantile Regression Estimators

Neutrosophic combined robust quantile regression estimators in stratified random sampling of a finite population mean ($\bar{Y}_{(\delta N)}$) are given as:

$$\hat{\bar{Y}}_{(Q_i).C_{12_{(\delta N)}}} = \bar{y}_{st.c12_{(\delta N)}} + \beta_{(q_j).c_{(\delta N)}}\left(\bar{X}_{(\delta N)} - \bar{x}_{st.c22_{(\delta N)}}\right) \tag{77}$$

(2) : Neutrosophic Separate Robust Quantile Regression Estimators

Neutrosophic separate robust quantile regression estimators in stratified random sampling of a finite population mean ($\bar{Y}_{(\delta N)}$) are given as:

$$\hat{\bar{Y}}_{(Q_i).S_{12_{(\delta N)}}} = \sum_{h=1}^{L} W_h\left[\bar{y}_{h.c12_{(\delta N)}} + \beta_{(q_j).h_{(\delta N)}}\left(\bar{X}_{h_{(\delta N)}} - \bar{x}_{h.c22_{(\delta N)}}\right)\right] \tag{78}$$

where $\beta_{(q_j).c_{(\delta N)}}$ and $\beta_{(q_i).h_{(\delta N)}}$ are computed from robust quantile regression tools for $j = 0.15, 0.25, 0.35, ..., 0.85$.

Properties of Neutrosophic Combined Robust Quantile Regression Estimators

Bias and MSE of combined neutrosophic regression estimator $\hat{\bar{Y}}_{(Q_i).C_{11_{(\delta N)}}}$ in case positive correlation given by

$$Bias(\hat{\bar{Y}}_{(Q_i).C_{11_{(\delta N)}}}) = -Cov\Big(\bar{x}_{st.c21_{(\delta N)}}, b_{c_{(\delta N)}}\Big) \tag{79}$$

$$\begin{aligned} MSE(\hat{\bar{Y}}_{(Q_i).C_{11_{(\delta N)}}}) \cong\ & MSE\Big(\hat{\bar{Y}}_{(Q_i).C_{0_{(\delta N)}}}\Big) - \sum_{h=1}^{L} W_h^2 \frac{2\lambda_h n_h}{N_h - 1}\Big(c_{1h_{(\delta N)}} - \hat{\beta}_{(q_j).c_{(\delta N)}} c_{2h_{(\delta N)}}\Big) \\ & \Big\{\Big(y_{h_{max_{(\delta N)}}} - y_{h_{min_{(\delta N)}}}\Big) - \hat{\beta}_{(q_j).c_{(\delta N)}}\Big(x_{h_{max_{(\delta N)}}} - x_{h_{min_{(\delta N)}}}\Big) \\ & -n_h\Big(c_{1h_{(\delta N)}} - \hat{\beta}_{(q_j).c_{(\delta N)}} c_{2h_{(\delta N)}}\Big)\Big\} \end{aligned} \tag{80}$$

where $\hat{\beta}_{(q_j).c_{(\delta N)}}$ is population regression coefficient.
Min MSE of $\hat{\bar{Y}}_{(Q_i).C_{11_{(\delta N)}}}$, is as follows:

$$\begin{aligned} MSE(\hat{\bar{Y}}_{(Q_i).C_{11_{(\delta N)}}})_{min} =\ & MSE(\hat{\bar{Y}}_{(Q_i).C_{0_{(\delta N)}}}) - \sum_{h=1}^{L} W_h^2 \frac{\lambda_h}{2(N_h - 1)} \\ & \Big\{\Big(y_{h_{max_{(\delta N)}}} - y_{h_{min_{(\delta N)}}}\Big) - \hat{\beta}_{(q_j).c_{(\delta N)}}\Big(x_{h_{max_{(\delta N)}}} - x_{h_{min_{(\delta N)}}}\Big)\Big\}^2 \end{aligned} \tag{81}$$

When there is negative correlation , the MSE of neutrosophic combined robust quantile regression estimator $\hat{\bar{Y}}_{(Q_i).C_{12_{(\delta N)}}}$, is as follows

$$\begin{aligned} MSE(\hat{\bar{Y}}_{(Q_i).C_{12_{(\delta N)}}}) \cong\ & MSE(\hat{\bar{Y}}_{(Q_i).C_{0_{(\delta N)}}}) - \sum_{h=1}^{L} W_h^2 \frac{2\lambda_h n_h}{N_h - 1}\Big(c_{1h_{(\delta N)}} + \hat{\beta}_{(q_j).c_{(\delta N)}} c_{2h_{(\delta N)}}\Big) \\ & \Big\{\Big(y_{h_{max_{(\delta N)}}} - y_{h_{min_{(\delta N)}}}\Big) + \hat{\beta}_{(q_j).c_{(\delta N)}}\Big(x_{h_{max_{(\delta N)}}} - x_{h_{min_{(\delta N)}}}\Big) \\ & -n_h\Big(c_{1h_{(\delta N)}} + \hat{\beta}_{(q_j).c_{(\delta N)}} c_{2h_{(\delta N)}}\Big)\Big\} \end{aligned} \tag{82}$$

Min MSE of $\hat{\bar{Y}}_{(Q_i).C_{12_{(\delta N)}}}$, is as follows:

$$\begin{aligned} MSE(\hat{\bar{Y}}_{(Q_i).C_{12_{(\delta N)}}})_{min} \cong\ & MSE(\hat{\bar{Y}}_{(Q_i).C_{0_{(\delta N)}}}) - \sum_{h=1}^{L} W_h^2 \frac{\lambda_h}{2(N_h - 1)} \\ & \Big\{\Big(y_{h_{max_{(\delta N)}}} - y_{h_{min_{(\delta N)}}}\Big) + \hat{\beta}_{(q_j).c_{(\delta N)}}\Big(x_{h_{max_{(\delta N)}}} - x_{h_{min_{(\delta N)}}}\Big)\Big\}^2 \end{aligned} \tag{83}$$

Equations (81) and (83)'s findings for the variances of combined regression estimators allow us to calculate the minimum mean square error of the combined regression estimator as follows.

$$\begin{aligned} MSE(\hat{\bar{Y}}_{(Q_i).C_{1_{(\delta N)}}})_{min} \cong\ & MSE(\hat{\bar{Y}}_{(Q_i).C_{0_{(\delta N)}}}) - \sum_{h=1}^{L} W_h^2 \frac{\lambda_h}{2(N_h - 1)} \\ & \Big\{\Big(y_{h_{max_{(\delta N)}}} - y_{h_{min_{(\delta N)}}}\Big) - |\hat{\beta}_{(q_j).c_{(\delta N)}}|\Big(x_{h_{max_{(\delta N)}}} - x_{h_{min_{(\delta N)}}}\Big)\Big\}^2 \end{aligned} \tag{84}$$

Properties of Neutrosophic Separate Robust Quantile Regression Estimators

Bias and MSE of separate neutrosophic robust quantile regression estimator $\hat{\bar{Y}}_{(Q_i).S_{11_{(\delta N)}}}$ in case positive correlation given by

$$Bias(\hat{\bar{Y}}_{(Q_i).S_{11_{(\delta N)}}}) = -\sum_{h=1}^{L} W_{h_{(\delta N)}} Cov\Big(\bar{x}_{h.c21_{(\delta N)}}, b_{h_{(\delta N)}}\Big) \tag{85}$$

$$MSE(\hat{\bar{Y}}_{(Q_i).S_{11_{(\delta N)}}}) \cong MSE(\hat{\bar{Y}}_{(Q_i).S_{0_{(\delta N)}}}) - \sum_{h=1}^{L} W_h^2 \frac{2\lambda_h n_h}{N_h - 1}(c_{1h_{(\delta N)}} - \hat{\beta}_{(q_i).h_{(\delta N)}} \left\{\left(y_{h_{max_{(\delta N)}}} - y_{h_{min_{(\delta N)}}}\right) - \hat{\beta}_{(q_i).h_{(\delta N)}}\left(x_{h_{max_{(\delta N)}}} - x_{h_{min_{(\delta N)}}}\right) - n_h(c_{1h_{(\delta N)}} - \hat{\beta}_{(q_i).h_{(\delta N)}})\right\} \quad (86)$$

where $\hat{\beta}_{(q_i).h_{(\delta N)}}$ is population regression coefficient in h^{th} stratum.
Min MSE of $\hat{\bar{Y}}_{(Q_i).S_{11_{(\delta N)}}}$, is as follows:

$$MSE(\hat{\bar{Y}}_{(Q_i).S_{11_{(\delta N)}}})_{min} = MSE(\hat{\bar{Y}}_{(Q_i).S_{0_{(\delta N)}}}) - \sum_{h=1}^{L} W_h^2 \frac{\lambda_h}{2(N_h - 1)} \left\{\left(y_{h_{max_{(\delta N)}}} - y_{h_{min_{(\delta N)}}}\right) - \hat{\beta}_{(q_i).h_{(\delta N)}}\left(x_{h_{max_{(\delta N)}}} - x_{h_{min_{(\delta N)}}}\right)\right\}^2 \quad (87)$$

When there is negative correlation , the MSE of neutrosophic separate robust quantile regression estimator $\hat{\bar{Y}}_{(Q_i).S_{12_{(\delta N)}}}$, is given by

$$MSE(\hat{\bar{Y}}_{(Q_i).S_{12_{(\delta N)}}}) \cong MSE(\hat{\bar{Y}}_{(Q_i).S_{0_{(\delta N)}}}) - \sum_{h=1}^{L} W_h^2 \frac{2\lambda_h n_h}{N_h - 1}(c_{1h_{(\delta N)}} + \hat{\beta}_{(q_i).h_{(\delta N)}} \left\{\left(y_{h_{max_{(\delta N)}}} - y_{h_{min_{(\delta N)}}}\right) + \hat{\beta}_{(rob_i).h_{(\delta N)}}\left(x_{h_{max_{(\delta N)}}} - x_{h_{min_{(\delta N)}}}\right) - n_h(c_{1h_{(\delta N)}} + \hat{\beta}_{(q_i).h_{(\delta N)}})\right\} \quad (88)$$

Min MSE of $\hat{\bar{Y}}_{(Q_i).S_{12_{(\delta N)}}}$, is as follows:

$$MSE(\hat{\bar{Y}}_{(Q_i).S_{12_{(\delta N)}}})_{min} = MSE(\hat{\bar{Y}}_{(Q_i).S_{0_{(\delta N)}}}) - \sum_{h=1}^{L} W_h^2 \frac{\lambda_h}{2(N_h - 1)} \left\{\left(y_{h_{max_{(\delta N)}}} - y_{h_{min_{(\delta N)}}}\right) + \hat{\beta}_{(q_i).h_{(\delta N)}}\left(x_{h_{max_{(\delta N)}}} - x_{h_{min_{(\delta N)}}}\right)\right\}^2 \quad (89)$$

Generally the Min MSE of Separate robust neutrosophic quantile regression estimator can be given as

$$MSE(\hat{\bar{Y}}_{(Q_i).S_{1_{(\delta N)}}})_{min} = MSE(\hat{\bar{Y}}_{(Q_i).S_{0_{(\delta N)}}}) - \sum_{h=1}^{L} W_h^2 \frac{\lambda_h}{2(N_h - 1)} \left\{\left(y_{h_{max_{(\delta N)}}} - y_{h_{min_{(\delta N)}}}\right) - |\hat{\beta}_{(q_i).h_{(\delta N)}}|\left(x_{h_{max_{(\delta N)}}} - x_{h_{min_{(\delta N)}}}\right)\right\}^2 \quad (90)$$

5. **Applications**

The use of weather data, notably dew point, emerges as a crucial and cutting-edge tactic in the context of our work, which is committed to improving sample survey data processing approaches. Assessing the impact of urban heat islands on locals' health in four important Pakistani cities—Karachi, Lahore, Peshawar, and Multan—is the central focus of our study. Our dataset has a total of 1460 data points, which corresponds to the four strata and each including 365 days of dew point data, with each city acting as a separate stratum. We take a unique strategy to improve estimation accuracy in the face of the complexity of outliers and imprecise findings, which are particularly prominent in humidity and temperature data.

The incorporation of dew point in the form of an auxiliary variable assumes a crucial role in this endeavor. Dew point offers an additional source of information that considerably improves our analysis as an indication of atmospheric moisture. This integration gives our investigation of the urban heat island phenomena a multidimensional dimension that is specially designed for it. Our analytical methodology overcomes the limitations imposed by single factors by comparing dew point alongside temperature and humidity data. The result of this complementary fusion is a notable improvement in estimating accuracy. Notably, the addition of dew point as an auxiliary variable enables us to stabilise our results, especially by reducing the likelihood of distortions caused by high values—a problem that outliers by their very nature provide.

The vital significance of dew point data is made clear as a result of this novel technique to navigating the complicated terrain of urban health dynamics. Beyond supporting the validity of our findings, this integration has larger implications that go across academic fields. The multidisciplinary scope includes areas like public health policy and urban planning, where precise data analysis is crucial.

Additionally, our system is built to cover every possible angle. Starting with combined as well as separate regression estimators accommodating both positive and negative data distributions, we meticulously explore a variety of estimate strategies. We guarantee the validity and usefulness of our findings by taking this heterogeneity into consideration. Next, we move on to the topic of robust regression algorithms, which are made to resist the effects of outliers and anomalies present in real-world data. Finally, robust quantile regression, which is a robust method that not only takes into consideration the existence of outliers but also offers insights into various quantiles of the data distribution, serves as the point where our trip comes to an end.

As we go forward, the mean squared error (MSE) becomes a crucial parameter for assessing the effectiveness of our suggested estimators. The MSE provides a thorough assessment of estimator accuracy by calculating the average squared difference between our estimators and the real values. This essential evaluation enables us to determine how effectively our novel approaches provide accurate estimates in the presence of outliers and erroneous observations. Interestingly, the study's least MSE value is displayed by our suggested neutrosophic estimators, further demonstrating their superiority in handling complicated data circumstances. Neutrosophic Summary of Weather Data (Dew Point) is given in table [1].

6. **Results Discussion**

The empirical examination of our suggested estimators, supplemented by Weather data (Dew Point), produced interesting findings that validate the efficacy of our novel technique. The evaluation includes a wide range of estimators, such ratio, product, robust regression, as well as robust quantile regression, each of which is evaluated for Mean Squared Error (MSE) efficiency.

(I): The MSE values for our estimators determined by dew point data are shown in Table [2]. Notable are the ratio estimators, indicated as $\hat{\hat{Y}}_{RC_{0_{(\delta N)}}}$, $\hat{\hat{Y}}_{RC_{1_{(\delta N)}}}$, $\hat{\hat{Y}}_{RS_{0_{(\delta N)}}}$, and $\hat{\hat{Y}}_{RS_{1_{(\delta N)}}}$, show results in the range [9.703, 10.218], demonstrating how well they did in capturing the fundamental demographic factors. Product estimators, $\hat{\hat{Y}}_{PC_{0_{(\delta N)}}}$, $\hat{\hat{Y}}_{PC_{1_{(\delta N)}}}$, $\hat{\hat{Y}}_{PS_{0_{(\delta N)}}}$, and $\hat{\hat{Y}}_{PS_{1_{(\delta N)}}}$, display MSE results in the range [835.224, 984.712], highlighting their resilience against data fluctuations. Additionally, the regression estimators, $\hat{\hat{Y}}_{lrC_{0_{(\delta N)}}}$, $\hat{\hat{Y}}_{lrC_{1_{(\delta N)}}}$, $\hat{\hat{Y}}_{lrS_{0_{(\delta N)}}}$, and $\hat{\hat{Y}}_{lrS_{1_{(\delta N)}}}$, demonstrate favorable MSE results in

the range [9.694, 9.965]. These findings demonstrate how well-suited these estimators are to handle scenarios involving complicated data.

We shifted to robust regression approaches for delving deeper into robustness, where
(II): The MSE results for the robust regression estimators are presented in Table [3]. A member of these $\hat{\bar{Y}}_{lrC_{1_{(\delta N)}}}$ and alternatives using the robustness methods Huber-M, Hampel-M, and Tukey-M provide MSE values that are comparable with the baseline regression estimators. This consistency highlights how well these estimators handle outliers and erroneous observations, leading to more accurate findings.

(III): Our study is expanded upon in Table [4] by robust quantile regression estimation tools. Particularly, when using the combined and separate regression methods, the MSE data across quantiles like $Q_{0.25}$, $Q_{0.35}$, $Q_{0.45}$, $Q_{0.55}$, $Q_{0.65}$, $Q_{0.75}$, and $Q_{0.85}$ stay closely grouped and within the [9.690, 9.696] range. This consistency underlines the robust quantile regression's capacity to offer consistent and trustworthy predictions across many quantile levels, assisting in the development of a more thorough knowledge of the data distribution.

The suggested neutrosophic estimators in the study show the lowest MSE values among the tested estimators, which is very noteworthy. This discovery highlights how effective our novel method is at improving estimation accuracy, especially when there are outliers and ambiguous observations. These findings confirm the potential of neutrosophic estimators to provide reliable statistical insights and more precise population mean estimations.

TABLE 1. Neutrosophic Summary of Weather Data (Dew Point).

Notations	**Stratum-1**	**Stratum-2**	**Stratum-3**	**Stratum-4**
n_h	[365]	[365]	[365]	[365]
$\bar{Y}_{h_{(\delta N)}}$	[76.03014, 102.4466]	[78.46849, 96.77534]	[70.90137, 77.46301]	[78.8, 81.94521]
$\bar{X}_{h_{(\delta N)}}$	[74.78356, 99.88767]	[73.59178, 96.74247]	[73.59452, 74.72055]	[75.45753, 81.88767]
$S^2_{hy_{(\delta N)}}$	[40246, 63944.32]	[40246.29, 64126.03]	[40185.34, 15747.04]	[40235.54, 15919.27]
$S^2_{hx_{(\delta N)}}$	[40591.26, 64155.49]	[37724.89, 63984.47]	[40214.41, 15831.21]	[37895.75, 15919.27]
$y_{h_{max_{(\delta N)}}}$	[2039]	[2041]	[2028]	[2041]
$y_{h_{min_{(\delta N)}}}$	[34]	[45]	[39]	[46]
$x_{h_{max_{(\delta N)}}}$	[2046]	[2043]	[2045]	[2048]
$x_{h_{min_{(\delta N)}}}$	[0]	[0]	[0]	[0]
$\hat{\beta}_{(rob_{(Huber-M)})_{(\delta N)}}$	[0.9912323, 0.9999753]	[0.9999594, 1.000504]	[0.9937949, 0.9950768]	[0.996666, 0.9984121]
$\hat{\beta}_{(rob_{(Hampel-M)})_{(\delta N)}}$	[0.9908839, 0.9993864]	[1.000555, 1.000763]	[0.9939584, 0.996543]	[0.9972101, 0.9977789]
$\hat{\beta}_{(rob_{(Tuckey-M)})_{(\delta N)}}$	[0.993372, 1.003817]	[1.000812, 1.000578]	[0.9938141, 0.9959412]	[0.997544, 0.9984946]
$\hat{\beta}_{(q_{(0.25)}).h_{(\delta N)}}$	[0.9959370, 0.9929364]	[1.0005076, 1.0015038]	[0.997544, 0.9984946]	[0.9985007, 1.0010030]
$\hat{\beta}_{(q_{(0.35)}).h_{(\delta N)}}$	[0.9939668, 0.9919355]	[1.000000, 1.0010081]	[0.9964485, 0.9938838]	[0.9974975, 1.0000000]
$\hat{\beta}_{(q_{(0.45)}).h_{(\delta N)}}$	[0.9929293, 1.00000]	[1.0010030, 1.0010147]	[0.9954292, 0.9919920]	[0.9965000, 0.9979859]
$\hat{\beta}_{(q_{(0.55)}).h_{(\delta N)}}$	[0.9915966, 0.9989909]	[1.00000, 1.0010010]	[0.9940090, 0.9939086]	[0.9955202, 0.9989960]
$\hat{\beta}_{(q_{(0.65)}).h_{(\delta N)}}$	[0.9903943, 1.0025432]	[1.0000000, 1.000000]	[0.9929613, 1.000000]	[0.9949900, 0.9979445]
$\hat{\beta}_{(q_{(0.75)}).h_{(\delta N)}}$	[0.9906863, 1.0015228]	[0.9990000, 0.9994926]	[0.9924887, 1.00000]	[0.9964877, 1.0000000]
$\hat{\beta}_{(q_{(0.85)}).h_{(\delta N)}}$	[0.9871668, 1.0015060]	[0.9974912, 0.9984864]	[0.9909684, 1.0020346]	[0.9945110, 0.9989889]

TABLE 2. Mean Square error of the Estimators based on Weather Data (Dew Point).

Estimators	**MSE**
Ratio Estimators	
$\hat{\hat{Y}}_{RC_{0_{(\delta N)}}}$	[9.9039479, 0.3649951]
$\hat{\hat{Y}}_{RC_{1_{(\delta N)}}}$	[9.8270814, 0.3342089]
$\hat{\hat{Y}}_{RS_{0_{(\delta N)}}}$	[10.2176925, 0.4133462]
$\hat{\hat{Y}}_{RS_{1_{(\delta N)}}}$	[10.0927650, 0.3727853]
Product Estimators	
$\hat{\hat{Y}}_{PC_{0_{(\delta N)}}}$	[975.3846, 984.7125]
$\hat{\hat{Y}}_{PC_{1_{(\delta N)}}}$	[975.3078, 984.6818]
$\hat{\hat{Y}}_{PS_{0_{(\delta N)}}}$	[974.6680, 983.6323]
$\hat{\hat{Y}}_{PS_{1_{(\delta N)}}}$	[835.2242, 897.9044]
Regression Estimators	
$\hat{\hat{Y}}_{lrC_{0_{(\delta N)}}}$	[9.7058299, 0.2982712]
$\hat{\hat{Y}}_{lrC_{1_{(\delta N)}}}$	[9.6942937, 0.2869683]
$\hat{\hat{Y}}_{lrS_{0_{(\delta N)}}}$	[9.9656934, 0.2974041]
$\hat{\hat{Y}}_{lrS_{1_{(\delta N)}}}$	[9.8900701, 0.2904405]

TABLE 3. Mean Square error of the Robust Regression Estimators based on Weather Data (Dew Point).

Estimators	**MSE**
Robust Combined Regression	
$\hat{\hat{Y}}_{lrC_{1_{(\delta N)}}}$	[9.6942937, 0.2869683]
$\hat{\hat{Y}}_{(P_{Huber-M}).C_{1_{(\delta N)}}}$	[9.6928511, 0.2871328]
$\hat{\hat{Y}}_{(P_{Hampel-M}).C_{1_{(\delta N)}}}$	[9.6927111, 0.2870227]
$\hat{\hat{Y}}_{(P_{Tuckey-M}).C_{1_{(\delta N)}}}$	[9.6926003, 0.2863628]
Robust Separate Regression	
$\hat{\hat{Y}}_{lrS_{1_{(\delta N)}}}$	[9.8900701, 0.2904405]
$\hat{\hat{Y}}_{(P_{Huber-M}).S_{1_{(\delta N)}}}$	[9.690895, 0.286713]
$\hat{\hat{Y}}_{(P_{Hampel-M}).S_{1_{(\delta N)}}}$	[9.6907605, 0.2864177]
$\hat{\hat{Y}}_{(P_{Tuckey-M}).S_{1_{(\delta N)}}}$	[9.6907406, 0.2877568]

TABLE 4. Mean Square error of the Robust Quantile Regression Estimators based on Weather Data (Dew Point).

Estimators	MSE
Robust Quantile Combined Regression	
$\hat{\hat{Y}}_{lrC_{1_{(\delta N)}}}$	[9.6942937, 0.2869683]
$\hat{\hat{Y}}_{(Q_{0.25}).C_{1_{(\delta N)}}}$	[9.6923676, 0.2896927]
$\hat{\hat{Y}}_{(Q_{0.35}).C_{1_{(\delta N)}}}$	[9.6923538, 0.2894481]
$\hat{\hat{Y}}_{(Q_{0.45}).C_{1_{(\delta N)}}}$	[9.6924074, 0.2879263]
$\hat{\hat{Y}}_{(Q_{0.55}).C_{1_{(\delta N)}}}$	[9.6929382, 0.2873903]
$\hat{\hat{Y}}_{(Q_{0.65}).C_{1_{(\delta N)}}}$	[9.6935143, 0.2862498]
$\hat{\hat{Y}}_{(Q_{0.75}).C_{1_{(\delta N)}}}$	[9.6934387, 0.2862297]
$\hat{\hat{Y}}_{(Q_{0.85}).C_{1_{(\delta N)}}}$	[9.6963570, 0.2862297]
Robust Quantile Separate Regression	
$\hat{\hat{Y}}_{lrS_{1_{(\delta N)}}}$	[9.8900701, 0.2904405]
$\hat{\hat{Y}}_{(Q_{0.25}).S_{1_{(\delta N)}}}$	[9.6915248, 0.2926856]
$\hat{\hat{Y}}_{(Q_{0.35}).S_{1_{(\delta N)}}}$	[9.6904203, 0.2928867]
$\hat{\hat{Y}}_{(Q_{0.45}).S_{1_{(\delta N)}}}$	[9.6904174, 0.2875686]
$\hat{\hat{Y}}_{(Q_{0.55}).S_{1_{(\delta N)}}}$	[9.691002, 0.287058]
$\hat{\hat{Y}}_{(Q_{0.65}).S_{1_{(\delta N)}}}$	[9.6916736, 0.2867495]
$\hat{\hat{Y}}_{(Q_{0.75}).S_{1_{(\delta N)}}}$	[9.6916991, 0.2866265]
$\hat{\hat{Y}}_{(Q_{0.85}).S_{1_{(\delta N)}}}$	[9.6946025, 0.2870798]

The importance of include Weather data, particularly dew point, in the analysis of sample survey data is highlighted by our study's thorough review, as shown by the MSE values. The innovative neutrosophic estimators, in particular, are frequently highlighted by the findings for their robustness, accuracy, and dependability. These results support our claim that novel techniques like these have the potential to revolutionise statistical analysis in the face of complicated data situations, resulting in more insightful and useful information.

7. **Conclusion**

In the analysis of sample survey data, this proposed study presents a novel approach that uses meteorological information—specifically, Dew point—as an auxiliary variable. This novel method overcomes the shortcomings of outliers and ambiguous observations, which frequently impair the precision of traditional estimating methods. The mean estimator is particularly highlighted in the study, which deliberately focuses on the significant influence of uncommon data on statistical estimators. It emphasises how important it is to keep outliers out of the results in order to maintain their reliability.

The study adopts a proactive approach to tackling this problem by exploring the integration of auxiliary factors and utilising the informative potential of dew point data to improve estimation accuracy. The research of the study also broadens to include problems brought on by anomalous results in a sample of survey data. It reveals how susceptible the mean estimator is to outliers, which might skew results. The study investigates approaches for auxiliary variable integration, such as transformations, to improve accuracy in order to counteract this. In particular, it recommends the use of known extreme

values as supplementary information to enhance estimation precision. The limitations of standard statistical approaches that rely on exact data and supplementary information becomes clear in situations characterised by outlier presence. To address this, the research employs novel neutrosophic estimators inside a designed effectively stratified random sampling system. This paradigm is ideally suited to dealing with uncertain along with imprecise data while also resisting the effect of outliers. Furthermore, the study broadens its scope by include dew point as a possible auxiliary variable, which significantly improves estimation accuracy.

The possible influence of dew point in combination with neutrosophic versions of robust regression techniques such as Huber-M, Hampel-M, and Tukey-M is examined at a pivotal point in the study. These approaches have been chosen purposefully to strengthen estimation accuracy against the disruptive character of outliers. This investigation is expanded to include robust quantile regression, offering a full and thorough estimate technique. A critical stage in the research is the thorough evaluation of the proposed methodologies, including the novel incorporation of dew point, using computer trials with real-world data. The results of these trials clearly demonstrate the superiority of the suggested techniques over standard estimators, especially in the face of outlier-ridden and ambiguous circumstances. This empirical confirmation is a compelling testimonial to the suggested techniques' durability, efficacy, and adaptability.

In the proposed study, an innovative approach for estimating the survey sample mean in the presence of outliers and ambiguous observations is presented. To get around the shortcomings associated with traditional estimating techniques, the strategy makes use of neutrosophic estimators using dew point as an auxiliary variable. The study emphasises how important it is to keep outliers out of the findings in order to retain the results' dependability and emphasises the major impact that rare data have on statistical estimators. The suggested approaches are thoroughly evaluated using computer simulations with real-world data, and the findings clearly show that they outperformed traditional estimators, even in the presence of outlier-filled and ambiguous conditions. Overall, the work has the potential to significantly improve the methodology of surveys and the discipline of statistics.

Data Availability Statement All the relevant data information is available within the manuscript.

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Quantifying Teaching Quality in University Chemistry Using Bipolar Neutrosophic Decision-Making Framework

Lina Sun[1], Feng Yi[2*]
[1]Sichuan College of Traditional Chinese Medicine, Mianyang, 621000, Sichuan, China
[2]Sichuan College of Traditional Chinese Medicine, Mianyang, 621000, Sichuan, China
*Corresponding author, E-mail: yfsczyy@163.com

Abstract
There is no doubt that the teaching of university Chemistry plays a critical role in enriching the students' knowledge with analytical and problem-solving skills that are necessary for achieving academic and professional progress in many fields. However, the process of evaluating the quality of Chemistry teaching remains a challenging task because it usually relies on subjective judgments, diversity of criteria, and variability in human perception, leading to inherent uncertainty. Bipolar Neutrosophic Sets (NS) provide appropriate representation for uncertainty setting ls with three levels of membership, each with positive and negative degrees. Inspired by that, this study contributes to the body of knowledge by proposing a Neutrosophic approach that can assess the quality of teaching of university psychics considering different factors and teaching scenarios. The weight of the Bipolar Neutrosophic number associated with each teaching criterion is determined using the IDOCRIW. Then, we introduce a multi-criteria decision-making (MCDM) that integrates a Minkowski-based TOPSIS method working on Bipolar NS. Proof-of-concept analysis is conducted to solve numerical case studies analyzing different teaching alternatives, and the results demonstrate the ability of our method to appropriately make the correct decision that maximizes providing valuable insights about the teaching scenario that can maximize the student benefit from the Chemistry course.

Keywords: Neutrosophic Theory, Bipolar Neutrosophic sets, Teaching Quality, MCDM, University Chemistry.

1. Introduction

In recent years, the quality of teaching in higher education has become a critical area of focus, especially in disciplines like Chemistry, where effective teaching plays a pivotal role in shaping students' understanding of complex concepts [1]. The evaluation of teaching quality remains an open challenge because it involves different subjective and objective factors, such as student satisfaction, instructional methods, learning outcomes, and course complexity [2]. Traditional evaluation techniques usually fail to take into account the inherent uncertainty, vagueness, and

indeterminacy in the feedback and data collected from diverse stakeholders [3]. Neutrosophic set (NS), was presented by Florentin Smarandache [4], as a kind of advanced mathematical framework that extends classical and fuzzy logic by incorporating three components: truth (T), indeterminacy (I), and falsity (F). Different from traditional approaches, NSs allow for a more flexible representation of information, enabling the simultaneous handling of uncertainty, vagueness, and contradictions. All of these three components are independent and might take any value in the range [0, 1], making the theory particularly effective in dealing with complex real-world problems characterized by incomplete or conflicting data. Bipolar NSs extend standard NSs by incorporating two poles for truth, indeterminacy, and falsity: one representing positive information and the other representing negative information [5]. This duality allows BNS to model situations where positive and negative aspects coexist and interact, providing a richer representation of complex phenomena [6], [7], [8]. For instance, in evaluating teaching quality, student feedback might simultaneously reflect positive aspects (e.g., engaging teaching style) and negative aspects (e.g., unclear explanations). The bipolar NS effectively captures this dual nature, offering a nuanced approach to decision-making and analysis in uncertain and contradictory environments. This makes it particularly well-suited for evaluating teaching quality, where diverse perspectives and subjective opinions play a significant role [5], [9].

To address these challenges, this paper proposes a novel approach to assessing teaching quality in university Chemistry using neutrosophic theory. This article presents a neutrosophic-based MCDM approach to model and analyze the different aspects, which may affect teaching quality in university Chemistry. Our framework analyzes both qualitative and quantitative data, which allows inclusive valuation for the effectiveness of the teaching process. Based on a conductive analysis of the university case study, we demonstrated the ability of the proposed approach to help decision-makers identify the primary strengths and the dimensions that need to be improved to further enhance the overall quality of teaching Chemistry. The proof-of-concept quantitative analysis shows that the Bi-polar NS provides actionable insights for educators and institutions, ultimately contributing to the enhancement of Chemistry education.

The remainder of this study is planned as follows. Section 2 summarizes the main concepts and definitions. Section 3 presents the proposed neutrosophic MCDM method. Section 4 discusses the results. Section 5 gives a conclusion about the findings.

2. Foundational Concepts

Inspired by the definition of NS, Bi-polar NSs were introduced [5] by extending each membership component to have positive as well as negative degrees. Thus, the Bipolar NS is defined as:

$$\overleftrightarrow{B} = \left\{c, \langle \mathcal{T}_{\overleftrightarrow{B}}^{+}(c), \mathcal{I}_{\overleftrightarrow{B}}^{+}(c), \mathcal{F}_{\overleftrightarrow{B}}^{+}(c), \mathcal{T}_{\overleftrightarrow{B}}^{-}(c), \mathcal{I}_{\overleftrightarrow{B}}^{-}(c), \mathcal{F}_{\overleftrightarrow{B}}^{-}(c) \rangle \middle| c \in C\right\}, \quad (1)$$

Where

$$\mathcal{T}_{\overleftrightarrow{B}}^{+}(c), \mathcal{I}_{\overleftrightarrow{B}}^{+}(c), \mathcal{F}_{\overleftrightarrow{B}}^{+}(c): C \to [0,1]$$

$$\mathcal{T}_{\vec{B}}^{-}(c), \mathcal{I}_{\vec{B}}^{-}(c), \mathcal{F}_{\vec{B}}^{-}(c): C \to [-1,0]$$

Given two bipolar NSs $S_1 = \left\{c, \langle \mathcal{T}_{\vec{B}_1}^{+}(c), \mathcal{I}_{\vec{B}_1}^{+}(c), \mathcal{F}_{\vec{B}_1}^{+}(c), \mathcal{T}_{\vec{B}_1}^{-}(c), \mathcal{I}_{\vec{B}_1}^{-}(c), \mathcal{F}_{\vec{B}_1}^{-}(c) \rangle \middle| c \in C \right\}$, and $S_2 = \left\{c, \langle \mathcal{T}_{\vec{B}_2}^{+}(c), \mathcal{I}_{\vec{B}_2}^{+}(c), \mathcal{F}_{\vec{B}_2}^{+}(c), \mathcal{T}_{\vec{B}_2}^{-}(c), \mathcal{I}_{\vec{B}_2}^{-}(c), \mathcal{F}_{\vec{B}_2}^{-}(c) \rangle \middle| c \in C \right\}$, The basic set of operations can be summarized as follows:

1) $S_1 \subseteq S_2$ if, and only if,

$$\mathcal{T}_{\vec{B}_1}^{+}(c) < \mathcal{T}_{\vec{B}_2}^{+}(c), \mathcal{F}_{\vec{B}_1}^{+}(c) < \mathcal{F}_{\vec{B}_2}^{+}(c), \mathcal{I}_{\vec{B}_1}^{+}(c) \geq \mathcal{I}_{\vec{B}_2}^{+}(c) \tag{2}$$

and

$$\mathcal{T}_{\vec{B}_1}^{-}(c) < \mathcal{T}_{\vec{B}_2}^{-}(c), \mathcal{F}_{\vec{B}_1}^{-}(c) < \mathcal{F}_{\vec{B}_2}^{-}(c), \mathcal{I}_{\vec{B}_1}^{-}(c) \geq \mathcal{I}_{\vec{B}_2}^{-}(c) \tag{3}$$

2) $S_1 = S_2$ if, and only if,

$$\mathcal{T}_{\vec{B}_1}^{+}(c) = \mathcal{T}_{\vec{B}_2}^{+}(c), \mathcal{F}_{\vec{B}_1}^{+}(c) = \mathcal{F}_{\vec{B}_2}^{+}(c), \mathcal{I}_{\vec{B}_1}^{+}(c) = \mathcal{I}_{\vec{B}_2}^{+}(c) > \tag{4}$$

and

$$\mathcal{T}_{\vec{B}_1}^{-}(c) = \mathcal{T}_{\vec{B}_2}^{-}(c), \mathcal{F}_{\vec{B}_1}^{-}(c) = \mathcal{F}_{\vec{B}_2}^{-}(c), \mathcal{I}_{\vec{B}_1}^{-}(c) = \mathcal{I}_{\vec{B}_2}^{-}(c) > \tag{5}$$

3) The intersection of both S_1 and S_2 is defined as:

$$S_1 \cap S_2 = \left\{ c, \min\left(\mathcal{T}_{\vec{B}_1}^{+}(c), \mathcal{T}_{\vec{B}_2}^{+}(c)\right), \frac{\mathcal{I}_{\vec{B}_1}^{+}(c) + \mathcal{I}_{\vec{B}_2}^{+}(c)}{2}, \max\left(\mathcal{F}_{\vec{B}_1}^{+}(c), \mathcal{F}_{\vec{B}_2}^{+}(c)\right), \max\left(\mathcal{T}_{\vec{B}_1}^{-}(c), \mathcal{T}_{\vec{B}_2}^{-}(c)\right), \frac{\mathcal{I}_{\vec{B}_1}^{-}(c) + \mathcal{I}_{\vec{B}_2}^{-}(c)}{2}, \min\left(\mathcal{F}_{\vec{B}_1}^{-}(c), \mathcal{F}_{\vec{B}_2}^{-}(c)\right) \middle| c \in C \right\} \tag{6}$$

4) The union of both S_1 and S_2 is defined as:

$$S_1 \cup S_2 = \left\{ c, \max\left(\mathcal{T}_{\vec{B}_1}^{+}(c), \mathcal{T}_{\vec{B}_2}^{+}(c)\right), \frac{\mathcal{I}_{\vec{B}_1}^{+}(c) + \mathcal{I}_{\vec{B}_2}^{+}(c)}{2}, \min\left(\mathcal{F}_{\vec{B}_1}^{+}(c), \mathcal{F}_{\vec{B}_2}^{+}(c)\right), \min\left(\mathcal{T}_{\vec{B}_1}^{-}(c), \mathcal{T}_{\vec{B}_2}^{-}(c)\right), \frac{\mathcal{I}_{\vec{B}_1}^{-}(c) + \mathcal{I}_{\vec{B}_2}^{-}(c)}{2}, \max\left(\mathcal{F}_{\vec{B}_1}^{-}(c), \mathcal{F}_{\vec{B}_2}^{-}(c)\right) \middle| c \in C \right\} \tag{7}$$

5) The complement of S_1 is computed as follows:

$$S_1^c = \left\{ \left(\{1^+\} - \mathcal{T}_{\vec{B}_1}^{+}(c)\right), \left(\{1^+\} - \mathcal{I}_{\vec{B}_1}^{+}(c)\right), \left(\{1^+\} - \mathcal{F}_{\vec{B}_1}^{+}(c)\right), \left(\{1^-\} - \mathcal{T}_{\vec{B}_1}^{-}(c)\right), \left(\{1^-\} - \mathcal{I}_{\vec{B}_1}^{-}(c)\right), \left(\{1^-\} - \mathcal{F}_{\vec{B}_1}^{-}(c)\right) \right\} \tag{8}$$

3. Research Approach

Herein, we provide our proposed approach for MCDM based on bi-polar NSs. Given a teaching scenario consisting of a number, N_A, of satisfactory alternatives $S = \{S_1, S_2, \cdots, S_{N_A}\}$ and a number, N_c, of criteria $C = \{C_1, C_2, \cdots, C_{N_c}\}$. Each of the above criteria is weighted using value $0 \leq \mathcal{w}_j \leq 1$ resulting in vector $\mathcal{W} = \{\mathcal{w}_1, \mathcal{w}_2, \cdots, \mathcal{w}_{N_c}\}$ in which, the total sum of weights is 1, such that $\sum_j \mathcal{w}_j$. Grounded on the original TOPSIS method [10], we formulate our proposed Bi-polar NSTOPSIS as a sequence of the following steps:

First, the decision matrix is formulated based on the alignment of values of S with respect to C alternative.

$$\mathfrak{D} = \left[\mathfrak{d}_{ij}\right]_{N_A \times N_c} = \begin{bmatrix} \mathfrak{d}_{11} & \mathfrak{d}_{12} & \cdots & \mathfrak{d}_{1N_c} \\ \mathfrak{d}_{21} & \mathfrak{d}_{22} & \cdots & \mathfrak{d}_{2N_c} \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ \mathfrak{d}_{N_A1} & \mathfrak{d}_{N_A2} & \cdots & \mathfrak{d}_{N_AN_c} \end{bmatrix}, \tag{9}$$

Second, based on the assumption that the decision-maker did not know the weights of the criteria,C_i, we employ IDOCRIW (Integrated Determination of Objective Criteria Weights) [11] to perform objective weighting. IDOCRIW method integrated the weights computed from ENTROPY and CILOS methods. Followingly, we calculate a single weight value indicating the significance for each criterion. This is achieved based on the following steps:

Step 1: compute entropy for each normalized teaching criteria:

$$E_{j,k} = -\frac{1}{\ln(N_A)} \sum_{i=1}^{N_A} \mathfrak{d}_{ij,k} \ln\left(\mathfrak{d}_{ij}, k\right), \tag{10}$$

where k refers to each component in the element Bi-polar NS.

Step 2: compute the level of variation for each criterion:

$$v_{j,k} = 1 - E_{j,k} \tag{11}$$

Step 3: compute the entropy weight as follows:

$$w_j = \frac{\sum_{k=1}^{6} v_{j,k}}{\sum_{j=1}^{N_c} \sum_{k=1}^{6} v_{j,k}}; \quad j = 1,2, \ldots, N_c \tag{12}$$

Step 4: Compute a square matrix in which the largest values for criteria in the decision matrix are nominated. This can be expressed as follows:

$$a_{max,j,k} = \max_i \mathfrak{d}_{ij,k}; \quad i = 1,2, \ldots, N_c; \; j = 1,2, \ldots, N_A$$
$$A_k[i,j] = \begin{cases} a_{max,j,k} & if\ i = j \\ \mathfrak{d}_{ij,k} & Otherwise \end{cases} \tag{13}$$

Step 5. We compute the lost influence matrix based on the following equation.

$$P_{ij,k} = \frac{-A_k[i,j] + \max(A_k[:,j])}{a_{max,j,k}}; \quad i,j = 1,2, \ldots, N_A; \; P_{jj} = 0 \tag{14}$$

Step 6. We compute the weight system as follows.

(15)

$$P_{ij,k}^{w} = \begin{cases} -\sum_{j=1}^{N_c} P_{ij,k} & if\ i = j \\ P_{ij,k} & Otherwise \end{cases}$$

Step 7. We compute the weights based on the criterion impact loss method (CILOS), in which each criterion's weight is represented as ψ_j, derived by solving the following linear equation system.

$$P_{ij,k}^{w}\psi^{T} = 0 \tag{16}$$

Where as ψ^{T} denote a vector of weights $\psi = [\psi_1, \psi_1, \cdots, \psi_{N_c}]$ such that $\sum_{j=1}^{N_c}\psi_j = 1$.

The final weights can be given as follows:

$$\mathcal{W}_j = \frac{\psi_j w_j}{\sum_{j=1}^{n}\psi_j w_j}\ j = 1,2,\ldots,N_c \tag{17}$$

Third, the decision matrix (based on bipolar NS),$\mathfrak{D}$, is computed based on the multiplication of weights, $\mathcal{W}$, of each criterion:

$$\mathfrak{D} \otimes \mathcal{W} = \left[\mathfrak{d}_{ij}^{w_j}\right]_{N_A \times N_c} = \begin{bmatrix} \mathfrak{d}_{11}^{w_1} & \mathfrak{d}_{12}^{w_2} & \cdots & \mathfrak{d}_{1N_c}^{w_{N_c}} \\ \mathfrak{d}_{21}^{w_1} & \mathfrak{d}_{22}^{w_2} & \cdots & \mathfrak{d}_{2N_c}^{w_{N_c}} \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ \mathfrak{d}_{N_A1}^{w_1} & \mathfrak{d}_{N_A2}^{w_2} & \cdots & \mathfrak{d}_{N_AN_c}^{w_{N_c}} \end{bmatrix}. \tag{18}$$

In the above formula,

$$\begin{aligned} \mathfrak{d}_{ij}^{w_j} &= \left\{\mathcal{T}_{ij}^{w_j+}, \mathcal{I}_{ij}^{w_j+}, \mathcal{F}_{ij}^{w_j+}, \mathcal{T}_{ij}^{w_j-}, \mathcal{I}_{ij}^{w_j-}, \mathcal{F}_{ij}^{w_j-}\right\} \\ &= \left\{1 - \left(1 - \mathcal{T}_{ij}^{+}\right)^{w_j}, \left(\mathcal{I}_{ij}^{+}\right)^{w_j}, \left(\mathcal{F}_{ij}^{+}\right)^{w_j}, -\left(-\mathcal{T}_{ij}^{-}\right)^{w_j}, -\left(-\mathcal{I}_{ij}^{-}\right)^{w_j}, -\left(\left(1 - \left(-\mathcal{F}_{ij}^{-}\right)\right)^{w_j}\right)\right\}, \end{aligned} \tag{19}$$

Fourth, calculate the bipolar NS relative positive ideal solution (BNSRPIS) according to the following expression:

$$BNSRPIS = \begin{cases} \langle {}^{+}\mathcal{T}_1^{w_1+}, {}^{+}\mathcal{I}_1^{w_1+}, {}^{+}\mathcal{F}_1^{w_1+}, {}^{+}\mathcal{T}_1^{w_1-}, {}^{+}\mathcal{I}_1^{w_1-}, {}^{+}\mathcal{F}_1^{w_1-}\rangle, \langle {}^{+}\mathcal{T}_2^{w_2+}, {}^{+}\mathcal{I}_2^{w_2+}, {}^{+}\mathcal{F}_2^{w_2+}, {}^{+}\mathcal{T}_2^{w_2-}, \mathcal{I}_2^{w_2-}, {}^{+}\mathcal{F}_2^{w_2-} \\ \ldots, \langle {}^{+}\mathcal{T}_{N_c}^{w_n+}, {}^{+}\mathcal{I}_{N_c}^{w_n+}, {}^{+}\mathcal{F}_{N_c}^{w_n+}, {}^{+}\mathcal{T}_{N_c}^{w_n-}, {}^{+}\mathcal{I}_{N_c}^{w_n-}, {}^{+}\mathcal{F}_{N_c}^{w_n-}\rangle \end{cases} \tag{20}$$

and calculate bipolar NS relative negative ideal solution (BNSRNIS) according to the following expression:

$$BNSRNIS = \begin{cases} \langle {}^{-}\mathcal{T}_1^{w_1+}, {}^{-}\mathcal{I}_1^{w_1+}, {}^{-}\mathcal{F}_1^{w_1+}, {}^{-}\mathcal{T}_1^{w_1-}, {}^{-}\mathcal{I}_1^{w_1-}, {}^{-}\mathcal{F}_1^{w_1-}\rangle, \langle {}^{-}\mathcal{T}_2^{w_2+}, {}^{-}\mathcal{I}_2^{w_2+}, {}^{-}\mathcal{F}_2^{w_2+}, {}^{-}\mathcal{T}_2^{w_2-}, {}^{-}\mathcal{I}_2^{w_2-}, {}^{-}\mathcal{F}_2^{w_2} \\ \ldots, \langle {}^{-}\mathcal{T}_n^{w_n+}, {}^{-}\mathcal{I}_n^{w_n+}, {}^{-}\mathcal{F}_n^{w_n+}, {}^{-}\mathcal{T}_n^{w_n-}, {}^{-}\mathcal{I}_n^{w_n-}, {}^{-}\mathcal{F}_n^{w_n-}\rangle \end{cases} \tag{21}$$

such that, for maximization criteria, $j = 1,2,\ldots,N_c$

$$\begin{aligned}\langle {}^{+}\mathcal{T}_j^{w_j+},{}^{+}\mathcal{I}_j^{w_j+},{}^{+}F_j^{w_j+},{}^{+}\mathcal{T}_j^{w_j-},{}^{+}\mathcal{I}_j^{w_j-},{}^{+}\mathcal{F}_j^{w_j-}\rangle &= \langle\max\left(\mathcal{T}_{ij}^{w_j+}\right),\min\left(\mathcal{I}_{ij}^{w_j+}\right),\min\left(F_{ij}^{w_j+}\right),\\ &\quad \min\left(\mathcal{T}_{ij}^{w_j-}\right),\max\left(\mathcal{I}_{ij}^{w_j-}\right),\max\left(\mathcal{F}_{ij}^{w_j-}\right)\rangle,\\ \langle {}^{-}\mathcal{T}_j^{w_j+},{}^{-}\mathcal{I}_j^{w_j+},{}^{-}\mathcal{F}_j^{w_j+},{}^{-}\mathcal{T}_j^{w_j-},{}^{-}\mathcal{I}_j^{w_j-},{}^{-}\mathcal{F}_j^{w_j-}\rangle &= \langle\min\left(\mathcal{T}_{ij}^{w_j+}\right),\max\left(\mathcal{I}_{ij}^{w_j+}\right),\max\left(\mathcal{F}_{ij}^{w_j+}\right),\\ &\quad \max\left(\mathcal{T}_{ij}^{w_j-}\right),\min\left(\mathcal{I}_{ij}^{w_j-}\right),\min\left(\mathcal{F}_{ij}^{w_j-}\right)\rangle.\end{aligned} \quad (22)$$

While minimization teaching criteria, $j = 1,2,\dots,N_c$

$$\begin{aligned}\langle {}^{+}\mathcal{T}_j^{w_j+},{}^{+}\mathcal{I}_j^{w_j+},{}^{+}\mathcal{F}_j^{w_j+},{}^{+}\mathcal{T}_j^{w_j-},{}^{+}\mathcal{I}_j^{w_j-},{}^{+}\mathcal{F}_j^{w_j-}\rangle &= \langle\min\left(\mathcal{T}_{ij}^{w_j+}\right),\max\left(\mathcal{I}_{ij}^{w_j+}\right),\max\left(\mathcal{F}_{ij}^{w_j+}\right),\\ &\quad \max\left(\mathcal{T}_{ij}^{w_j-}\right),\min\left(\mathcal{I}_{ij}^{w_j-}\right),\min\left(\mathcal{F}_{ij}^{w_j-}\right)\rangle,\\ \langle {}^{-}\mathcal{T}_j^{w_j+},{}^{-}\mathcal{I}_j^{w_j+},{}^{-}\mathcal{F}_j^{w_j+},{}^{-}\mathcal{T}_j^{w_j-},{}^{-}\mathcal{I}_j^{w_j-},{}^{-}\mathcal{F}_j^{w_j-}\rangle &= \langle\max\left(\mathcal{T}_{ij}^{w_j+}\right),\min\left(\mathcal{I}_{ij}^{w_j+}\right),\min\left(\mathcal{F}_{ij}^{w_j+}\right),\\ &\quad \min\left(\mathcal{T}_{ij}^{w_j-}\right),\max\left(\mathcal{I}_{ij}^{w_j-}\right),\max\left(\mathcal{F}_{ij}^{w_j-}\right)\rangle.\end{aligned} \quad (23)$$

Fifth, normalized distance of each alternative from BNSRPIS,

$$D_{mins}(A_i, BNSRPIS) = \left(\frac{1}{6N_c}\sum_{j=1}^{N_c}\left\{\begin{array}{l}\left|\mathcal{T}_{ij}^{w_j+}-{}^{+}\mathcal{T}_j^{w_j+}\right|^p+\left|\mathcal{I}_{ij}^{w_j+}-{}^{+}\mathcal{I}_j^{w_j+}\right|^p+\left|\mathcal{F}_{ij}^{w_j+}-{}^{+}F_j^{w_j+}\right|^p+\\ \left|\mathcal{T}_{ij}^{w_j-}-{}^{+}\mathcal{T}_j^{w_j-}\right|^p+\left|\mathcal{I}_{ij}^{w_j-}-{}^{+}\mathcal{I}_j^{w_j-}\right|^p+\left|\mathcal{F}_{ij}^{w_j-}-{}^{+}\mathcal{F}_j^{w_j-}\right|^p\end{array}\right\}\right)^{\frac{1}{p}}, \quad (24)$$

and normalized distance of each alternative from BNSRPIS,

$$D_{mins}(A_i, BNSRNIS) = \left(\frac{1}{6N_c}\sum_{j=1}^{N_c}\left\{\begin{array}{l}\left|\mathcal{T}_{ij}^{w_j+}-{}^{-}\mathcal{T}_j^{w_j+}\right|^p+\left|\mathcal{I}_{ij}^{w_j+}-{}^{-}\mathcal{I}_j^{w_j+}\right|^p+\left|\mathcal{F}_{ij}^{w_j+}-{}^{-}\mathcal{F}_j^{w_j+}\right|^p+\\ \left|\mathcal{T}_{ij}^{w_j-}-{}^{-}\mathcal{T}_j^{w_j-}\right|^p+\left|\mathcal{I}_{ij}^{w_j-}-{}^{-}\mathcal{I}_j^{w_j-}\right|^p+\left|F_{ij}^{w_j-}-{}^{-}\mathcal{F}_j^{w_j-}\right|^p\end{array}\right\}\right)^{\frac{1}{p}}. \quad (25)$$

Sixth, we compute the degree of closeness, $\propto_i$, for each alternative w.r.t BNSRPIS as expressed as:

$$\propto(A_i) = \frac{D_{mins}(A_i, BNSRNIS)}{\max\{D_{mins}(A_i, BNSRNIS)\}} - \frac{D_{mins}(A_i, BNSRPIS)}{\min\{D_{mins}(A_i, BNSRPIS)\}}, i = 1,2,\dots,N_A. \quad (26)$$

Based on the above degrees, we compute the inferior ratio, $\varphi_{IR}(i)$, for each alternative as formulated below:

$$\varphi_{IR}(i) = \frac{\propto(A_i)}{\min\limits_{1\le i\le m}\left(\propto(A_i)\right)}. \quad (27)$$

where $\varphi_{IR}(i)$ lies in the closed unit interval [0,1].

Finally, we can rank alternatives based on the mounting order of values of $\varphi_{IR}(i)$, where the best alternative is selected based on the lowest value.

4. Results and Discussions

Many universities are concerned about proceeding with their Chemistry courses, aiming to satisfy the different needs of students in different fields such as engineering, communication [12], Artificial intelligence (AI) [13], Medicine, environmental studies, space exploration, etc. Suppose that a student seeks to study a course with appropriate teaching scenarios. The student can choose the course that has a comparatively better teaching scenario, thereby having high popularity among students in the field. By screening many online universities, we found 10 main teaching scenarios vary in different dimensions. These scenarios include Interactive Lecture-Based Teaching (A1), Project-Based Learning (A2), Flipped Classroom Approach (A3), Gamified Learning (A4) [14], Simulation-Based Teaching (A5), and Collaborative Learning (A6). For each scenario, the following criteria are considered as follows: Lecture Clarity (C1), Student Engagement (C2), Course Difficulty Perception (C3), Learning Improvement (C4), and Student Complaints (C5).

Step 1. Based on the university Chemistry courses, we formulae decision matrix in the shape of bipolar NS is displayed in Table 1.

Table 1. Decision matrix based on Bipolar NS representation.

Criteria / ALT	C1	C2	C3	C4	C5
A1	(0.58, 0.34, 0.15, -0.28, -0.34, -0.25)	(0.7, 0.27, 0.41, -0.7, -0.27, -0.69)	(0.57, 0.32, 0.16, -0.67, -0.32, -0.34)	(0.54, 0.26, 0.39, -0.37, -0.05, -0.28)	(0.76, 0.38, 0.68, -0.52, -0.48, -0.42)
A2	(0.67, 0.35, 0.36, -0.67, -0.35, -0.47)	(0.51, 0.14, 0.27, -0.51, -0.17, -0.5)	(0.64, 0.12, 0.13, -0.74, -0.31, -0.44)	(0.57, 0.23, 0.14, -0.57, -0.23, -0.32)	(0.69, 0.17, 0.39, -0.7, -0.17, -0.39)
A3	(0.74, 0.33, 0.37, -0.74, -0.33, -0.38)	(0.5, 0.1, 0.2, -0.3, -0.1, -0.3)	(0.71, 0.22, 0.21, -0.31, -0.22, -0.19)	(0.64, 0.18, 0.38, -0.34, -0.18, -0.34)	(0.52, 0.46, 0.27, -0.45, -0.46, -0.35)
A4	(0.79, 0.45, 0.27, -0.69, -0.05, -0.16)	(0.87, 0.16, 0.52, -0.57, -0.06, -0.45)	(0.35, 0.35, 0.24, -0.35, -0.35, -0.31)	(0.57, 0.26, 0.19, -0.47, -0.16, -0.36)	(0.61, 0.34, 0.25, -0.54, -0.34, -0.19)
A5	(0.71, 0.16, 0.65, -0.71, -0.16, -0.47)	(0.48, 0.16, 0.17, -0.18, -0.25, -0.11)	(0.62, 0.5, 0.69, -0.51, -0.5, -0.56)	(0.31, 0.22, 0.4, -0.28, -0.22, -0.26)	(0.5, 0.14, 0.37, -0.6, -0.34, -0.15)
A6	(0.71, 0.17, 0.58, -0.71, -0.17, -0.64)	(0.54, 0.41, 0.46, -0.54, -0.41, -0.46)	(0.94, 0.15, 0.54, -0.94, -0.15, -0.86)	(0.22, 0.1, 0.11, -0.22, -0.1, -0.17)	(0.22, 0.45, 0.15, -0.22, -0.45, -0.13)

Step 2. The quantitative results of the IDOCRIW method are presented for each criterion in Table 2.

Table 2. IDOCRIW weights for different Chemistry teaching criteria.

Criteria	C1	C2	C3	C4	C5
Weights	0.1774	0.2354	0.2442	0.1591	0.1837

Step 3. The quantitative results of weighted bipolar NS are displayed in Table 3.

Table 3. IDOCRIW weighted decision matrix.

Criteria / ALT	C1	C2	C3	C4	C5
A1	(0.143, 0.826, 0.714, -0.798, -0.826, -0.95)	(0.247, 0.735, 0.811, -0.919, -0.735, -0.759)	(0.186, 0.757, 0.639, -0.907, -0.757, -0.903)	(0.116, 0.807, 0.861, -0.854, -0.621, -0.949)	(0.231, 0.837, 0.932, -0.887, -0.874, -0.905)
A2	(0.179, 0.83, 0.834, -0.931, -0.83, -0.893)	(0.155, 0.629, 0.735, -0.853, -0.659, -0.849)	(0.221, 0.596, 0.608, -0.929, -0.751, -0.868)	(0.126, 0.791, 0.731, -0.914, -0.791, -0.94)	(0.194, 0.722, 0.841, -0.937, -0.722, -0.913)
A3	(0.213, 0.821, 0.838, -0.948, -0.821, -0.919)	(0.151, 0.581, 0.685, -0.753, -0.581, -0.919)	(0.261, 0.691, 0.683, -0.751, -0.691, -0.95)	(0.15, 0.761, 0.857, -0.842, -0.761, -0.936)	(0.126, 0.867, 0.786, -0.864, -0.867, -0.924)
A4	(0.242, 0.868, 0.793, -0.936, -0.588, -0.97)	(0.381, 0.65, 0.857, -0.876, -0.516, -0.869)	(0.1, 0.774, 0.706, -0.774, -0.774, -0.913)	(0.126, 0.807, 0.768, -0.887, -0.747, -0.931)	(0.159, 0.82, 0.775, -0.893, -0.82, -0.962)
A5	(0.197, 0.722, 0.926, -0.941, -0.722, -0.893)	(0.143, 0.65, 0.659, -0.668, -0.722, -0.973)	(0.211, 0.844, 0.913, -0.848, -0.844, -0.818)	(0.057, 0.786, 0.864, -0.817, -0.786, -0.953)	(0.12, 0.697, 0.833, -0.91, -0.82, -0.971)
A6	(0.197, 0.73, 0.908, -0.941, -0.73, -0.834)	(0.167, 0.811, 0.833, -0.865, -0.811, -0.865)	(0.497, 0.629, 0.86, -0.985, -0.629, -0.619)	(0.039, 0.693, 0.704, -0.786, -0.693, -0.971)	(0.045, 0.864, 0.706, -0.757, -0.864, -0.975)

Step 4. The ideal solutions are displayed in Table 4.

Table 4. Ideals solutions from different scenarios.

BNSRPIS	BNSRNIS
(0.242, 0.722, 0.714, -0.948, -0.588, -0.834), (0.381, 0.582, 0.659, -0.919, -0.516, -0.759), (0.100, 0.844, 0.913, -0.751, -0.844, -0.950), (0.150, 0.693, 0.704, -0.914, -0.621, -0.931), (0.045, 0.867, 0.932, -0.757, -0.874, -0.975)	(0.143, 0.868, 0.926, -0.797, -0.830, -0.969), (0.143, 0.811, 0.857, -0.667, -0.811, -0.973), (0.497, 0.596, 0.607, -0.985, -0.629, -0.619), (0.038, 0.807, 0.864, -0.786, -0.794, -0.971), (0.231, 0.697, 0.706, -0.937, -0.722, -0.905)

Step 5: The quantitative results of Minkowski distances for each alternative w.r.t BNSRPISs BNSRNISs are displayed in Table 5.

Table 5. Minkowski distances for each alternative w.r.t BNSRPIS BNSRNIS

Alternatives	$D_{mins}(A_i, BNSRPISs)$	$D_{mins}(A_i, BNSRNISs)$
A1	0.1114	0.1187
A2	0.1252	0.0987
A3	0.1071	0.1179
A4	0.0820	0.1434
A5	0.1140	0.1164
A6	0.1431	0.0962

Step 6: The quantitative results of computing the inferior ratio, and the dependent ranking of Chemistry teaching scenarios are displayed in Table 6.

Table 2. Inferior ratio and ranking of different Chemistry teaching criteria.

		φ_{IR}	Rank ↓
A1	-0.531	0.495	3
A2	-0.837	0.781	5
A3	-0.483	0.451	2
A4	0.000	0.000	1
A5	-0.578	0.539	4
A6	-1.073	1.000	6

5. Conclusions

This research introduced a Bipolar neutrosophic framework for assessing the quality of teaching of university psychics by taking into account diverse factors and teaching setups. The weight of different decision alternatives (along with different teaching criteria) is computed based on an adapted version of Integrated Determination of Objective Criteria Weights. Following, we present a Minkowski-based TOPSIS technique employed on Bipolar NS. Proof-of-concept analysis is directed to solve quantitative case studies by analyzing various teaching paradigms, and the results show the aptitude of our framework to correctly make the precise decision that exploits offering valuable insights about different teaching scenarios that could assist students benefit from the Chemistry course.

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Fermatean Neutrosophic Fuzzy Graphs: A Study on the Winner Index with Enhancing Election Analysis

Wadei Fares AL-omeri [1], Kaviyarasu M [2] and M. Rajeshwari [3*]

[1] Department of Mathematics, Faculty of Science and Information Technology Jadara University, Irbid, Jordan; wadeimoon1@hotmail.com;

[2]Department of Mathematics, Vel Tech Rangarajan Dr Sagunthala R & D Institute of Science and Technology, Chennai, Tamilnadu-600062, India; kavitamilm@gmail.com;

[3*]School of Engineering, Presidency University, Bangalure, India; rajeakila@gmail.com;

[*]wadeimoon1@hotmail.com. rajeakila@gmail.com

Abstract. In this article, we discuss the fermatean neutrosophic graph of Wiener index, which is an essential topological index formed according to geodesical distance of vertices. The Wiener index is an important factor to describe the structure of a graph and we defined it in relation to fermatean neutrosophic graphs and computed it for some specific fermatean neutrosophic graph structures including complete fermatean neutrosophic graphs, cycles and trees. Subsequently, the Wiener index is compared with the connectivity index, a core-degree based parameter, using a sequence of theorems. As an application the study responds to the difficulties in election analysis in democratic environments where voter choices are often nuanced an unpredictable and the methods of measurement are not sensitive enough to capture these changes. To improve the modeling of election data, this work employs fermatean neutrosophic graphs (FNGs) and the Wiener index, which distinguish nodes that represent leadership qualities, policy suggestions, and public commitment as well as the relationship between these nodes. This approach manages uncertainty and indeterminacy well and provides a sound method of enhancing the measurability and credibility of analytical techniques in managing complicated events like elections.

Keywords: Fuzzy graph, $\mathcal{FNG}$, Winner Index, Uncertainty Modeling, Decision Making.

1. Introduction

Rosenfeld first proposed the idea of fuzzy graphs in 1975 [18], where Zadeh presented the notion of fuzzy sets in 1965 [27]. Since then, a great deal of study has been done on fuzzy graphs, examining both their features and potential uses. Computing degree-based and distance-based indices in fuzzy graphs has been one topic of interest. These indices provide graphs numerical values, which serve as a helpful standard for comparing graphs with an equal number of vertices.

Arif, W.et al. [6] used some indices of picture fuzzy graphs and their applications are also given. Atanassov first presented the intuitionistic fuzzy set theory [7]. Smarandache [21] developed fuzzy theory in 1995, which resulted in the creation of neutrosophic sets. As a result of this growth, neutrosophic graphs were created, bringing new mathematical ideas to this area. Neutro hyper algebra and anti hyper algebra [22], energy and spectrum analysis [9], supply chain issue resolution [2], and decision-making [1] are just a few of the uses and characteristics of neutrosophic graphs that have been presented by academics in recent years. Ghods and Rostami focus on the application of topological indices in neutrosophic graphs, particularly in the realm of connectivity. Topological indices in neutrosophic graphs, provide an overview of their significance and potential applications, with specific reference to behavioral sciences [13]. In [14] connectivity indices in neutrosophic trees, they present an algorithm to find its maximum size. Finally, [15] shows the importance of summation and the integration index in behavioral sciences while discussing their presentation in the form of a neutrosophic graph. This work contributes to the understanding and use of topological indices in the context of neutersophic diagrams, providing insight into their uses and mathematical methods. Kaviayarasu [[17]] studied regular r-edge neutrosophic graphs. Borzooei and Rashmanlou established new ideas in the interpolation intervals of interest to them [19] and then discussed negative graphs and their applications in linear systems [20]. Sadati, Rashmanlou, and Talebi introduce new notions for m-polar interval-valued intuitionistic fuzzy graphs, advancing this field of research [25]. Omeri, Kaviyarasu, and Rajeshwari use the max product of complement in neutrosophic graphs to discover online streaming services [26], where Al-Omeri W.F. researches mixed b-fuzzy topological spaces [4] and later they discussed virtually e-I-continuous functions [5]. Borzooei and Rashmanlou address cayley interval-valued fuzzy graphs, contributing to the existing corpus of study on the subject [12].

In the context of graph theory, offers an introduction to the theory and applications of fermatean neutrosophic graphs, offering insights into this innovative paradigm [10]. Using the neutrosophic kano technique and extending the idea to fermatean neutrosophic topological spaces, probably investigate the uses of these spaces in symmetry research [16]. The interval-valued fermatean neutrosophic shortest path issue is addressed [11] offers a score function-based solution approach that may have ramifications for optimization and decision-making. A novel interpretation of fermatean neutrosophic dombi fuzzy graphs [23], indicates a step forward in the comprehension and utilization of fuzzy graphs in neutrosophic frameworks. A fermatean fuzzy multi-criteria decision making (MCDM) approach for ranking and selection issues, illustrating real-world uses in the field of decision science [8]. Offering a fresh viewpoint on rough set theory in decision analysis, [24] develop fermatean fuzzy covering-based rough sets and investigate their applications in multi-attribute decision-making. In [3] a significant contribution

to transportation planning optimization approaches by creating an enhanced data envelopment analysis (DEA) method for solving multi-objective transportation problems using fermatean fuzzy sets. Broumi et al. [31] presented an original development on the interval-valued Fermatean neutrosophic shortest path problem. In a similar manner, Sasikala and Divya [32] developed a new approach to the Fermatean neutrosophic Dombi fuzzy graphs, increasing the appearance of uncertainty of the graph structure. In medical field, Dhanalakshmi [33] used Rough Fermatean Neutrosophic Sets to enhance the medical diagnosis where handling imprecise and incomprehensive patient information is a common issue.

In section two, we provide preliminary information on the fermatean neutrosophic graphs (FNGs) which models these relationships. In Section 3, we generalize the wiener index to FNGs in order to implement a framework for evaluating the connectivity and characteristics of these graphs under conditions characterized by uncertainty. The innovation in this study is the extension of the wiener index for fermatean neutrosophic graphs to enhance the understanding of systems. In Section 4, we provide description of this extended wiener index in the framework in election results analysis. Section 5 analyzes the new wiener index against the background of other indices presented . Last but not the least, Section 6 presents the conclusion and offers some recommendations for the research that follows.

1.1. *Motivation*

The goal of the research is to fill in the knowledge gaps about the behavior of the well-known topological metric wiener index in the context of fermatean neutrosophic networks. By exploring this uncharted area, the study aims to provide fresh perspectives on the structural characteristics of these graphs and how they affect other fields of study. Meets the requirements of complex network analysis. The need for sophisticated analytical tools that can capture the nuances of network topology increases as real-world networks become more complex. Focusing on the wiener index, which measures the distance between network vertices, the work aims to make a significant addition to the analytical toolbox for fermate neutrosophic networks.

1.2. *Novelty*

This research offers a fresh viewpoint on networks by extending wiener to fermatean neutrosophic networks. wiener extension to neutrosophic networks stratigraphic: Through an examination of the wiener application of the distance-based feature index, this research offers a fresh viewpoint on network analysis and delves into the nature of these intricate systems. The neutrosophic network of Fermat. This study enhances network science and opens up new avenues for future research and applications by expanding the wiener index into previously uncharted terrain. Using the connection index for comparison: Through a comparison of the

wiener index and the connectedness index, this study offers a thorough knowledge of the structural features of fermatean neutrosophic network. By clarifying the similarities and differences between the two metrics, this research offers significant insights into the distinctive properties of Fermat neutrosophic networks and their implications for network analysis and design.

1.3. *Goal*

wiener index definition and computation for fermatean neutrosophic graphs: Establishing a framework for calculating the wiener index in fermatean neutrosophic networks throughout a range of graph topologies, such as cycles, trees, and full graphs, is the main goal of the study. The study intends to provide a fuller understanding of the distance-based features of fermatean neutrosophic networks by creating reliable methods for calculating this index. Comparative analysis with connectedness index: Using pertinent theorems, the study not only computes the wiener index but also attempts to compare it with another significant measure, the connectedness index. By comparing these two measures, we want to clarify their unique properties and ramifications in the context of fermatean neutrosophic networks, strengthening the theoretical basis for network research in this field.

1.4. *Key Differences and Contributions of This Manuscript*

This paper rectifies the aforementioned gaps by introducing the wiener index of fermatean neutrosophic graphs (FNGs) and applying it to election analysis. Novel contributions include:

(1) FNGs, which have a higher order of membership than traditional neutrosophic structures, are used to model relations between voters, candidates, and regions.
(2) A new measure referred to as the Winner Index is introduced using the Wiener index to identify both separatist and convert candidate relationships in election graphs.
(3) The applicability of the methodology is demonstrated using election networks with the identification of influential candidates and areas and, as a result, improving decision-making in election strategies.

2. **Preliminary**

In this section, we consider to provide the instruments that made this idea come to our mind. To start, we suppose that (Ω, F, Q) is the probability measure space where Ω is a sample space, F is an event space and Q is a probability function and we work in this space. In the following, we will define the Exchange and Lookback options and show the price equation. To this end, we consider the following two risky assets:

Definition 2.1. [29] Let the universe be $\mathcal{R}$. A truth membership function ($T_{\mathcal{S}}$), an indeterminacy membership function ($I_{\mathcal{S}}$), and a falsity membership function ($F_{\mathcal{S}}$), where $T_{\mathcal{S}}$, $I_{\mathcal{S}}$,

and $F_{\mathcal{S}}$ are real standard elements of $[0,1]$, characterize a Neutrosophic set (NS) $\mathcal{S}$ in $\mathcal{R}$. $\mathcal{S} = \{\langle \varkappa, T_{\mathcal{S}}(\varkappa), I_{\mathcal{S}}(\varkappa), F_{\mathcal{S}}(\varkappa)\rangle ; \varkappa \in \mathcal{R}, T_{\mathcal{S}}, I_{\mathcal{S}}, F_{\mathcal{S}} \in]^-, 1^+[\}$. is one way to write it.
The sum of $T_{\mathcal{S}}(\varkappa), I_{\mathcal{S}}(\varkappa), F_{\mathcal{S}}(\varkappa)$ is unrestricted, and as a result, $0^- \leq T_{\mathcal{S}}(\varkappa), I_{\mathcal{S}}(\varkappa), F_{\mathcal{S}}(\varkappa) \leq 3^+$.

Definition 2.2. [30] A Fermatean fuzzy set $\mathcal{S}$ on a universe of discourse $\mathcal{R}$ is a structure having the form as: $\mathcal{S} = \{\langle \varkappa, T_{\mathcal{S}}(\varkappa), F_{\mathcal{S}}(\varkappa)\rangle : \varkappa \in \mathcal{R}\}$ where $T_{\mathcal{S}}(\varkappa) : \mathcal{R} \to [0,1]$ indicates the degree of membership, and $F_{\mathcal{S}}(\varkappa) : \mathcal{R} \to [0,1]$ indicates the degree of non-membership of the element $\varkappa \in \mathcal{R}$ to the set A, respectively, with the constraints : $0 \leq (T_{\mathcal{S}}(\varkappa))^3 + (F_{\mathcal{S}}(\varkappa))^3 \leq 1$

Definition 2.3. [28] The pair $\mathcal{G} = (\mathcal{W}, \mathcal{H})$ represents $\mathcal{FNG}$ defined on a universal set $\mathcal{U}$, where $\mathcal{W}$ denotes a fermatean neutrosophic set on $\mathcal{R}$ and $\mathcal{H}$ a fermatean neutrosophic relation on $\mathcal{R}$.

$$1.\mathcal{M}_{\mathcal{W}}(\eta,\tau) \leq min\{\mathcal{M}_{\mathcal{H}}(\eta), \mathcal{M}_{\mathcal{H}}(\tau)\}$$
$$2.I\mathcal{M}_{\mathcal{W}}(\eta,\tau) \geq max\{I\mathcal{M}_{\mathcal{H}}(\eta), I\mathcal{M}_{\mathcal{H}}(\tau)\}$$
$$3.N\mathcal{M}_{\mathcal{W}}(\eta,\tau) \geq max\{N\mathcal{M}_{\mathcal{H}}(\eta), N\mathcal{M}_{\mathcal{H}}(\tau)\}$$

and $0 \leq \mathcal{M}^3_{\mathcal{W}}(\eta,\tau) + I\mathcal{M}^3_{\mathcal{W}}(\eta,\tau) + N\mathcal{M}^3_{\mathcal{W}}(\eta,\tau) \leq 2$, for all $\eta, \tau \in \mathcal{R}$, where $\mathcal{M}_{\mathcal{W}} : \mathcal{R} \times \mathcal{R} \to [0,1], I\mathcal{M}_{\mathcal{W}} : \mathcal{R} \times \mathcal{R} \to [0,1]$ and $N\mathcal{M}_{\mathcal{W}} : \mathcal{R} \times \mathcal{R} \to [0,1]$. signifying the degree of membership(M), indeterminacy(IM), and non-membership(NM).

Definition 2.4. [28] The single valued $\mathcal{FNG}$ $\mathcal{G} = (\mathcal{W}, \mathcal{H})$ satisfies the following condition

$$1.\mathcal{M}_{\mathcal{W}}(\eta,\tau) \leq min\{\mathcal{M}_{\mathcal{H}}(\eta), \mathcal{M}_{\mathcal{H}}(\tau)\}$$
$$2.I\mathcal{M}_{\mathcal{W}}(\eta,\tau) \leq min\{I\mathcal{M}_{\mathcal{H}}(\eta), I\mathcal{M}_{\mathcal{H}}(\tau)\}$$
$$3.N\mathcal{M}_{\mathcal{W}}(\eta,\tau) \leq max\{N\mathcal{M}_{\mathcal{H}}(\eta), N\mathcal{M}_{\mathcal{H}}(\tau)\}$$

3. **Wineer Index of $\mathcal{FNG}$**

Definition 3.1. Let $\mathcal{G} = (\mathcal{W}, \mathcal{H})$ be the connected $\mathcal{FNG}$.

$$1.CON_{MG}(\eta,\tau) = max\{min\ \mathcal{M}_{\mathcal{H}}(\eta.\tau)\}$$
$$2.CON_{IMG}(\eta,\tau) = max\{min\ I\mathcal{M}_{\mathcal{H}}(\eta,\tau)\}$$
$$3.CON_{NMG}(\eta,\tau) = min\{max\ N\mathcal{M}_{\mathcal{H}}(\eta,\tau)\}$$

where (η, τ) is the path between them, then the partial connectivity index is denoted as

$$PCI_M(\mathcal{G}) = \sum_{\eta,\tau \in N} \mathcal{M}_{\mathcal{H}}(\eta)\mathcal{M}_{\mathcal{H}}(\tau)CON_{MG}(\eta,\tau)$$
$$PCI_{IM}(\mathcal{G}) = \sum_{\eta,\tau \in N} I\mathcal{M}_{\mathcal{H}}(\eta)I\mathcal{M}_{\mathcal{H}}(\tau)CON_{IMG}(\eta,\tau)$$
$$PCI_{NM}(\mathcal{G}) = \sum_{\eta,\tau \in N} N\mathcal{M}_{\mathcal{H}}(\eta)N\mathcal{M}_{\mathcal{H}}(\tau)CON_{NMG}(\eta,\tau)$$

The total connectivity index is denoted as,
$TWI(\mathcal{G}) = \frac{8+4PCI_M(\mathcal{G})-4PCI_{NM}(\mathcal{G})-3PCI_{IM}(\mathcal{G})-4PCI_{NM}(\mathcal{G})}{12}$

Definition 3.2. The $\mathcal{FNG}$ is represented as $\mathcal{G} = (\mathcal{W}, \mathcal{H})$, where W denotes the set of vertices and H denotes the set of edges. Let vertices in W be τ_1 and τ_2. If there is not a stronger path with a shorter length connecting τ_1 and τ_2 is referred to as a fermatean neutrosophic geodesic. It is worth that, according to the definition provided, the shortest strong path for each of the M, NM, and IM components of the $\mathcal{FNG}$ should be found separately.

Definition 3.3. The total of the weighted geodesic distance between vertices for each aspect of M, IM, and NM yields the partial wineer index of $\mathcal{FNG}$.

$$PWI_M(\mathcal{G}) = \sum_{\eta,\tau \in N} \mathcal{M}_{\mathcal{H}}(\eta)\mathcal{M}_{\mathcal{H}}(\tau)ds_M(\eta,\tau)$$
$$PWI_{NM}(\mathcal{G}) = \sum_{\eta,\tau \in N} IN_{\mathcal{H}}(\eta)I\mathcal{M}_{\mathcal{H}}(\tau)ds_{NM}(\eta,\tau)$$
$$PWI_{IM}(\mathcal{G}) = \sum_{\eta,\tau \in N} N\mathcal{M}_{\mathcal{H}}(\eta)N\mathcal{M}_{\mathcal{H}}(\tau)ds_{IM}(\eta,\tau)$$

The total winner index of fermatean neutrosophic fuzzy graph is
$TWI(\mathcal{G}) = \frac{8+4PWI_M(\mathcal{G})-4PWI_{NM}(\mathcal{G})-3PWI_{IM}(\mathcal{G})}{12}$

Example 3.4. In Figure 1, we will analyze the $\mathcal{FNG}$. The collection of vertex set $V = \{a, b, c, d\}$ and their fermatean neutrosophic membership are as follows

$$(\mathcal{M}_{\mathcal{H}}, I\mathcal{M}_{\mathcal{H}}, N\mathcal{M}_{\mathcal{H}})(a) = (0.5, 0.3, 0.2)$$
$$(\mathcal{M}_{\mathcal{H}}, I\mathcal{M}_{\mathcal{H}}, N\mathcal{M}_{\mathcal{H}})(b) = (0.8, 0.1, 0.1)$$
$$(\mathcal{M}_{\mathcal{H}}, I\mathcal{M}_{\mathcal{H}}, N\mathcal{M}_{\mathcal{H}})(c) = (0.6, 0.2, 0.1)$$
$$(\mathcal{M}_{\mathcal{H}}, I\mathcal{M}_{\mathcal{H}}, N\mathcal{M}_{\mathcal{H}})(d) = (0.7, 0.2, 0.3)$$

These are the fermatean neutrosophic memberships of edge set M.

$$(\mathcal{M}_{\mathcal{H}}, I\mathcal{M}_{\mathcal{H}}, N\mathcal{M}_{\mathcal{H}})(a,b) = (0.4, 0.1, 0.1)$$
$$(\mathcal{M}_{\mathcal{H}}, I\mathcal{M}_{\mathcal{H}}, N\mathcal{M}_{\mathcal{H}})(a,c) = (0.5, 0.2, 0.1)$$
$$(\mathcal{M}_{\mathcal{H}}, I\mathcal{M}_{\mathcal{H}}, N\mathcal{M}_{\mathcal{H}})(a,d) = (0.4, 0.2, 0.1)$$
$$(\mathcal{M}_{\mathcal{H}}, I\mathcal{M}_{\mathcal{H}}, N\mathcal{M}_{\mathcal{H}})(b,d) = (0.5, 0.1, 0.1)$$
$$(\mathcal{M}_{\mathcal{H}}, I\mathcal{M}_{\mathcal{H}}, N\mathcal{M}_{\mathcal{H}})(c,d) = (0.5, 0.2, 0.1)$$

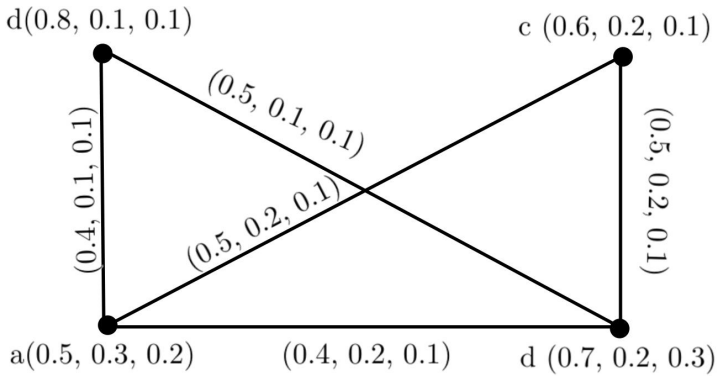

FIGURE 1. Fermatean neutrosophic graph

Edge	$\mathbf{d_{SM}}(\eta,\tau)$	$\mathbf{d_{SIM}}(\eta,\tau)$	$\mathbf{d_{SNM}}(\eta,\tau)$
(a,b)	1.5	0.5	0.1
(a,c)	0.5	0.2	0.1
(a,d)	1	0.2	0.1
(b,c)	1	0.3	0.2
(b,d)	0.5	0.1	0.1
(c,d)	0.5	0.2	0.1

TABLE 1. Total edge weights between η and τ in geodesic paths for $\mathcal{FNG}$

$PWI_M(\mathcal{G}) = 2.07, PWI_{IM}(\mathcal{G}) = 0.055$ and $PWI_{NM}(\mathcal{G}) = 0.018$. $TWI(\mathcal{G}) = 1.3369$.

Theorem 3.5. *Let $\mathcal{G} = (\mathcal{W}, \mathcal{H})$ be the complete $\mathcal{FNG}$ with τ such that $m_1 \leq m_2 \leq \ldots\ldots \leq m_n, im_1 \geq im_2 \geq \ldots\ldots \geq m_n$ and $nm_1 \geq nm_2 \geq \ldots\ldots \geq nm_n$. where*

$m_j = \mathcal{M}_{\mathcal{H}}(\tau_j), (im)_j = I\mathcal{M}_{\mathcal{H}}(\tau_j)$ *and* $(nm)_j = N\mathcal{M}_{\mathcal{H}}(\tau_j)$ *for* $1 \leq j \leq n$. *Then*

$$PWI_M(\mathcal{G}) = \sum_{j=1}^{n-1} m_j^4 \sum_{k=j+1}^{n} m_k,$$
$$PWI_{NM}(\mathcal{G}) = \sum_{j=1}^{n-1} nm_j^4 \sum_{k=j+1}^{n} nm_k,$$
$$andPWI_{IM}(\mathcal{G}) = \sum_{j=1}^{n-1} im_j^4 \sum_{k=j+1}^{n} im_k.$$

proof. Consider the $\mathcal{FNG}$ $\mathcal{G} = (\mathcal{W}, \mathcal{H})$ under the constraints stated in the theorem. According to the definition, the Wiener Index.

$$PWI_M(\mathcal{G}) = \sum_{\eta,\tau \in N} T_N(\eta) T_N(\tau) ds_M(\eta, \tau) \tag{1}$$

In a frematean neutrosophic graph, any two vertices can have a direct relationship. We want to show that this straight line is a geodesic path. Let $u = \tau_1$. Then for any $2 \leq i \leq n$,we have $m_1 \leq m_i$. This insight ensures

$$d_{SM}(\tau_1, \tau_i) = m_1^3, \quad 2 \leq i \leq n,$$
$$d_{SM}(\tau_2, \tau_i) = m_2^3, \quad 3 \leq i \leq n,$$
$$.....$$
$$.....$$
$$.....$$
$$d_{SM}(\tau_{n-1}, \tau_n) = m_{n-1}^3, \quad 2 \leq i \leq n.$$

Now, we obtain by putting the above values in equ 1.

$$\begin{aligned} PWI_M(\mathcal{G}) &= \mathcal{M}_{\mathcal{H}}(\tau_1)\mathcal{M}_{\mathcal{H}}(\tau_2)M_1^3 + ... + \mathcal{M}_{\mathcal{H}}(\tau_1)\mathcal{M}_{\mathcal{H}}(\tau_n)M_1^3 + \mathcal{M}_{\mathcal{H}}(\tau_2)\mathcal{M}_{\mathcal{H}}(\tau_3)M_2^3 + ... \\ &+ \mathcal{M}_{\mathcal{H}}(\tau_2)\mathcal{M}_{\mathcal{H}}(\tau_n)M_2^3 + ... + \mathcal{M}_{\mathcal{H}}(\tau_k)\mathcal{M}_{\mathcal{H}}(\tau_n)M_k^3 + ... + \mathcal{M}_{\mathcal{H}}(\tau_{n-1})\mathcal{M}_{\mathcal{H}}(\tau_n)M_{n-1}^3 \\ &= m_1 m_2 m_1^3 + ... + m_1 m_n m_1^3 + m_2 m_3 m_2^3 + ... + m_2 m_n m_2^3 + ... + m_k m_n m_k^3 + ... + m_{n-1} m_n m_{n-1}^3 \\ &= m_1^4(m_2 + ... + m_n) + m_2^4(m_3 + ... + m_n) + ... + m_k^4(m_{k+1} + ... + m_n) + ... + m_{n-1}^4 m_n \\ PWI_M(\mathcal{G}) &= \sum_{j=1}^{n-1} m_j^4 \sum_{k=j+1}^{n} m_k \end{aligned}$$

In a similar way, $PWI_{IM}(\mathcal{G})$ and $PWI_{NM}(\mathcal{G})$ are provable.

Corollary 3.6. *The complete $\mathcal{FNG}$ $\mathcal{G} = (\mathcal{W}, \mathcal{H})$ meeting the criteria mentioned in the previous theorem, the next connections become clear;*

$$\begin{aligned} PWI_M(\mathcal{G}) &= PCI_M(\mathcal{G}) \\ PWI_{IM}(\mathcal{G}) &= PCI_{IM}(\mathcal{G}) \\ PWI_{NM}(\mathcal{G}) &= PCI_{NM}(\mathcal{G}) \\ Also, TWI(\mathcal{G}) &= TCI(\mathcal{G}) \end{aligned}$$

proof. The above theorem is evident based on this first theorem.

Theorem 3.7. *Let $\mathcal{G}$ be the $\mathcal{FNG}$ where G^* is a tree and $|N^*| = n$. If every $\eta\tau \in M, G-\eta\tau$ has two connected components, w_1&w_2 then it has l and k vertices, respectively, so that $l+k = n$.*

$$PWI_M(\mathcal{G}) = \sum_{\eta\tau \in G} \mathcal{M}_{\mathcal{W}}(\eta) \sum_{i=1}^{l} \mathcal{M}_{\mathcal{H}}(\eta_i) \sum_{j=1}^{k} \mathcal{M}_{\mathcal{H}}(\tau_j)$$

$$PWI_{IM}(\mathcal{G}) = \sum_{\eta\tau \in G} (IM)_{\mathcal{W}}(\eta\tau) \sum_{i=1}^{l} (IM)_{\mathcal{H}}(\eta_i) \sum_{j=1}^{k} (IM)_{\mathcal{H}}(\tau_j)$$

$$PWI_{NM}(\mathcal{G}) = \sum_{\eta\tau \in G} (NM)_{\mathcal{W}}(\eta\tau) \sum_{i=1}^{l} (NM)_{\mathcal{H}}(\eta_i) \sum_{j=1}^{k} (NM)_{\mathcal{H}}(\tau_j)$$

proof.A $\mathcal{FNG}$ $\mathcal{G} = (\mathcal{W}, \mathcal{H})$, where $|N^*| = n$. and G^* is a tree. upon the elimination the chosen edge $\eta\tau$ from $\mathcal{G}$, where $\eta\tau \in M$, the graph is divided into two linked component, w_1 and w_2. The graph w_1 has vertices with a length of l, where w_2 contains vertices with a length of $k = n - 1$. In the event where $l = 1, k = n - 1$ and $\tau_1 \in w_1$, then

$$\begin{aligned} PWI_M(\eta) &= \sum_{\eta,\tau \in N} \mathcal{M}_{\mathcal{H}}(\eta)\mathcal{M}_{\mathcal{H}}(\tau) d_{SM}(\eta, \tau) \\ &= \mathcal{M}_{\mathcal{H}}(\tau_1)\mathcal{M}_{\mathcal{H}}(\tau_2)\mathcal{M}_{\mathcal{W}}(\eta\tau) + \mathcal{M}_{\mathcal{H}}(\tau_1)\mathcal{M}_{\mathcal{H}}(\tau_3)(\mathcal{M}_{\mathcal{W}}(\eta\tau) + e_1) + ...+ \\ &\mathcal{M}_{\mathcal{H}}(\tau_1)\mathcal{M}_{\mathcal{H}}(\tau_n)(\mathcal{M}_{\mathcal{W}}(\eta\tau) + ... + e_m) + \sum_{\eta,\tau \in N-\tau_1} \mathcal{M}_{\mathcal{H}}(\eta)\mathcal{M}_{\mathcal{H}}(\tau) d_{SM}(\eta, \tau) \end{aligned}$$

Let us examine the scenario in which $e_i \in M$ and $e_i \neq M$. Repeat the same for $\sum_{\eta,\tau \in N-\tau_1} \mathcal{M}_{\mathcal{H}}(\eta)\mathcal{M}_{\mathcal{H}}(\tau) d_{SM}(\eta, \tau)$. This technique should be respect until w_2 has just one vertex left. The indented outcome can then be obtained by factoring and adding the two components total number of vertices. Likewise, it is possible to demonstrate $PWI_{IM}(\mathcal{G})$ and $PWI_{NM}(\mathcal{G})$ by employing comparable techniques.

Theorem 3.8. *The $\mathcal{FNG}$ $\mathcal{G} = (\mathcal{W}, \mathcal{H})$ has a unique strong spanning tree. Then $PWI_M(\mathcal{G}) = PWI_M(T), PWI_{IM}(\mathcal{G}) = PWI_{IM}(T), PWI_{NM}(\mathcal{G}) = PWI_{NM}(T)$ and $TWI(\mathcal{G}) = TWI(T)$.*

proof. Suppose connected $\mathcal{FNG}$ $\mathcal{G}$, and it is a unique tree strong spanning tree is T. A strong spanning tree is defined as, for any two vertices u and τ in graph $\mathcal{G}$, the distance measure

in graph $\mathcal{G}$ are equal to those in strong spanning tree T.

The partial winner index of the $\mathcal{FNG}$ $\mathcal{G}$ is equivalent to the corresponding values in the strong spanning tree T, as can be seen from the aforementioned relations. Therefore, it is possible to deduced that $TWI(\mathcal{G}) = IWI(T)$.

Theorem 3.9. *Let $\mathcal{G}$ be the $\mathcal{FNG}$ with $G^* = G$. Then. For $n = 2m, m \in N$.*

$$PWI_M(\mathcal{G}) = \sum_{k=1}^{\frac{n}{2}-1} k(m)(\sum_{j=1}^{n} \mathcal{M}_{\mathcal{H}}(\eta_j) * \mathcal{M}_{\mathcal{H}}(\eta_{j+k})) + \frac{n}{2}m\sum_{j=1}^{\frac{n}{2}}(\mathcal{M}_{\mathcal{H}}(\eta_i) * \mathcal{M}_{\mathcal{H}}(\eta_{i+\frac{n}{2}}))$$

$$PWI_{IM}(\mathcal{G}) = \sum_{k=1}^{\frac{n}{2}-1} k(im)(\sum_{j=1}^{n} I\mathcal{M}_{\mathcal{H}}(\eta_j) * I\mathcal{M}_{\mathcal{H}}(\eta_{j+k})) + \frac{n}{2}(im)\sum_{j=1}^{\frac{n}{2}}(I\mathcal{M}_{\mathcal{H}}(\eta_i) * I\mathcal{M}_{\mathcal{H}}(\eta_{i+\frac{n}{2}}))$$

$$PWI_{NM}(\mathcal{G}) = \sum_{k=1}^{\frac{n}{2}-1} k(nm)(\sum_{j=1}^{n} N\mathcal{M}_{\mathcal{H}}(\eta_j) * N\mathcal{M}_{\mathcal{H}}(\eta_{j+k})) + \frac{n}{2}(nm)\sum_{j=1}^{\frac{n}{2}}(N\mathcal{M}_{\mathcal{H}}(\eta_i) * N\mathcal{M}_{\mathcal{H}}(\eta_{i+\frac{n}{2}}))$$

For $n = 2m + 1, m \in N$.

$$PWI_M(\mathcal{G}) = \sum_{k=1}^{\frac{n-1}{2}} k(m)(\sum_{j=1}^{n} \mathcal{M}_{\mathcal{H}}(\eta_j) * \mathcal{M}_{\mathcal{H}}(\eta_{j+k}))$$

$$PWI_{IM}(\mathcal{G}) = \sum_{k=1}^{\frac{n}{2}-1} k(im)(\sum_{j=1}^{n} I\mathcal{M}_{\mathcal{H}}(\eta_j) * I\mathcal{M}_{\mathcal{H}}(\eta_{j+k}))$$

$$PWI_{NM}(\mathcal{G}) = \sum_{k=1}^{\frac{n}{2}-1} k(nm)(\sum_{j=1}^{n} N\mathcal{M}_{\mathcal{H}}(\eta_j) * N\mathcal{M}_{\mathcal{H}}(\eta_{j+k}))$$

For $j + k > n, \eta_{j+k} = \eta d$, which means $j + k = d(mode\ n)$. In addition, for $(G - \eta\tau)$, we have

$$PWI_M(G - \eta\tau) = \sum_{k=1}^{n-1} k(m)(\sum_{j=1}^{n-k} \mathcal{M}_{\mathcal{H}}(\eta_j) * \mathcal{M}_{\mathcal{H}}(\eta_{j+k}))$$

$$PWI_{IM}(G - \eta\tau) = \sum_{k=1}^{n-1} k(im)(\sum_{j=1}^{n-k} I\mathcal{M}_{\mathcal{H}}(\eta_j) * I\mathcal{M}_{\mathcal{H}}(\eta_{j+k}))$$

$$PWI_{NM}(G - \eta\tau) = \sum_{k=1}^{n-1} k(nm)(\sum_{j=1}^{n-k} N\mathcal{M}_{\mathcal{H}}(\eta_j) * N\mathcal{M}_{\mathcal{H}}(\eta_{j+k}))$$

The function $M = (m, im, nm)$ is constant.

proof. In $\mathcal{FNG}$ $\mathcal{G} = (\mathcal{W}, \mathcal{H})$, where $\mathcal{W}$ is a constant function, $\mathcal{G}^*$ is an even-length cycle, and $\mathcal{G}$ is neutral for each edge for $\mathcal{G}$, the maximum length of $\mathcal{FNG}$ is $\frac{n}{2}$. Consider a situation where $\frac{n}{2}$ is the distance between two vertices.

Assume that distance between two vertices η and v is equal to k, which is smaller than $\frac{n}{2}$. The geodesic length between two vertices is defined as p_k, where $(\eta, \tau) \in \mathcal{H}^* X \mathcal{H}^*$.

Conversely, there are $\frac{n}{2}$ pairs of vertices (η, τ) such that $\frac{n}{2}(M, IM, NM)$ is the geodesic length

between them. For each of these pairs, we may calculated the product of $\mathcal{M}_{\mathcal{H}}(\eta)$ in $\mathcal{M}_{\mathcal{H}}(\tau)$, and then add up all of these pairs of η and τ to get the desired outcome.

$$\frac{n}{2}m\sum_{i=1}^{\frac{n}{2}}\mathcal{M}_{\mathcal{H}}(\eta_i)\mathcal{M}_{\mathcal{H}}(\eta_{l+\frac{n}{2}}) \tag{2}$$

Let us return to the previous condition where $1 \leq k \leq \frac{n}{2}$. This is a vertex with distance km from every vertex in the cycle c_n, such as η. Assuming $k = 1$, we obtain $\mathcal{M}_{\mathcal{H}}(\eta_1)\mathcal{M}_{\mathcal{H}}(\eta_2) + \mathcal{M}_{\mathcal{H}}(\eta_2)\mathcal{M}_{\mathcal{H}}(\eta_3) + ... + \mathcal{M}_{\mathcal{H}}(\eta_j)\mathcal{M}_{\mathcal{H}}(\eta_{j+1}) + ... + \mathcal{M}_{\mathcal{H}}(\eta_n)\mathcal{M}_{\mathcal{H}}(\eta_{n+1})$.
Since $n + 1 = 1(modn)$, hence $\mathcal{M}_{\mathcal{H}}(\eta_n)\mathcal{M}_{\mathcal{H}}(\eta_{n+1}) = \mathcal{M}_{\mathcal{H}}(\eta_n)\mathcal{M}_{\mathcal{H}}(\eta_1)$. Then for $k = 1$, we have

$$1 \times m \times \sum_{j=i}^{n}\mathcal{M}_{\mathcal{H}}(\eta_j)\mathcal{M}_{\mathcal{H}}(\eta_{j+1}),$$

for k=2

$$2 \times m \times \sum_{j=i}^{n}\mathcal{M}_{\mathcal{H}}(\eta_j)\mathcal{M}_{\mathcal{H}}(\eta_{j+2}),$$

for $k = r, r < \frac{n}{2}$

$$r \times m \times \sum_{j=i}^{n}\mathcal{M}_{\mathcal{H}}(\eta_j)\mathcal{M}_{\mathcal{H}}(\eta_{j+r})$$

containing along these lines and adding up k gives us

$$\sum_{k=1}^{\frac{n}{2}-1} km(\sum_{j=i}^{n}\mathcal{M}_{\mathcal{H}}(\eta_j)\mathcal{M}_{\mathcal{H}}(\eta_{j+r})) \tag{3}$$

utilizing 2 and 3

$$\begin{aligned} PWI_M(\mathcal{G}) &= (1) + (2) \\ &= \sum_{k=1}^{\frac{n}{2}-1} km(\sum_{j=1}^{n}\mathcal{M}_{\mathcal{H}}(\eta_j)\mathcal{M}_{\mathcal{H}}(\eta_{j+r})) \\ &+ \sum_{l=1}^{\frac{n}{2}}\frac{n}{2}m(\sum_{i=l}^{n}\mathcal{M}_{\mathcal{H}}(\eta_l)\mathcal{M}_{\mathcal{H}}(\eta_{l+r})) \end{aligned}$$

The fact that there is a maximum distance of $\frac{n-1}{2}$ between the vertices η and τ indicates that is odd. The proof continuation is comparable to the scenario in which n is even.

Theorem 3.10. *Let $\mathcal{G}$ be a fermatean neutrosophic tree (FNT) with $|N^*| \geq 3$. Subsequently $PCI_M(\mathcal{G}) < PWI_M(\mathcal{G}), PCI_{IM}(\mathcal{G}) < PWI_{IM}(\mathcal{G})$ and*
$PCI_{NM}(\mathcal{G}) < PWI_{NM}(\mathcal{G})$. Its not necessary for $TCI(\mathcal{G})$ to be smaller than or equal to $TFI(\mathcal{G})$.

Let $\mathcal{G} = (\mathcal{W}, \mathcal{H})$ be a FNT. Single for all vertices η and τ in the FNT, there is a single strong path connecting them. Therefore, this path is the only one that is the strongest from η to τ. For every η and τ. $d_{SM}(\eta, \tau)$ is the true of the membership values on the other hand, the membership values of the weakest edges of the $(\eta-\tau)$-path are represented as $CONN_{MG}(\eta, \tau)$. Consequently, $CONN_{MG}(\eta, \tau) \leq d_{SM}(\eta, \tau)$. Equality arises in the relationship above when $\eta\tau$ is a strong edge. If not $CONN_{MG}(\eta, \tau) < d_{SM}(\eta, \tau)$ then, we have $PCI_M(\mathcal{G}) < PWI_M(\mathcal{G})$. Likewise, it is possible to show $PWI_{IM}(\mathcal{G})$ and $PWI_{NM}(\mathcal{G})$.

Example 3.11. The FNT $\mathcal{G} = (\mathcal{W}, \mathcal{H})$ is shown in Figure 2.

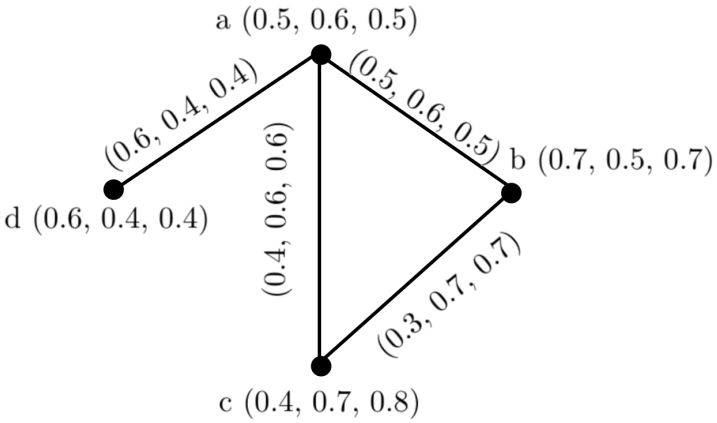

FIGURE 2. Fermatean neutrosophic tree.

It should be observed that bc is a weak edge. $PWI_M(\mathcal{G}) = 1.551, PWI_{IM}(\mathcal{G}) = 1.771$ and

Edges	$\mathbf{d_{SM}}(\eta, \tau)$	$\mathbf{d_{SIM}}(\eta, \tau)$	$\mathbf{d_{SNM}}(\eta, \tau)$
$(\tilde{a}, \tilde{b})$	0.7	1.3	1.3
$(\tilde{a}, \tilde{c})$	0.4	0.6	0.6
$(\tilde{a}, \tilde{d})$	0.6	0.4	0.4
$(\tilde{b}, \tilde{c})$	0.3	0.7	0.7
$(\tilde{b}, \tilde{d})$	1.3	1.7	1.7
$(\tilde{c}, \tilde{d})$	1.4	1.6	1.5

TABLE 2. Total edge weights between η and τ in geodesic paths for fermatean neutrosophic tree.

$PWI_{NM}(\mathcal{G}) = 2.123$. $TWI(\mathcal{G}) = 0.0485$.

$PCI_M(\mathcal{G}) = 0.853, PCI_{IM}(\mathcal{G}) = 0.989$ and $PCI_{NM}(\mathcal{G}) = 1.052$. $TCI(\mathcal{G}) = 0.35308$. As demonstrated by this instance $PCI_M(\mathcal{G}) < PWI_M(\mathcal{G})$,
$PCI_{IM}(\mathcal{G}) < PWI_{IM}(\mathcal{G})$ and $PCI_{NM}(\mathcal{G}) < PWI_{NM}(\mathcal{G})$. But $TCI(\mathcal{G}) > TWI(\mathcal{G})$. In addition, the $\mathcal{FNG}$ depicted in figure 3, below is a tree where $PCI_M(\mathcal{G}) < PWI_M(\mathcal{G}), PCI_{IM}(\mathcal{G}) <$

Edges	$CON_{MG}(\eta,\tau)$	$CON_{IMG}(\eta,\tau)$	$CON_{NMG}(\eta,\tau)$
$(\tilde{a},\tilde{b})$	0.5	0.6	0.4
$(\tilde{a},\tilde{c})$	0.4	0.6	0.6
$(\tilde{a},\tilde{d})$	0.6	0.5	0.4
$(\tilde{b},\tilde{c})$	0.4	0.7	0.6
$(\tilde{b},\tilde{d})$	0.5	0.4	0.4
$(\tilde{c},\tilde{d})$	0.4	0.4	0.6

TABLE 3. Strength of connectedness for for each pair of vertices η,τ

$PWI_{IM}(\mathcal{G})$ and $PCI_{NM}(\mathcal{G}) < PWI_{NM}(\mathcal{G})$.
Moreovr $TCI(\mathcal{G}) < TWI(\mathcal{G})$.

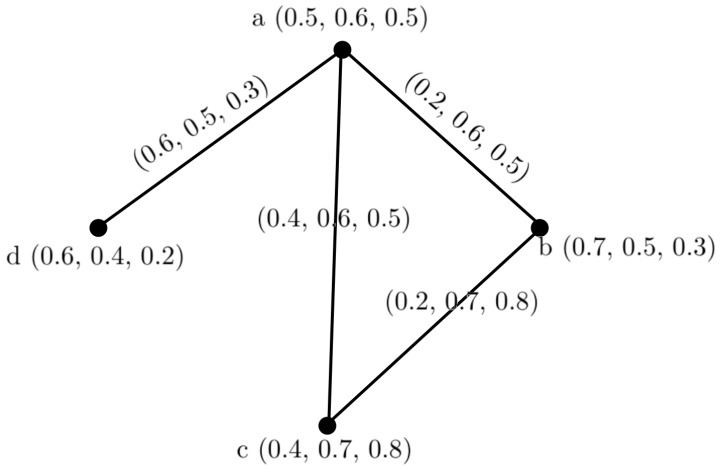

FIGURE 3. Illustration of fermatean neutrosophic tree nodes and relationship

Edges	$d_{SM}(\eta,\tau)$	$d_{SIM}(\eta,\tau)$	$d_{SNM}(\eta,\tau)$
$\tilde{a},\tilde{b}$	0.2	0.6	0.5
$\tilde{a},\tilde{c}$	0.5	0.6	0.5
$\tilde{a},\tilde{d}$	0.6	0.5	0.3
$\tilde{b},\tilde{c}$	0.2	0.7	0.8
$\tilde{b},\tilde{d}$	0.9	1.1	0.8
$\tilde{c},\tilde{d}$	1.4	1.6	1.5

TABLE 4. Total edge weights between η and τ in geodesic paths

$PWI_M(\mathcal{G}) = 1.024, PWI_{IM}(\mathcal{G}) = 1.325$ and $PWI_{NM}(\mathcal{G}) = 1.325. TWI(\mathcal{G}) = 0.4364$. $PCI_M(\mathcal{G}) = 0.5666, PCI_{IM}(\mathcal{G}) = 1.037$ and $PCI_{NM}(\mathcal{G}) = 0.491$. $TCI(\mathcal{G}) = 0.4324$. As demonstrated by this instance $PCI_M(\mathcal{G}) < PWI_M(\mathcal{G}), PCI_{IM}(\mathcal{G}) < PWI_{IM}(\mathcal{G})$ and $PCI_{NM}(\mathcal{G}) < PWI_{NM}(\mathcal{G})$. But $TCI(\mathcal{G}) < TWI(\mathcal{G})$.

Edges	$CON_{MG}(\eta,\tau)$	$CON_{IMG}(\eta,\tau)$	$CON_{NMG}(\eta,\tau)$
$\tilde{a},\tilde{b}$	0.2	0.6	0.5
$\tilde{a},\tilde{c}$	0.4	0.6	0.5
$\tilde{a},\tilde{d}$	0.6	0.5	0.3
$\tilde{b},\tilde{c}$	0.2	0.7	0.5
$\tilde{b},\tilde{d}$	0.2	0.5	0.3
$\tilde{c},\tilde{d}$	0.4	0.5	0.3

TABLE 5. Strength of connectedness

Theorem 3.12. *In a FNT $\mathcal{G}$, where the underlying set is N and the membership function is M, with a constraint that the cardinality of the neutral element set $|N^*| \leq 3$ and G^* is a star, suppose M is a constant function. Assume τ, is the center vertex and $\tau_2, \tau_3, ...\tau_n$ are the vertices connecting to it,*

$$\begin{aligned} PWI_M(\mathcal{G}) &= 2m\sum_{j=1}^{n-1}\mathcal{M}_{\mathcal{H}}(\tau_j)\sum_{k=j+1}^{n}\mathcal{M}_{\mathcal{H}}(\tau_k) \\ &\quad - m\mathcal{M}_{\mathcal{H}}(\tau_1)\sum_{j=2}^{n}M(\tau_j) \\ PWI_{IM}(\mathcal{G}) &= 2(im)\sum_{j=1}^{n-1}I\mathcal{M}_{\mathcal{H}}(\tau_j),\sum_{k=j+1}^{n}I\mathcal{M}_{\mathcal{H}}(\tau_k) \\ &\quad - (im)*(IM)_N(\tau_1)\sum_{j=2}^{n}IM(\tau_j), \end{aligned}$$

$$\begin{aligned} PWI_{NM}(\mathcal{G}) &= 2(nm)\sum_{j=1}^{n-1}N\mathcal{M}_{\mathcal{H}}(\tau_j),\sum_{k=j+1}^{n}N\mathcal{M}_{\mathcal{H}}(\tau_k) \\ &\quad - (nm)*(NM)_N(\tau_1)\sum_{j=2}^{n}NM(\tau_j). \end{aligned}$$

proof. Assume that $\mathcal{G} = (\mathcal{W}, \mathcal{H})$ is a FNT with $|N^*| > 3$. A star can be represented by the symbol $G*$. Since $M = (m, im, nm)$ is a constant function and τ_i is the center vertex, for each $\tau_i, 1 \leq i \leq n$ we obtain $d_{SM}(\tau,\tau_i) = m^3, d_{S(IM)}(\tau,\tau_i) = (im)^3$, and $d_{S(NM)}(\tau,\tau_i) = (nm)^3$. where τ_i and τ_j are not equal to one, then $d_{SM}(\tau_j,\tau_i) = 2m^3, d_{S(IM)}(\tau_j,\tau_i) = 2(im)^3$, and

$d_{S(NM)}(\tau_j, \tau_i) = 2(nm)^3$. Then

$$\begin{aligned} PWI_M(\mathcal{G}) &= \sum \tau_i, \tau_j \in N \mathcal{M}_{\mathcal{H}}(\tau_i)\mathcal{M}_{\mathcal{H}}(\tau_j) d_{SM}(\tau_i, \tau_j) \\ &= \sum_{\tau_j \in N} \mathcal{M}_{\mathcal{H}}(\tau_i)\mathcal{M}_{\mathcal{H}}(\tau_j) d_{SM}(\tau_i, \tau_j) \\ &+ \sum_{\tau_i, \tau_j \in N(i \neq 1)} \mathcal{M}_{\mathcal{H}}(\tau_i)\mathcal{M}_{\mathcal{H}}(\tau_j) d_{SM}(\tau_i, \tau_j) \\ &= m^3 \mathcal{M}_{\mathcal{H}}(\tau_1) \sum_{j=2}^{n} \mathcal{M}_{\mathcal{H}}(\tau_j) + (2m)^3 \sum_{j=2}^{n-1} \mathcal{M}_{\mathcal{H}}(\tau_j) \\ &\sum_{k=j+1}^{n} \mathcal{M}_{\mathcal{H}}(\tau_k) - (2m)^3 \mathcal{M}_{\mathcal{H}}(\tau_1) \sum_{j=2}^{n} M(\tau_j) \\ &= (2m)^3 \sum_{j=1}^{n-1} \mathcal{M}_{\mathcal{H}}(\tau_j) \sum_{k=j+1}^{n} \mathcal{M}_{\mathcal{H}}(\tau_k) \\ &- m^3 \mathcal{M}_{\mathcal{H}}(\tau_1) \sum_{j=2}^{n} \mathcal{M}_{\mathcal{H}}(\tau_j). \end{aligned}$$

Likewise it is possible to show that $PWI_{IM}(\mathcal{G})$ and $PWI_{NM}(\mathcal{G})$.

4. **Application of Improving Election Analysis with the Winner Index**

Elections are important processes in democratic countries to make decisions on options through votes and depend on the vague aspirations of the voting population and so on. However, as Mao and Rosen may imply, evaluating the results of the elections can be quite a puzzle. Due to innate indeterminacy and variability of peoples perceptions and their choices. Unfortunately, conventional methods often do not allow for capturing the subtleties of voters emotions sufficiently. Therefore, the utilization of such sophisticated models as FNGs alongside with the winner index can be viewed as a possible way of enhancing the election analysis due to handling uncertainty and indeterminacy in a more effective manner. To solve the problem of modeling and interpreting of connections using data which is fuzzy, FNGs can be used. In this model, nodes represent entities (e. g. , Leadership Traits, Policy Recommendations, Dedication to the Public Interest, while the lines linking them, represent inter connectivity between them (vertices are And Vision and Goals).

They involve the use fermatean neutrosophic numbers to denote the level of $\mathcal{M}$, $\mathcal{NM}$,and the level of $\mathcal{IM}$ about a connection, to make the representation of fuzzy data more realistic. The winner index is one of the essential parameters derived from FNGs namely representing the total favorableness or approval regarding some object/individual inside the graph. In addition to the degree of $\mathcal{M}$, $\mathcal{NM}$ and $\mathcal{IM}$ of each relation is also considered as providing the full degree of the positions of entities. While analyzing the election data, the concepts

of FNGs are beneficial as they can help to model interactions between voters and applicants. Every voter has a baked-in preference for politicians and this choice depends on several factors such as party affiliation, popularity of the politicians, and promises made among others. We may clarify the rather vague and uncertain voter attitudes by representing these preferences as FNGs. The Winner Index may then perhaps be applied to identify which candidate had the most support among the poll-going populace. In contrast with previous methodologies that rely on completely based on precise figures, the Winner Index reflects the vagueness and uncertainty of a voter inputs leading to a kind of study of the election therefore brings out a rather balanced perception of election processes and the choices made regarding them. We define a FNG in the following sequence, depending on each candidate: For first candidate we have G_1, for second candidate G_2, for third candidate G_3 and for the fourth candidate G_4.
$PWI_M(\mathcal{G}) = 1.287, PWI_{IM}(\mathcal{G}) = 0.704$ and $PWI_{NM}(\mathcal{G}) = 0.516$. $TWI(\mathcal{G}) = 0.90606$.

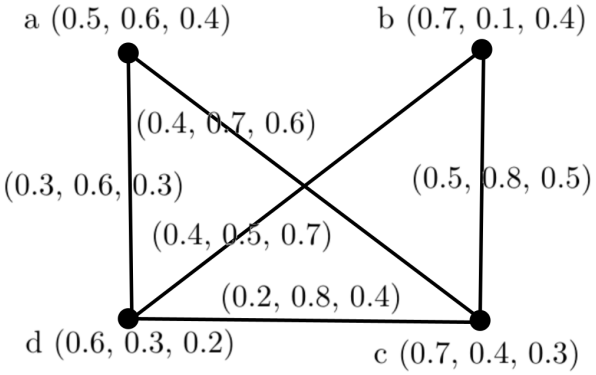

FIGURE 4. Fermatean neutrosophic graph G_1

Edges	$d_{SM}(\eta, \tau)$	$d_{SIM}(\eta, \tau)$	$d_{SNM}(\eta, \tau)$
(a, b)	0.3	1.5	1.0
(a, c)	0.5	0.7	0.6
(a, d)	1	1.5	1.1
(b, c)	0.2	0.8	1.2
(b, d)	0.7	1.6	0.7
(c, d)	0.6	0.8	0.5

TABLE 6. Total Edge weights between η and τ in geodesic paths fermatean neutrosophic G_1

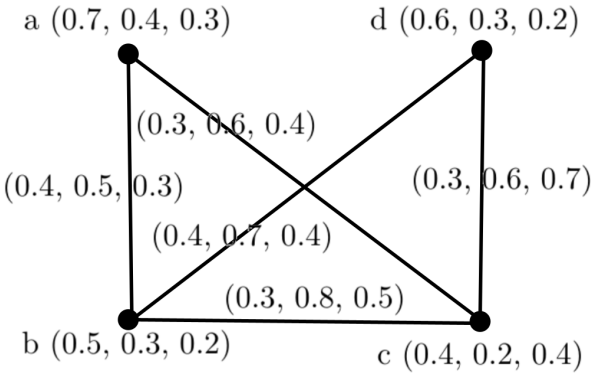

FIGURE 5. Fermatean neutrosophic graph G_2

Edges	$d_{SM}(\eta,\tau)$	$d_{SIM}(\eta,\tau)$	$d_{SNM}(\eta,\tau)$
(a,b)	0.6	1.4	0.9
(a,c)	0.3	0.6	0.4
(a,d)	0.6	1.2	0.8
(b,c)	0.6	0.8	0.5
(b,d)	0.6	1.4	0.9
(c,d)	0.3	1.5	0.9

TABLE 7. Total Edge weights between $\eta$and τ in geodesic paths for fermatean neutrosophic G_2

$PWI_M(\mathcal{G}) = 0.918, PWI_{IM}(\mathcal{G}) = 0.624$ and $PWI_{NM}(\mathcal{G}) = 0.298$. $TWI(\mathcal{G}) = 0.7173$.

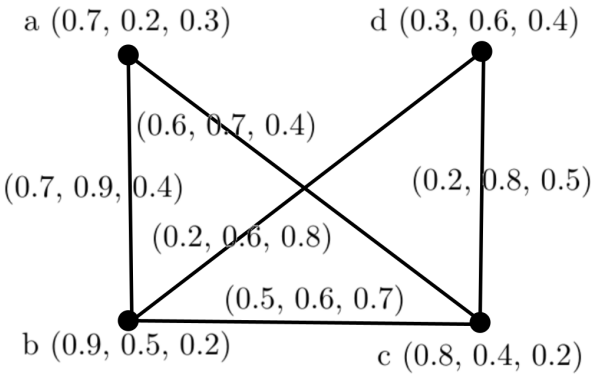

FIGURE 6. Fermatean neutrosophic graph G_3

$PWI_M(\mathcal{G}) = 1.587, PWI_{IM}(\mathcal{G}) = 0.882$ and $PWI_{NM}(\mathcal{G}) = 0.392$. $TWI(\mathcal{G}) = 0.8445$.
$PWI_M(\mathcal{G}) = 1.421, PWI_{IM}(\mathcal{G}) = 0.9$ and $PWI_{NM}(\mathcal{G}) = 0.589$. $TWI(\mathcal{G}) = 0.719$.

Edges	$d_{SM}(\eta,\tau)$	$d_{SIM}(\eta,\tau)$	$d_{SNM}(\eta,\tau)$
(a,b)	1.1	0.9	0.4
(a,c)	0.6	1.5	0.4
(a,d)	0.8	1.5	0.9
(b,c)	0.4	0.6	1.3
(b,d)	0.2	0.6	0.8
(c,d)	0.2	0.8	1.5

TABLE 8. Total Edge weights between η and τ in geodesic paths Fermatean neutrosophic graph G_3

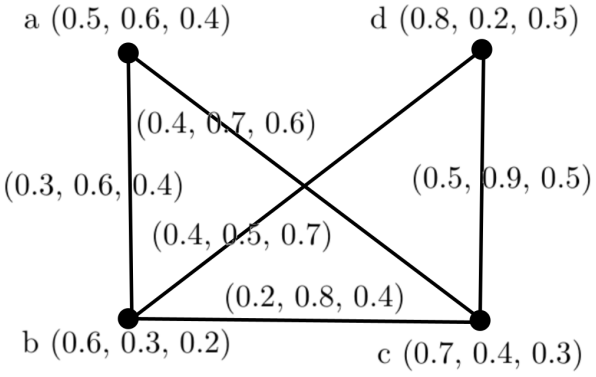

FIGURE 7. Fermatean neutrosophic graph G_4

Edges	$d_{SM}(\eta,\tau)$	$d_{SIM}(\eta,\tau)$	$d_{SNM}(\eta,\tau)$
(a,b)	0.3	1.5	1
(a,c)	0.5	0.7	0.6
(a,d)	1	1.6	1.1
(b,c)	0.2	0.8	1.2
(b,d)	0.7	1.7	0.7
(c,d)	0.6	0.9	0.5

TABLE 9. Total Edge weights between η and τ in geodesic paths fermatean neutrosophic graphG_4

Using the winner index in FNG plots has various advantages in election analysis. This allows policymakers, analysts, and candidates to make informed judgments that take into account the complexity and ambiguity inherent in voters' preferences. It also increases the openness and integrity of the political process, creating a reliable method for objective analysis of election results.

4.1. *Comparative Analysis*

For this reason, the wiener awards are accurate and exhaustive when mapping and illustrating indeterminacy in distance measurement. The current study of the wiener Index in Fermatean Neutrosophic Fuzzy Graphs is found to be more efficient wiener index in neutrosophic graphs. The fermatean neutrosophic components support the FNGs to represent and study the uncertainties in a deeper manner and hence it can be useful to analyze systems which involves highly complex uncertainties. In addition, the wiener index FNGs has captured the direct shortest between vertices and it has also directly dealt with indeterminacy thereby providing a better shot at distance, and a more understanding of the configuration of the network. For academics and practitioners which need to study systems with high level of non-linearity and level of uncertainties, it is the approach of choice because of the its applicability areas as for example biological networks or a complex social system. Hence, wiener index for FNGs is an ideal solution as offers more analytical proficiency to handle with the uncertainty and accurate mesh distances.

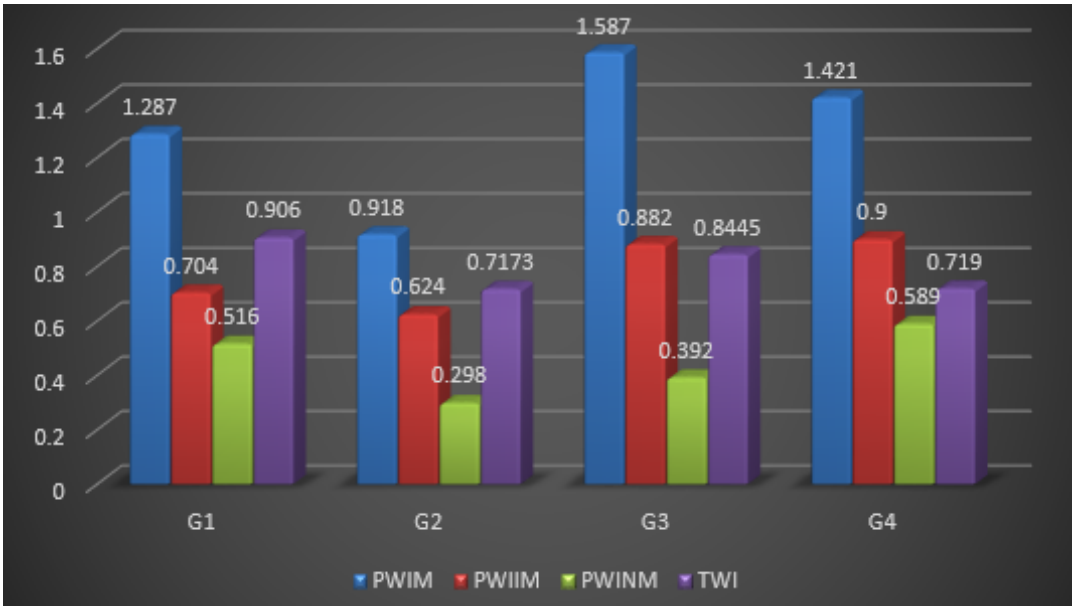

FIGURE 8. Graphical Representation

5. **Conclusion**

This paper aims to explore the applicability of the wiener index, a basic parameter in the area of graph theory, in this article is set within the context of Fermatean neutrosophic networks. The wiener index offers details on the discrimination of distances separating the vertices in these networks can be undertaken by studying the shortest path between the vertices. This work also aims and provide the definition and computation of the wiener index, making an

analysis of several kinds of FNGs including trees, cycles, and full graphs. Using pertinent theories, the paper also looks at the relation of the wiener index to another useful parameter, connectedness. Finally, the wiener index is defined for election analysis using FNGs. In FNGs, the wiener index is an effective way to enhance election analyses in democratic states. It provides an increased understanding of the voters mood in the polling booth through the use of better and truer methodologies.

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Comprehensive Evaluation of Green Construction in Building Projects Using an Integrated MCDM and Plithogenic Set Approach

Yuqian Liu[1*], Xuan Yu[1], Shuangquan Zheng[2]
[1]Gingko College of Hospitality Management, Chengdu, 611743, Sichuan, China
[2]Sichuan Airport and Road Engineering Co.,Ltd, Chengdu, 610081, Sichuan, China
*Corresponding author: E-mail: hemi0926@163.com

Abstract: Green construction aims to minimize the overall negative impact of the built environment on both human health and the environment by efficiently using energy, water, and other resources. It helps reduce waste, pollution, and environmental impact while enhancing worker productivity and protecting occupant health. This study introduces a multi-criteria decision-making (MCDM) approach to evaluate green construction in building projects. The MCDM approach is integrated into the plithogenic set framework to address uncertainty and vague information. The simple additive weighting (SAW) method is employed to rank the alternatives. Three experts evaluated nine criteria and eight alternatives to determine the criteria weights and rank the alternatives. Sensitivity analysis was conducted to assess the stability of the rankings, with results showing that the ranks of the alternatives remain stable under varying criteria weights.

Keywords: Sustainability; Green construction; building projects; Decision-making; MCDM Approach.

1. Introduction

Depending on the expert, different people have different ideas about green building. It denotes clean construction made of natural materials for eco-builders. They believe that a building's ability to accommodate people must come first, with the welfare of its residents serving as its most valuable asset. These proponents of green building denounce the industrial production of building materials using hazardous ingredients. Using

cutting-edge technologies, energy conservation experts seek to minimize the adverse environmental effects of human habitation and lower the energy consumption of homes, buildings, and flats. They advise using cutting-edge building methods and improved thermal insulation. Eco-builders take a building's lifespan into account. In addition to including energy conservation, they also examine the source of the materials[1], [2].

Eco-construction, sometimes known as green building or sustainable construction, suggests several ways to lessen the influence that buildings have on the environment. Green building is not a single construction methodology; rather, it is a collection of methods, materials, and technologies that, when properly applied to a project, help to improve its environmental performance[3], [4]. In a perfect scenario, eco-construction maximizes the use of recyclable, non-toxic, and recycled materials while minimizing water usage and energy consumption. Additionally, it produces as little waste as feasible both during building and after it is occupied[5], [6].

The procedures used to create the structural elements of a green building are resource-efficient and environmentally conscious. This approach is becoming more popular and supports the traditional goals of creating comfortable, sustainable, and energy-efficient buildings. A green building is a sustainable, clean structure that is easy to maintain, made of natural materials, uses little energy—and renewable energy—and is reasonably priced [7].

A green building can effectively use sustainable materials (recycled, recyclable, repurposed, or derived from renewable resources) in its construction, provide a healthy interior environment with minimal pollutants, and have a water-efficient landscape design that uses native vegetation that grows well without extra watering[7], [8]. Evaluating green construction is a multi-criteria decision-making (MCDM) approach. However the evaluation of this problem contains uncertainty.

Uncertain, ambiguous, and incomplete information is a flaw that can result in a less-than-ideal choice, as is the case with many assessments and decision-making issues. Therefore, more accurate evaluation findings should be obtained by combining the plithogenic set with the triple components of the neutrosophic set (truth-membership, falsity-membership, and indeterminacy-membership). Plithogenic sets improve decision-making efficiency and accuracy. Smarandache developed plithogeny, which is a generalization of neutrosophy[9], [10].

The plithogenic set is a collection of elements where each element x is distinguished by an appurtenance degree d(x, v) of element x to the plithogenic set and attribute values v that have a matching contradiction degree c(v, D) between them and a dominant attribute value D[11], [12].

MCDM focuses on organizing and resolving planning and decision-making issues that involve several criteria. Some prevalent features of MCDM difficulties include the existence of disparate alternatives, multiple non-commensurable and competing criteria, and differing units of measurement among the criteria. The MCDM problem's criteria can all be divided into two groups. Benefit criteria are those that should be maximized. On the other hand, the cost criteria group includes factors that should be minimized. An example of an MCDM problem with m alternatives[13], [14].

The most straightforward, well-known, and widely applied MCDM technique is most likely the SAW approach. The weighted total of all attribute values determines a candidate solution's overall score in SAW. The three main processes of the SAW technique are determining the overall score for each alternative, assigning the weight vector W, and normalizing the decision matrix X[15].

2. Methods

In this work, the MCDM approach is applied to compute the criteria weights and rank the alternatives. The MCDM approach is applied under the plithogenic sets. The

plithogenic set is used to solve the uncertainty in the MCDM approach to evaluate the alternatives.

2.1 Basic concepts of the plithogenic set

Plithogeny is a generalization of neutrosophy established by Smarandache that refers to the creation, improvement, and advancement of new items from syntheses of competing or nonconflicting multiple old ones. Plithogenic intersection (∧p), plithogenic union (∨p), plithogenic complement (¬p), plithogenic inclusion (→), and plithogenic equality (↔) are the operations on plithogenic sets[16], [17].

Because of its two primary characteristics, the degree of appurtenance and the degree of contradiction, the plithogenic set offers a high consideration of information uncertainty to produce more accurate findings. Each attribute value is distinguished from the dominant (most desired) attribute value using the contradiction (dissimilarity) degree function c(v,D). The following axioms are maintained by the attribute value contradiction degree function c(v1, v2), which is c: V×V → [0, 1][18], [19].

Let two plithogenic numbers such as $a = (a_1, a_2, a_3)\ and\ b = (b_1, b_2, b_3)$ and plithogenic operators can be computed as:

$$\big((a_{i1}, a_{i2}, a_{i3}), 1 \le i \le n\big) \wedge p\big((b_{i1}, b_{i2}, b_{i3}), 1 \le i \le n\big)$$

$$= \big((a_{i1} \wedge_F b_{i1}, 0.5 * (a_{i2} \wedge_F b_{i2}) + 0.5(a_{i2} \vee_F b_{i2}), a_{i3} \vee_F b_{i3}\ \ \big)\big), 1 \le i < n$$

$$\big((a_{i1}, a_{i2}, a_{i3}), 1 \le i \le n\big) \vee p\big((b_{i1}, b_{i2}, b_{i3}), 1 \le i \le n\big)$$

$$= \big((a_{i1} \vee_F b_{i1}, 0.5 * (a_{i2} \wedge_F b_{i2}) + 0.5(a_{i2} \vee_F b_{i2}), a_{i3} \wedge_F b_{i3}\ \ \big)\big), 1 \le i < n$$

2.2 SAW Approach

SAW is the MCDM approach and the simplest approach. The overall score can be computed by the weighted normalized matrix. The SAW approach has three main stages:

normalizing the decision matrix, assigning the weight vector, and computing the overall score[15]. The steps of the SAW approach are detailed such as:

Step 1. Build the decision matrix.

The decision matrix is constructed between the criteria and alternatives by the opinions of experts.

Step 2. Normalize the decision matrix.

A normalization process should be used to convert the original data into equivalent values. Many different normalizing algorithms have been devised for the SAW approach. The most used normalizing technique is most likely the Max approach. It is necessary to convert cost criteria into benefit criteria during normalization. The following is the normalization process:

$$r_{ij} = \frac{x_{ij}}{\max x_{ij}}$$

$$r_{ij} = \frac{\min x_{ij}}{x_{ij}}$$

Step 3. Assign weight to each criterion.

Step 4. Compute the final rank.

$$S_i = \sum_{j=}^{n} w_j x_{ij}$$

Where r_{ij} is the ith alternative's normalized performance concerning the jth criterion, wj is the weight of the jth criterion, and Si is the ith alternative's ranking score. The ranking score Si in the SAW method indicates the overall performance of the ith alternative, and the option with the highest Si value is the one that is rated highest.

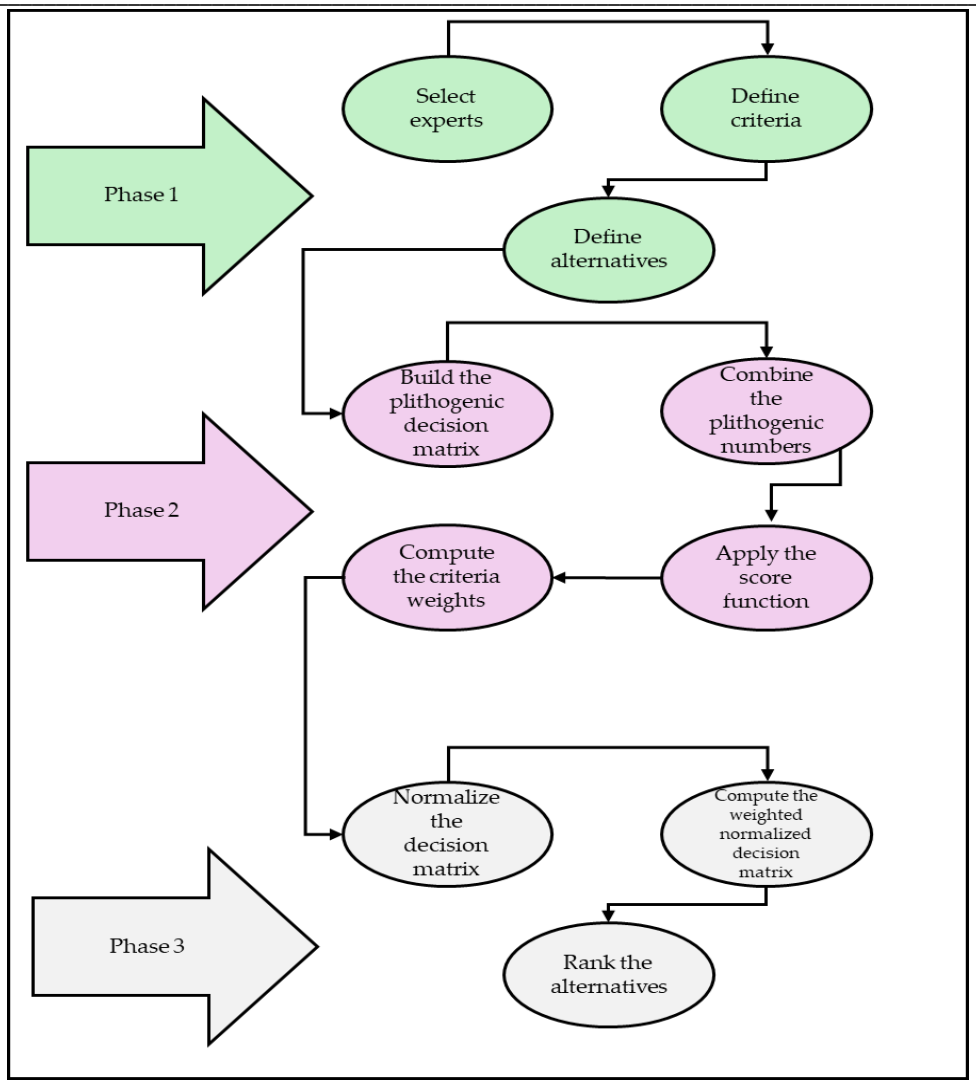

Figure 1. Phases of proposed method.

3. Proposed Framework

This study proposed a combined framework to evaluate the performance of Green Construction in building projects under the uncertainty environment. We compute the criteria weights to affect the rank of alternatives. Then we applied the SAW approach to rank the alternatives.

This strategy is crucial for managing the significant degree of uncertainty brought on by the lack of subject-matter expertise. By considering the truth-membership function, falsity-membership function, and indeterminacy-membership function, Plithogenic

makes it effective at managing ambiguous judgments. Furthermore, the plithogenic set operations' properties yield more precise outcomes. To provide a more accurate assessment, this framework makes use of the benefits of the SAW technique and plithogenic set operation. The steps of the suggested framework are shown in Figure 1 and explained below:

Phase 1: A panel of decision makers is established to obtain valuable information on evaluation issues. We define a set of criteria to measure the performance of Green Construction in building projects and the alternatives to evaluate a set of projects.

In this study, the proposed approach examines the performance of Green Construction in building projects based on nine criteria and eight alternatives as shown in Figure 2.

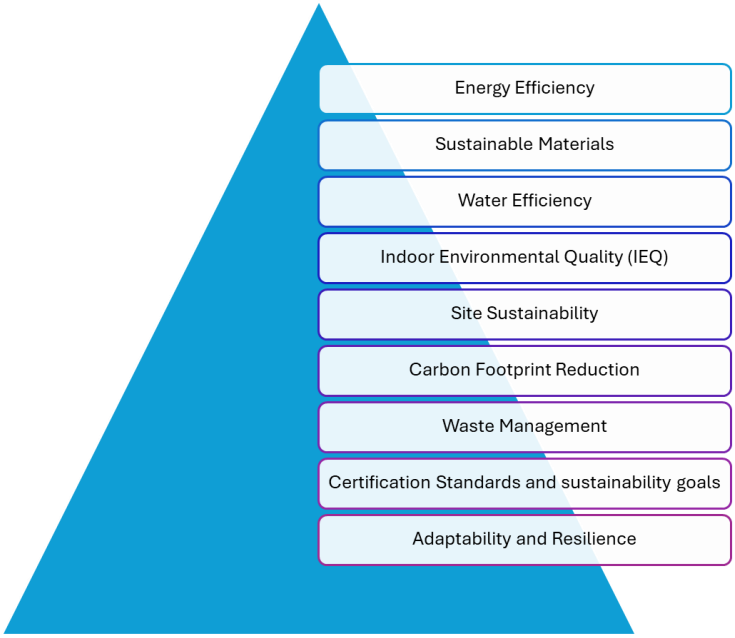

Figure 2. Assessment criteria of Green Construction in building projects.

Phase 2: Compute the criteria weights.

- Step 1. Build the plithogenic decision matrix between the criteria. Experts use the plithogenic terms to evaluate the criteria and alternatives.

- Step 2. Combine the plithogenic numbers into one matrix using the plithogenic operators.
- Step 3. Apply the score function to obtain the crisp values between the criteria and alternatives.
- Step 4. Compute the criteria weights by normalizing the crisp values.

Phase 3: rank the alternatives using the SAW approach.

- Step 1: Normalize the decision matrix between the criteria and alternatives.

- Step 2: Compute the weighted normalized decision matrix.

- Step 3: Rank the alternatives.

4. Case Study

The comprehensive evaluation of green construction in building projects focuses on assessing the overall implementation of sustainable practices during the construction process. It aims to ensure that the project aligns with environmental protection goals, resource efficiency, and sustainable development principles. Green construction emphasizes reducing environmental impact, promoting innovation, and adopting energy-saving technologies to create environmentally friendly and efficient construction processes. This evaluation serves as a systematic approach to reviewing the integration of green practices, ensuring the project meets sustainability requirements. It highlights the importance of minimizing waste, conserving natural resources, and protecting the surrounding ecosystem. Additionally, the evaluation encourages the adoption of advanced construction methods and technologies that align with long-term environmental and economic goals. By conducting a comprehensive evaluation, stakeholders can ensure that green construction principles are effectively implemented, contributing to sustainable urban development and eco-friendly building practices. The proposed approach based on plithogenic set is used to measure the green construction in

building projects performance. The evaluation matrix is obtained by three experts and decision-makers.

Phase 1: The performance of Green Construction in building projects is based on nine criteria and eight alternatives. Evaluate eight projects based on their performance.

Phase 2: To compute the criteria weights, three experts have evaluated the criteria and alternatives. Then we evaluate these criteria and alternatives using the plithogenic numbers as shown in Table 1. Then we combine these numbers using the plithogenic operators. Then we apply the score function to obtain crisp values. Then we compute the criteria weights as shown in Figure 3.

Table 1. The decision matrix.

	C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8	C_9
A_1	(0.25, 0.60, 0.80)	(0.80, 0.10, 0.30)	(0.95, 0.05, 0.05)	(0.80, 0.10, 0.30)	(0.65, 0.30, 0.45)	(0.50, 0.40, 0.60)	(0.40, 0.70, 0.50)	(0.25, 0.60, 0.80)	(0.10, 0.75, 0.85)
A_2	(0.80, 0.10, 0.30)	(0.25, 0.60, 0.80)	(0.80, 0.10, 0.30)	(0.65, 0.30, 0.45)	(0.50, 0.40, 0.60)	(0.25, 0.60, 0.80)	(0.25, 0.60, 0.80)	(0.80, 0.10, 0.30)	(0.25, 0.60, 0.80)
A_3	(0.80, 0.10, 0.30)	(0.80, 0.10, 0.30)	(0.25, 0.60, 0.80)	(0.50, 0.40, 0.60)	(0.25, 0.60, 0.80)	(0.80, 0.10, 0.30)	(0.80, 0.10, 0.30)	(0.80, 0.10, 0.30)	(0.80, 0.10, 0.30)
A_4	(0.95, 0.05, 0.05)	(0.80, 0.10, 0.30)	(0.80, 0.10, 0.30)	(0.25, 0.60, 0.80)	(0.80, 0.10, 0.30)	(0.80, 0.10, 0.30)	(0.80, 0.10, 0.30)	(0.95, 0.05, 0.05)	(0.80, 0.10, 0.30)
A_5	(0.95, 0.05, 0.05)	(0.95, 0.05, 0.05)	(0.80, 0.10, 0.30)	(0.80, 0.10, 0.30)	(0.80, 0.10, 0.30)	(0.95, 0.05, 0.05)	(0.95, 0.05, 0.05)	(0.95, 0.05, 0.05)	(0.95, 0.05, 0.05)
A_6	(0.10, 0.75, 0.85)	(0.95, 0.05, 0.05)	(0.95, 0.05, 0.05)	(0.80, 0.10, 0.30)	(0.95, 0.05, 0.05)	(0.95, 0.05, 0.05)	(0.95, 0.05, 0.05)	(0.10, 0.75, 0.85)	(0.95, 0.05, 0.05)
A_7	(0.80, 0.10, 0.30)	(0.10, 0.75, 0.85)	(0.95, 0.05, 0.05)	(0.95, 0.05, 0.05)	(0.95, 0.05, 0.05)	(0.10, 0.75, 0.85)	(0.10, 0.75, 0.85)	(0.65, 0.30, 0.45)	(0.10, 0.75, 0.85)
A_8	(0.95, 0.05, 0.05)	(0.65, 0.30, 0.45)	(0.10, 0.75, 0.85)	(0.95, 0.05, 0.05)	(0.10, 0.75, 0.85)	(0.10, 0.75, 0.85)	(0.95, 0.05, 0.05)	(0.80, 0.10, 0.30)	(0.10, 0.75, 0.85)
	C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8	C_9
A_1	(0.80, 0.10, 0.30)	(0.80, 0.10, 0.30)	(0.95, 0.05, 0.05)	(0.80, 0.10, 0.30)	(0.65, 0.30, 0.45)	(0.50, 0.40, 0.60)	(0.40, 0.70, 0.50)	(0.25, 0.60, 0.80)	(0.10, 0.75, 0.85)
A_2	(0.10, 0.75, 0.85)	(0.10, 0.75, 0.85)	(0.80, 0.10, 0.30)	(0.65, 0.30, 0.45)	(0.50, 0.40, 0.60)	(0.40, 0.70, 0.50)	(0.10, 0.75, 0.85)	(0.10, 0.75, 0.85)	(0.80, 0.10, 0.30)
A_3	(0.95, 0.05, 0.05)	(0.95, 0.05, 0.05)	(0.80, 0.10, 0.30)	(0.50, 0.40, 0.60)	(0.80, 0.10, 0.30)	(0.65, 0.30, 0.45)	(0.80, 0.10, 0.30)	(0.10, 0.75, 0.85)	(0.10, 0.75, 0.85)
A_4	(0.80, 0.10, 0.30)	(0.80, 0.10, 0.30)	(0.10, 0.75, 0.85)	(0.80, 0.10, 0.30)	(0.10, 0.75, 0.85)	(0.80, 0.10, 0.30)	(0.10, 0.75, 0.85)	(0.80, 0.10, 0.30)	(0.95, 0.05, 0.05)
A_5	(0.50, 0.40, 0.60)	(0.65, 0.30, 0.45)	(0.95, 0.05, 0.05)	(0.10, 0.75, 0.85)	(0.95, 0.05, 0.05)	(0.10, 0.75, 0.85)	(0.95, 0.05, 0.05)	(0.10, 0.75, 0.85)	(0.80, 0.10, 0.30)
A_6	(0.65, 0.30, 0.45)	(0.95, 0.05, 0.05)	(0.80, 0.10, 0.30)	(0.95, 0.05, 0.05)	(0.80, 0.10, 0.30)	(0.95, 0.05, 0.05)	(0.80, 0.10, 0.30)	(0.95, 0.05, 0.05)	(0.80, 0.10, 0.30)
A_7	(0.80, 0.10, 0.30)	(0.80, 0.10, 0.30)	(0.40, 0.70, 0.50)	(0.80, 0.10, 0.30)	(0.80, 0.10, 0.30)	(0.80, 0.10, 0.30)	(0.80, 0.10, 0.30)	(0.80, 0.10, 0.30)	(0.10, 0.75, 0.85)
A_8	(0.95, 0.05, 0.05)	(0.65, 0.30, 0.45)	(0.50, 0.40, 0.60)	(0.40, 0.70, 0.50)	(0.65, 0.30, 0.45)	(0.80, 0.10, 0.30)	(0.65, 0.30, 0.45)	(0.80, 0.10, 0.30)	(0.95, 0.05, 0.05)
	C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8	C_9
A_1	(0.65, 0.30, 0.45)	(0.65, 0.30, 0.45)	(0.95, 0.05, 0.05)	(0.80, 0.10, 0.30)	(0.65, 0.30, 0.45)	(0.50, 0.40, 0.60)	(0.40, 0.70, 0.50)	(0.25, 0.60, 0.80)	(0.10, 0.75, 0.85)
A_2	(0.50, 0.40, 0.60)	(0.50, 0.40, 0.60)	(0.80, 0.10, 0.30)	(0.65, 0.30, 0.45)	(0.65, 0.30, 0.45)	(0.65, 0.30, 0.45)	(0.65, 0.30, 0.45)	(0.10, 0.75, 0.85)	(0.95, 0.05, 0.05)
A_3	(0.40, 0.70, 0.50)	(0.40, 0.70, 0.50)	(0.65, 0.30, 0.45)	(0.50, 0.40, 0.60)	(0.50, 0.40, 0.60)	(0.65, 0.30, 0.45)	(0.50, 0.40, 0.60)	(0.65, 0.30, 0.45)	(0.80, 0.10, 0.30)
A_4	(0.50, 0.40, 0.60)	(0.50, 0.40, 0.60)	(0.50, 0.40, 0.60)	(0.40, 0.70, 0.50)	(0.40, 0.70, 0.50)	(0.50, 0.40, 0.60)	(0.40, 0.70, 0.50)	(0.50, 0.40, 0.60)	(0.65, 0.30, 0.45)
A_5	(0.40, 0.70, 0.50)	(0.65, 0.30, 0.45)	(0.40, 0.70, 0.50)	(0.50, 0.40, 0.60)	(0.50, 0.40, 0.60)	(0.40, 0.70, 0.50)	(0.50, 0.40, 0.60)	(0.40, 0.70, 0.50)	(0.50, 0.40, 0.60)
A_6	(0.40, 0.70, 0.50)	(0.50, 0.40, 0.60)	(0.50, 0.40, 0.60)	(0.40, 0.70, 0.50)	(0.40, 0.70, 0.50)	(0.50, 0.40, 0.60)	(0.40, 0.70, 0.50)	(0.50, 0.40, 0.60)	(0.40, 0.70, 0.50)
A_7	(0.80, 0.10, 0.30)	(0.40, 0.70, 0.50)	(0.40, 0.70, 0.50)	(0.40, 0.70, 0.50)	(0.25, 0.60, 0.80)	(0.40, 0.70, 0.50)	(0.40, 0.70, 0.50)	(0.40, 0.70, 0.50)	(0.50, 0.40, 0.60)
A_8	(0.95, 0.05, 0.05)	(0.50, 0.40, 0.60)	(0.50, 0.40, 0.60)	(0.40, 0.70, 0.50)	(0.25, 0.60, 0.80)	(0.10, 0.75, 0.85)	(0.95, 0.05, 0.05)	(0.40, 0.70, 0.50)	(0.40, 0.70, 0.50)

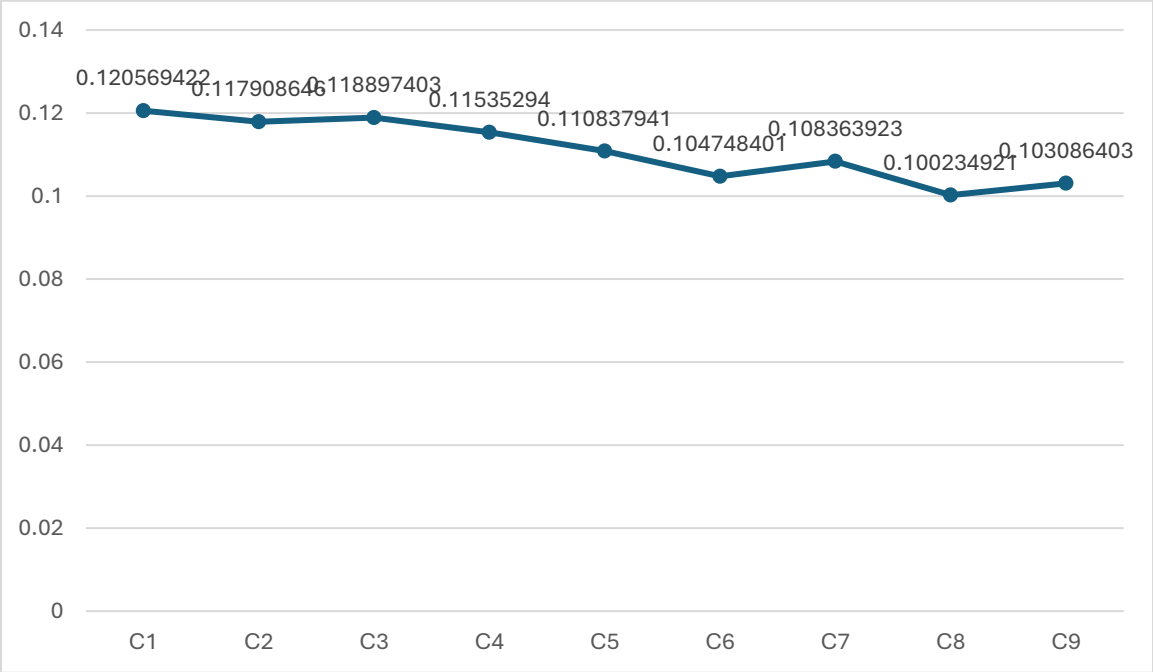

Figure 3. The criteria weights.

Phase 3: In this study, we used the SAW approach to rank the alternatives. We normalize the decision matrix between the criteria and alternatives as shown in Table 2. Then we compute the weighted normalized decision matrix. Then we compute the overall score values for each alternative as shown in Table 3.

Table 2. Normalization matrix

	C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8	C_9
A_1	0.744448	0.958415	1	1	0.870142	0.692136	0.530394	0.375488	0.150534
A_2	0.609572	0.416999	0.947288	0.844584	0.770544	0.58486	0.49574	0.577873	0.952107
A_3	0.874474	0.926567	0.744448	0.689572	0.734245	0.889986	0.926635	0.712359	0.721315
A_4	0.898979	0.926635	0.609572	0.676368	0.611119	0.926635	0.595375	1	1
A_5	0.741959	0.902123	0.874474	0.643492	1	0.628981	1	0.660325	0.946898
A_6	0.517645	1	0.898979	0.923134	0.933336	1	0.909291	0.742954	0.903912
A_7	0.947288	0.6303	0.676803	0.905922	0.942983	0.6303	0.6303	0.86626	0.352625
A_8	1	0.802556	0.494959	0.714464	0.47078	0.478109	0.944536	0.922553	0.677153

Table 3. The rank of alternatives.

	Score	Rank
A_1	0.716589	2
A_2	0.68918	1
A_3	0.803524	6
A_4	0.800782	5
A_5	0.822914	7
A_6	0.868076	8
A_7	0.735526	4
A_8	0.723355	3

To show the stability of the proposed approach we performed the sensitivity analysis by changing criteria weights and then ranking the alternatives. We proposed ten cases in changing the criteria weights as shown in Figure 4. Then we apply the SAW approach to these ten cases. Then we rank the alternatives based on these cases as shown in Table 4. The results show that alternative 6 is the best and alternative 2 is the worst.

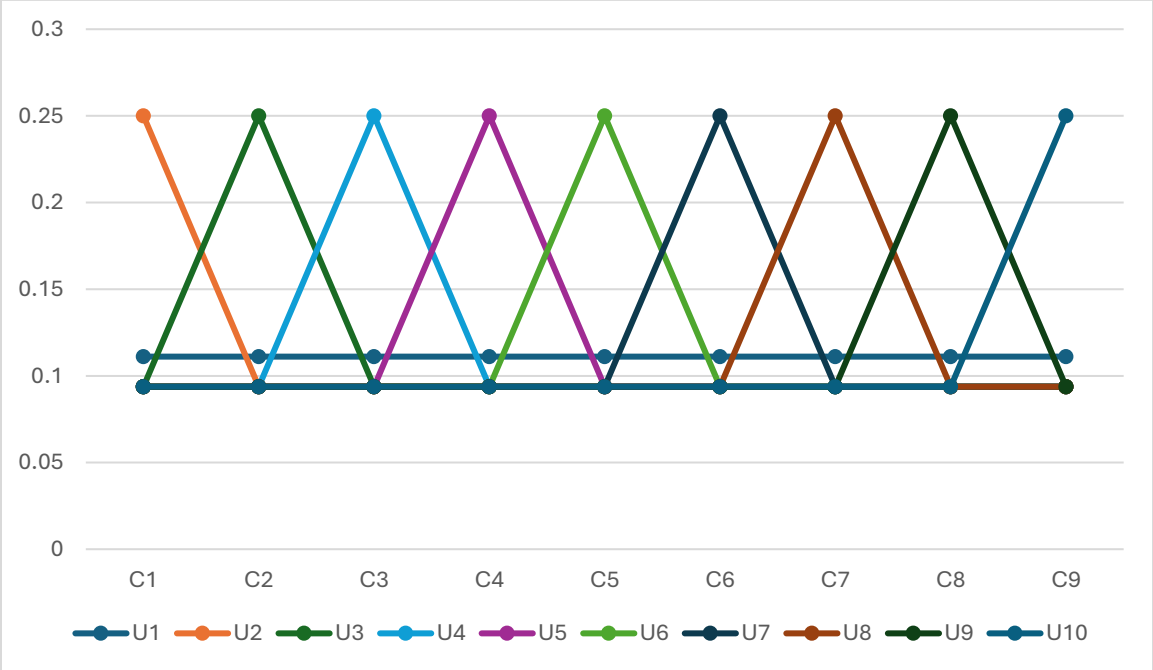

Figure 4. The change in criteria weights.

Table 4. The results of sensitivity analysis.

	U_1	U_2	U_3	U_4	U_5	U_6	U_7	U_8	U_9	U_{10}
A_1	2	2	4	4	3	3	3	2	1	1
A_2	1	1	1	3	1	2	1	1	2	4
A_3	5	6	5	6	5	6	6	6	5	5
A_4	6	8	6	5	6	5	7	5	7	6
A_5	7	5	7	7	7	7	5	7	6	7
A_6	8	7	8	8	8	8	8	8	8	8
A_7	4	3	2	2	4	4	4	3	3	2
A_8	3	4	3	1	2	1	2	4	4	3

5. Conclusions

This study uses a hybrid SAW approach to formulate the problem of performance evaluation of green construction as an MCDM. The majority of evaluation difficulties have shortcomings because there are disparate decision-makers, options, and standards. For this reason, the plithogenic set serves as the foundation for the suggested structure. The plithogenic hypothesis yields extremely accurate answers for information that is ambiguous, ambiguous, inconsistent, and lacking in real-world assessments. In the meantime, it considers each evaluation's degrees of truth, indeterminacy, and falsity.

Three experts have evaluated the criteria and alternatives. They used plithogenic numbers to evaluate the decision matrix. Eight alternatives and nine criteria are used to build the decision matrix. The SAW approach is a MCDM approach used to rank alternatives. The results show that alternative 6 is the best and alternative 2 is the worst.

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TOPSIS approach for MADM based on Quadripartitioned single valued neutrosophic refined Hamacher aggregation operations

Arokia Pratheesha S V[1], Annapoorna M S[2],Radha R[3],Parul Arora[4],Ritika Wason[5],Divya Baliga B[6]

[1]Assistant Professor,Department of Mathematics,Dr.N.G.P Arts and Science College,Coimbatore; pratheesha555@gmail.com

[2] Assistant Professor, Department of Mathematics, BMS Institute of Technology and Management, Bengaluru, Karnataka, India, ;anuraga@bmsitin

[3]Assistant Professor,Department of Science and Humanities,Karpagam College of Engineering,Coimbatore;radharmat2020@gmail.com

[4]Associate Professor, Department of Computer Science and Applications, Bharati Vidyapeeth's Institute of Computer Applications and Management (BVICAM), Delhi, India;paruldevsum@gmail.com

[5]Deputy Director, Department of Computer Science and Applications, Bharati Vidyapeeth's Institute of Computer Applications and Management (BVICAM), Delhi, India;rit_2282@yahoo.co.in

[6]Assistant Professor, Department of Mathematics, New Horizon College, Bangalore, Karnataka, India;divyabaligab@gmail.com

*Correspondence: pratheesha555@gmail.com;

Abstract. Hamacher operators are extensively utilized in multicriteria attribute group decision-making (MAGDM) problems due to their remarkable adaptability provided by an adjustable parameter. Here, the Hamacher T-norm and T-conorm operations for two $\mathcal{QSVNRN}$s are formulated.Using these Hamacher operations, we present the quadripartitioned single-valued neutrosophic refined Hamacher weighted averaging ($\mathcal{QSVNRHWA}$) operators within the $\mathcal{QSVNR}$ framework and analyze their properties.Finally, we explore a TOPSIS-based approach for multi-attribute decision-making problems that employs the $\mathcal{QSVNRHWA}$ operators, demonstrating its application in evaluating practical scenarios related to converting solid waste into energy.

Keywords: Quadripartitioned Single Valued Neutrosophic Refined Set,Quadripartitioned Single Valued Neutrosophic Refined Number,Multi attribute decision making, Aggregation Operator, TOPSIS.

1. Introduction

In real life, we often encounter issues involving inconsistent, indeterminate, and incomplete information that cannot be accurately represented by precise numbers. In response to this,

Lotfi Aliasker Zadeh[34], the pioneer of fuzzy logic, introduced fuzzy sets in 1965 as a way to effectively address these challenges. Later, in 1986, Krassimir Atanassov[1] expanded upon Zadeh's concept by introducing intuitionistic fuzzy sets, which incorporate varying degrees of both membership and non-membership.

Florentin Smarandache[23,24] became familiar with the neutrosophic set, which underpins contemporary mathematical theories. Neutrosophic sets can generate intuitionistic fuzzy sets(IF), fuzzy sets(FS), and classical sets. The term "neutrosophic" originates from Neutrosophy, a philosophy that encourages the exploration of neutral thought. The main distinction among fuzzy, intuitionistic fuzzy, and neutrosophic logic or sets is this concept of neutrality. This idea is crucial for differentiating the three associated functions in the neutrosophic framework: the truth membership function (T), the indeterminate membership function (I), and the false membership function (F). Additionally, inspired by practical needs, Haibin Wang and his team created single-valued neutrosophic sets by streamlining the neutrosophic set for real-world applications.

Yager[31] pioneered the theory of bags, which extends classical set theory by permitting the repetition of elements, thus introducing the foundation of the multiset. This concept, allowing multiple occurrences of the same element within a set, was later rigorously defined by Blizard [4] and Calude et al.[8].To synthesize the principles of fuzzy multisets with IFS, Shinoj et al. [25] developed the notion of intuitionistic fuzzy multisets(IFMS). Nevertheless, both fuzzy and intuitionistic fuzzy multisets exhibit limitations in addressing higher-order uncertainties, often required in complex data analyses.To overcome these constraints, Chatterjee et al.[21] introduced the single-valued neutrosophic multiset, which refined the representation of uncertainty by incorporating distinct truth, indeterminacy, and falsity values. Building on this framework, Smarandache [26] advanced neutrosophic logic in 2013 by proposing n-valued refined neutrosophic logic. This sophisticated model decomposes the neutrosophic components T_1,T_2,...,T_m and I_1,I_2,...,I_p and F_1,F_2,...,F_r allowing for a more granular depiction of uncertainty across diverse applications.Deli et al.[12]expanded on these ideas by investigating the fundamental properties of neutrosophic refined sets, which generalize both fuzzy and intuitionistic fuzzy multisets, thereby offering enhanced flexibility in representing complex information. Additionally, Ye et al.[33] developed sophisticated distance and similarity measures tailored for single-valued neutrosophic multisets. These metrics were subsequently applied to fields such as medical diagnostics, where data often present ambiguities, contradictions, and incompleteness, illustrating the model's practical utility in handling intricate, uncertain datasets.

In four-valued logic, Belnap [6] created a new concept in which any information is represented by four parameters: T, F, none, and both. These parameters stand for true, false,

neither true nor false, and both true and false, respectively. Smarandache [24] presented four numerical valued neutrosophic logic, which separates indeterminacy into two terms: Contradiction (C) and Unknown (U). Belnap's four-valued logic provides the foundation of this reasoning.Thus, the Quadripartitioned Single Valued Neutrosophic Set (QSVNS), which has four components T, C, U, and F in the real unit interval [0,1], was established by Rajashi Chatterjee et al.[22].

The TOPSIS method integrates a range of valuable and effective models, including the ideal positive and negative solutions, discrimination measures, and closeness measures, to determine rankings based on proximity to ideal solutions. Initially introduced by researchers in 1981, the foundation of the TOPSIS concept was laid by Hwang and Yoon [15]. Yoon [32] then presented the formal theory in 1987, and Hwang et al. provided further evaluation in 1993 [16].For example, imagine a person shopping for a new mobile phone, assessing options based on features like RAM, storage, display size, battery life, and price. Overwhelmed by the variety, the customer may find it difficult to choose the most suitable option. TOPSIS provides an effective solution by ranking the options based on the weight and significance of each feature.Later,Chu and Lin [9] introduced the fuzzy TOPSIS technique, and Wang and Elhag [30] applied a TOPSIS method with FS theory for bridge risk assessments. Several other scholars, including Chen and Tsao [10], Sun and Lin [28], Dymova et al. [12], Ashtiani et al. [3], and Memari et al. [20], applied TOPSIS within FS theory contexts. Furthermore, Shen et al. [28] proposed a TOPSIS method for A-IF set theory, and Joshi and Kumar [18] incorporated entropy measures in TOPSIS for A-IF information. Additional applications include Liu [19] developing a TOPSIS approach for physical education issues, while Zulqarnain and Dayan [35] applied TOPSIS to address challenges in the automotive industry.

Many important norms have been identified by researchers, but the Hamacher t-norms, introduced by Hamacher in 1970, have had a significant and well-regarded influence on the field of aggregation operators in computing. Additionally, Bellman and Zadeh [5] contributed to the development of Hamacher aggregation theory, particularly in decision-making with fuzzy information. Huang [14] explored the use of Hamacher aggregation operators for A-IF sets and their role in decision-making, while Garg [13] proposed Hamacher aggregation operators and entropy measures specifically for A-IF data. More recently, Cakir and Ulukan [7] made advancements in the theory of A-IF Hamacher aggregation operators. These aggregation operators, based on Hamacher norms, are particularly powerful due to their adjustable parameter range of $0 \leq \alpha \leq \infty$.

The paper is structured as follows: Section 2 introduces fundamental concepts related to neutrosophic sets, neutrosophic refined sets, quadripartitioned single-valued neutrosophic sets,

and Hamacher operations. Section 3 focuses on enhancing the Hamacher operations for quadripartitioned single-valued neutrosophic refined numbers. In Section 4, quadripartitioned single-valued neutrosophic refined Hamacher weighted aggregation operators are introduced, along with their essential properties. Section 5 explores the TOPSIS method using QSVNRHWA information and demonstrates a MADM procedure to assess uncertain and unreliable data. This method utilizes the TOPSIS technique and derived operators are developed to enhance the accuracy and significance of the information.Section 7 concludes the paper and discusses future research opportunities.

2. **Preliminaries**

Definition 2.1. [22] Let $\mathcal{A}^*_{\mathcal{QR}}$ be a universe. A quadripartitioned single valued neutrosophic set $\mathcal{QSVNS}$ $\mathcal{K}_{\mathcal{QNR}}$ on $\mathcal{A}^*_{\mathcal{QR}}$ is defined by by a truth-membership function $\breve{\mathcal{T}}_{\mathcal{K}}(\mathfrak{a}_1)$, a contradiction membership function $\breve{\mathcal{D}}_{\mathcal{K}}(\mathfrak{a}_1)$,an unknown membership function $\breve{\mathcal{Y}}_{\mathcal{K}}(\mathfrak{a}_1)$ and a falsity membership function $\breve{\mathcal{F}}_{\mathcal{K}}(\mathfrak{a}_1)$ such that for each $\mathfrak{a}_1 \in \mathcal{A}^*_{\mathcal{QR}}$, $\breve{\mathcal{T}}_{\mathcal{K}}$,$\breve{\mathcal{D}}_{\mathcal{K}}$,$\breve{\mathcal{Y}}_{\mathcal{K}}$,$\breve{\mathcal{F}}_{\mathcal{K}} \in [0,1]$ and $0 \leq \breve{\mathcal{T}}_{\mathcal{K}}(\mathfrak{a}_1)+\breve{\mathcal{T}}_{\mathcal{K}}(\mathfrak{a}_1)+\breve{\mathcal{T}}_{\mathcal{K}}(\mathfrak{a}_1)+\breve{\mathcal{T}}_{\mathcal{K}}(\mathfrak{a}_1) \leq 4$.

Definition 2.2. [2] Let $\mathcal{A}^*_{\mathcal{QR}}$ be a universe. A Quadripartitioned Single Valued Neutrosophic Refined Set(briefly, $\mathcal{QSVNRS}$) $\mathcal{K}_{\mathcal{QNR}}$ on $\mathcal{A}^*_{\mathcal{QR}}$ is defined by

$$\mathcal{K}_{\mathcal{QNR}} = \{\langle\, \mathfrak{a}_1,(\breve{\mathcal{T}}^1_{\mathcal{K}_{\mathcal{QNR}}}(\mathfrak{a}_1),\breve{\mathcal{T}}^2_{\mathcal{K}_{\mathcal{QNR}}}(\mathfrak{a}_1),....,\breve{\mathcal{T}}^{\mathfrak{p}}_{\mathcal{K}_{\mathcal{QNR}}}(\mathfrak{a}_1)),(\breve{\mathcal{D}}^1_{\mathcal{K}_{\mathcal{QNR}}}(\mathfrak{a}_1),\breve{\mathcal{D}}^2_{\mathcal{K}_{\mathcal{QNR}}}(\mathfrak{a}_1),....,\breve{\mathcal{D}}^{\mathfrak{p}}_{\mathcal{K}_{\mathcal{QNR}}}(\mathfrak{a}_1)),$$

$$(\breve{\mathcal{Y}}^1_{\mathcal{K}_{\mathcal{QNR}}}(\mathfrak{a}_1),\breve{\mathcal{Y}}^2_{\mathcal{K}_{\mathcal{QNR}}}(\mathfrak{a}_1),....,\breve{\mathcal{Y}}^{\mathfrak{p}}_{\mathcal{K}_{\mathcal{QNR}}}(\mathfrak{a}_1)),(\breve{\mathcal{F}}^1_{\mathcal{K}_{\mathcal{QNR}}}(\mathfrak{a}_1),\breve{\mathcal{F}}^2_{\mathcal{K}_{\mathcal{QNR}}}(\mathfrak{a}_1),....,\breve{\mathcal{F}}^{\mathfrak{p}}_{\mathcal{K}_{\mathcal{QNR}}}(\mathfrak{a}_1))\rangle : \mathfrak{a}_1 \in \mathcal{A}^*_{\mathcal{QR}}\}$$

where $\breve{\mathcal{T}}^1_{\mathcal{K}_{\mathcal{QNR}}}(\mathfrak{a}_1),\breve{\mathcal{T}}^2_{\mathcal{K}_{\mathcal{QNR}}}(\mathfrak{a}_1),....,\breve{\mathcal{T}}^{\mathfrak{p}}_{\mathcal{K}_{\mathcal{QNR}}}(\mathfrak{a}_1) : \mathcal{A}^*_{\mathcal{QR}} \to [0,1]$,
$\breve{\mathcal{D}}^1_{\mathcal{K}_{\mathcal{QNR}}}(\mathfrak{a}_1)$, $\breve{\mathcal{D}}^2_{\mathcal{K}_{\mathcal{QNR}}}(\mathfrak{a}_1),....,\breve{\mathcal{D}}^{\mathfrak{p}}_{\mathcal{K}_{\mathcal{QNR}}}(\mathfrak{a}_1) : \mathcal{A}^*_{\mathcal{QR}} \to [0,1]$,
$\breve{\mathcal{Y}}^1_{\mathcal{K}_{\mathcal{QNR}}}(\mathfrak{a}_1),\breve{\mathcal{Y}}^2_{\mathcal{K}_{\mathcal{QNR}}}(\mathfrak{a}_1),....,\breve{\mathcal{Y}}^{\mathfrak{p}}_{\mathcal{K}_{\mathcal{QNR}}}(\mathfrak{a}_1) : \mathcal{A}^*_{\mathcal{QR}} \to [0,1]$ and
$\breve{\mathcal{F}}^1_{\mathcal{K}_{\mathcal{QNR}}}(\mathfrak{a}_1),\breve{\mathcal{F}}^2_{\mathcal{K}_{\mathcal{QNR}}}(\mathfrak{a}_1),....,\breve{\mathcal{F}}^{\mathfrak{p}}_{\mathcal{K}_{\mathcal{QNR}}}(\mathfrak{a}_1) : \mathcal{A}^*_{\mathcal{QR}} \to [0,1]$ such that
$0 \leq \breve{\mathcal{T}}^{\mathfrak{j}}_{\mathcal{K}_{\mathcal{QNR}}}(\mathfrak{a}_1)+ \breve{\mathcal{D}}^{\mathfrak{j}}_{\mathcal{K}_{\mathcal{QNR}}}(\mathfrak{a}_1)+ \breve{\mathcal{Y}}^{\mathfrak{j}}_{\mathcal{K}_{\mathcal{QNR}}}(\mathfrak{a}_1)+ \breve{\mathcal{F}}^{\mathfrak{j}}_{\mathcal{K}_{\mathcal{QNR}}}(\mathfrak{a}_1) \leq 4$ ($\mathfrak{j}$=1,2,...$\mathfrak{p}$) for any $\mathfrak{a}_1 \in \mathcal{A}^*_{\mathcal{QR}}$.
Here $\breve{\mathcal{T}}^1_{\mathcal{K}_{\mathcal{QNR}}}(\mathfrak{a}_1),\breve{\mathcal{T}}^2_{\mathcal{K}_{\mathcal{QNR}}}(\mathfrak{a}_1),....,\breve{\mathcal{T}}^{\mathfrak{p}}_{\mathcal{K}_{\mathcal{QNR}}}(\mathfrak{a}_1),\breve{\mathcal{D}}^1_{\mathcal{K}_{\mathcal{QNR}}}(\mathfrak{a}_1),\breve{\mathcal{D}}^2_{\mathcal{K}_{\mathcal{QNR}}}(\mathfrak{a}_1),....,\breve{\mathcal{D}}^{\mathfrak{p}}_{\mathcal{K}_{\mathcal{QNR}}}(\mathfrak{a}_1)$,
$\breve{\mathcal{Y}}^1_{\mathcal{K}_{\mathcal{QNR}}}(\mathfrak{a}_1),\breve{\mathcal{Y}}^2_{\mathcal{K}_{\mathcal{QNR}}}(\mathfrak{a}_1),....,\breve{\mathcal{Y}}^{\mathfrak{p}}_{\mathcal{K}_{\mathcal{QNR}}}(\mathfrak{a}_1),\breve{\mathcal{F}}^1_{\mathcal{K}_{\mathcal{QNR}}}(\mathfrak{a}_1),\breve{\mathcal{F}}^2_{\mathcal{K}_{\mathcal{QNR}}}(\mathfrak{a}_1),....,\breve{\mathcal{F}}^{\mathfrak{p}}_{\mathcal{K}_{\mathcal{QNR}}}(\mathfrak{a}_1)$ is the truth ,a contradiction, an unknown and a falsity membership sequence of the element $\mathfrak{a}_1$, respectively.And $\mathfrak{p}$ is often referred to as $\mathcal{QSVNRS}(\mathcal{K}_{\mathcal{QNR}})$ dimension.

Definition 2.3. [14] Let $\mathcal{K}_{\mathcal{QNR}} = (\breve{\mathcal{T}}_{\mathcal{K}},\breve{\mathcal{F}}_{\mathcal{K}})$ be a collection of $\mathcal{IFN}$s and Let $\mathcal{IFHWA}:\mathcal{Q}^{\backslash} \to \mathcal{Q}$,if

$$\mathcal{IFHWA}_w = (\mathfrak{a}_1,\mathfrak{a}_2,....,\mathfrak{a}_n) = \bigoplus_{j=1}^{n} (w_j\mathfrak{a}_j)$$

where w = $(w_1,w_2,....,w_n)^T$ be the weight vector of $\mathfrak{a}_n$(j=1,2...$\mathfrak{n}$) and $w_j > 0, \sum_{j=1}^{n} w_j = 1$,then $\mathcal{IFHWA}$ is called the $\mathcal{IFHWA}$ operator.

3. **Hamacher Weighted Aggregation Operators of** $\mathcal{QSVNRN}$**s**

Definition 3.1. Let $k_1^* = \langle \breve{\mathfrak{T}}_1^j, \breve{\mathfrak{D}}_1^j, \breve{\mathfrak{y}}_1^j, \breve{\mathfrak{F}}_1^j : \mathrm{j} = 1,2,...\mathfrak{p}\rangle$ and $k_2^* = \langle \breve{\mathfrak{T}}_2^j, \breve{\mathfrak{D}}_2^j, \breve{\mathfrak{y}}_2^j, \breve{\mathfrak{F}}_2^j : \mathrm{j} = 1,2,...\mathfrak{p}\rangle$ be two $\mathcal{QSVNRN}$s and $\breve{\eta} > 0$ for any real number, then we define Hamacher $\mathbb{T}$-norm and Hamacher $\mathbb{T}$-conorm with $\breve{\beta} > 0$

i) $$k_1^* \oplus k_2^* = \left\langle \frac{\breve{\mathfrak{T}}_1^j + \breve{\mathfrak{T}}_2^j - \breve{\mathfrak{T}}_1^j\breve{\mathfrak{T}}_2^j - (1-\breve{\beta})\breve{\mathfrak{T}}_1^j\breve{\mathfrak{T}}_2^j}{1-(1-\breve{\beta})\breve{\mathfrak{T}}_1^j\breve{\mathfrak{T}}_2^j}, \frac{\breve{\mathfrak{D}}_1^j + \breve{\mathfrak{D}}_2^j - \breve{\mathfrak{D}}_1^j\breve{\mathfrak{D}}_2^j - (1-\breve{\beta})\breve{\mathfrak{D}}_1^j\breve{\mathfrak{D}}_2^j}{1-(1-\breve{\beta})\breve{\mathfrak{D}}_1^j\breve{\mathfrak{D}}_2^j}, \right.$$

$$\left. \frac{\breve{\mathfrak{y}}_1^j\breve{\mathfrak{y}}_2^j}{\breve{\beta}+(1-\breve{\beta})(\breve{\mathfrak{y}}_1^j + \breve{\mathfrak{y}}_2^j - \breve{\mathfrak{y}}_1^j\breve{\mathfrak{y}}_2^j}, \frac{\breve{\mathfrak{F}}_1^j\breve{\mathfrak{F}}_2^j}{\breve{\beta}+(1-\breve{\beta})(\breve{\mathfrak{F}}_1^j + \breve{\mathfrak{F}}_2^j - \breve{\mathfrak{F}}_1^j\breve{\mathfrak{F}}_2^j)} \right\rangle.$$

ii) $$k_1^* \otimes k_2^* = \left\langle \frac{\breve{\mathfrak{T}}_1^j\breve{\mathfrak{T}}_2^j}{\breve{\beta}+(1-\breve{\beta})(\breve{\mathfrak{T}}_1^j + \breve{\mathfrak{T}}_2^j - \breve{\mathfrak{T}}_1^j\breve{\mathfrak{T}}_2^j}, \frac{\breve{\mathfrak{D}}_1^j\breve{\mathfrak{D}}_2^j}{\breve{\beta}+(1-\breve{\beta})(\breve{\mathfrak{D}}_1^j + \breve{\mathfrak{D}}_2^j - \breve{\mathfrak{D}}_1^j\breve{\mathfrak{D}}_2^j}, \right.$$

$$\left. \frac{\breve{\mathfrak{y}}_1^j + \breve{\mathfrak{y}}_2^j - \breve{\mathfrak{y}}_1^j\breve{\mathfrak{y}}_2^j - (1-\breve{\beta})\breve{\mathfrak{y}}_1^j\breve{\mathfrak{y}}_2^j}{1-(1-\breve{\beta})\breve{\mathfrak{y}}_1^j\breve{\mathfrak{y}}_2^j}, \frac{\breve{\mathfrak{F}}_1^j + \breve{\mathfrak{F}}_2^j - \breve{\mathfrak{F}}_1^j\breve{\mathfrak{F}}_2^j - (1-\breve{\beta})\breve{\mathfrak{F}}_1^j\breve{\mathfrak{F}}_2^j}{1-(1-\breve{\beta})\breve{\mathfrak{F}}_1^j\breve{\mathfrak{F}}_2^j} \right\rangle.$$

iii) $$\breve{\eta}k_1^* = \left\langle \frac{(1+(\breve{\beta}-1)\breve{\mathfrak{T}}_1^j)^{\breve{\eta}}) - (1-\breve{\mathfrak{T}}_1^j)^{\breve{\eta}}}{(1+(\breve{\beta}-1)\breve{\mathfrak{T}}_1^j)^{\breve{\eta}} + (\breve{\beta}-1)(1-\breve{\mathfrak{T}}_1^j)^{\breve{\eta}}}, \frac{(1+(\breve{\beta}-1)\breve{\mathfrak{D}}_1^j)^{\breve{\eta}}) - (1-\breve{\mathfrak{D}}_1^j)^{\breve{\eta}}}{(1+(\breve{\beta}-1)\breve{\mathfrak{D}}_1^j)^{\breve{\eta}} + (\breve{\beta}-1)(1-\breve{\mathfrak{D}}_1^j)^{\breve{\eta}}}, \right.$$

$$\left. \frac{\breve{\beta}(\breve{\mathfrak{y}}_1^j)^{\breve{\eta}}}{(1+(\breve{\beta}-1)(1-\breve{\mathfrak{y}}_1^j))^{\breve{\eta}} + (\breve{\beta}-1)(\breve{\mathfrak{y}}_1^j)^{\breve{\eta}}}, \frac{\breve{\beta}(\breve{\mathfrak{F}}_1^j)^{\breve{\eta}}}{(1+(\breve{\beta}-1)(1-\breve{\mathfrak{F}}_1^j))^{\breve{\eta}} + (\breve{\beta}-1)(\breve{\mathfrak{F}}_1^j)^{\breve{\eta}}} \right\rangle.$$

iv) $$(k_1^*)^{\breve{\eta}} = \left\langle \frac{(\breve{\mathfrak{T}}_1^j)^{\breve{\eta}}}{(1+(\breve{\beta}-1)(1-\breve{\mathfrak{T}}_1^j))^{\breve{\eta}} + (\breve{\beta}-1)(\breve{\mathfrak{T}}_1^j)^{\breve{\eta}}}, \frac{(\breve{\mathfrak{D}}_1^j)^{\breve{\eta}}}{(1+(\breve{\beta}-1)(1-\breve{\mathfrak{D}}_1^j))^{\breve{\eta}} + (\breve{\beta}-1)(\breve{\mathfrak{D}}_1^j)^{\breve{\eta}}}, \right.$$

$$\left. \frac{(1+(\breve{\beta}-1)\breve{\mathfrak{y}}_1^j)^{\breve{\eta}} - (1-\breve{\mathfrak{y}}_1^j)^{\breve{\eta}}}{(1+(\breve{\beta}-1)\breve{\mathfrak{y}}_1^j)^{\breve{\eta}} + (\breve{\beta}-1)(1-\breve{\mathfrak{y}}_1^j)^{\breve{\eta}}}, \frac{(1+(\breve{\beta}-1)\breve{\mathfrak{F}}_1^j)^{\breve{\eta}} - (1-\breve{\mathfrak{F}}_1^j)^{\breve{\eta}}}{(1+(\breve{\beta}-1)\breve{\mathfrak{F}}_1^j)^{\breve{\eta}} + (\breve{\beta}-1)(1-\breve{\mathfrak{F}}_1^j)^{\breve{\eta}}} \right\rangle.$$

Example 3.2. Let two $\mathcal{QSVNRN}$s are $k_1^* = (\langle 0.5,0.3,0.2,0.2\rangle, \langle 0.5,0.6,0.3,0.3\rangle, \langle 0.4,0.7,0.2,0.3\rangle)$ and $k_2^* = (\langle 0.3,0.6,0.3,0.1\rangle, \langle 0.5,0.5,0.2,0.3\rangle, \langle 0.6,0.9,0.3,0.3\rangle$ and $\breve{\eta} = 2$.Then for $\breve{\beta} = 3$.

$k_1^* \oplus k_2^* = \langle 0.949,0.919,0.500,0.455\rangle$

$k_1^* \otimes k_2^* = \langle 0.371,0.246,0.972,0.969\rangle$

$\breve{\eta}k_1^* = \langle 0.941,0.821,0.381,0.280\rangle$
$(k_1^*)^{\breve{\eta}} = \langle 0.496,0.381,0.986,0.931\rangle$

Proposition 3.3. Let k_1^* and k_2^* be two $\mathcal{QSVNRN}$s and $\breve{\beta},\breve{\beta}' \geq 0$.

a $k_1^* \oplus k_2^* = k_2^* \oplus k_1^*$.
b $k_1^* \otimes k_2^* = k_2^* \otimes k_1^*$.
c $\breve{\beta}(k_1^* \oplus k_2^*) = \breve{\beta}\ k_1^* \oplus \breve{\beta}\ k_2^*$.
d $\breve{\beta}\ k_1^* \oplus \breve{\beta}'\ k_1^* = (\breve{\beta}+\breve{\beta}')k_1^*$.
e $(k_1^* \otimes k_2^*)^{\breve{\beta}} = (k_1^*)^{\breve{\beta}} \otimes (k_2^*)^{\breve{\beta}}$.
f $(k_1^*)^{\breve{\beta}} \otimes (k_1^*)^{\breve{\beta}'} = (k_1^*)^{\breve{\beta}+\breve{\beta}'}$

Proof. They are easily seen from the formulas in Definition 3.1, hence omitted □

Definition 3.4. Let $k_r^* = \langle\ \breve{\mathfrak{T}}_r^j,\ \breve{\mathfrak{D}}_r^j,\ \breve{\mathfrak{y}}_r^j,\breve{\mathfrak{F}}_r^j\ :\ \mathrm{j} = 1,2,...,\mathfrak{p}\rangle$ (r = 1,2,...,$\mathfrak{n}$) is a collection of $\mathcal{QSVNRN}$s.Then the Quadripartitioned single valued neutrosophic refined hamacher weighted average operator($\mathcal{QSVNRHWA}$) can be defined as follows:

$$\mathcal{QSVNRHWA}_{\ddot{\varpi}}(k_1^*,k_2^*,...,k_{\mathfrak{n}}^*) = \bigoplus_{r=1}^{\mathfrak{n}}(\ddot{\varpi}_r k_r^*).$$

where $\ddot{\varpi} = (\ddot{\varpi}_1,\ \ddot{\varpi}_2,....,\ddot{\varpi}_{\mathfrak{n}})$be a weighted vector of k_r^* such that $\ddot{\varpi}_r \in [0,1]$ and $\sum_{r=1}^{\mathfrak{n}}(\ddot{\varpi}_r) = 1$.

Theorem 3.5. Let $k_r^* = \langle\ \breve{\mathfrak{T}}_r^j,\ \breve{\mathfrak{D}}_r^j,\ \breve{\mathfrak{y}}_r^j,\breve{\mathfrak{F}}_r^j\ :\ \mathrm{j} = 1,2,...,\mathfrak{p}\rangle$ (r = 1,2,...,$\mathfrak{n}$) is a collection of $\mathcal{QSVNRN}$s then their aggregated value of $\mathcal{QSVNRHWA}$ operator is again a $\mathcal{QSVNRN}$s can be defined as follows

$$\mathcal{QSVNRHWA}_{\ddot{\varpi}}(k_1^*,k_2^*,...,k_{\mathfrak{n}}^*) = \bigoplus_{r=1}^{\mathfrak{n}}{}_{\mathcal{H}}(\ddot{\varpi}_r k_r^*)$$

$$= \left\langle \frac{\prod_{r=1}^{\mathfrak{n}}(1+(\breve{\beta}-1)\breve{\mathfrak{T}}_r^j)^{\ddot{\varpi}_r}-\prod_{r=1}^{\mathfrak{n}}(1-\breve{\mathfrak{T}}_r^j)^{\ddot{\varpi}_r}}{\prod_{r=1}^{\mathfrak{n}}(1+(\breve{\beta}-1)\breve{\mathfrak{T}}_r^j)^{\ddot{\varpi}_r}+(\breve{\beta}-1)\prod_{r=1}^{\mathfrak{n}}(1-\breve{\mathfrak{T}}_r^j)^{\ddot{\varpi}_r}}, \frac{\prod_{r=1}^{\mathfrak{n}}(1+(\breve{\beta}-1)\breve{\mathfrak{D}}_r^j)^{\ddot{\varpi}_r}-\prod_{r=1}^{\mathfrak{n}}(1-\breve{\mathfrak{D}}_r^j)^{\ddot{\varpi}_r}}{\prod_{r=1}^{\mathfrak{n}}(1+(\breve{\beta}-1)\breve{\mathfrak{D}}_r^j)^{\ddot{\varpi}_r}+(\breve{\beta}-1)\prod_{r=1}^{\mathfrak{n}}(1-\breve{\mathfrak{D}}_r^j)^{\ddot{\varpi}_r}}, \right.$$

$$\left. \frac{\breve{\beta}\prod_{r=1}^{\mathfrak{n}}(\breve{\mathfrak{y}}_r^j)^{\ddot{\varpi}_r}}{\prod_{r=1}^{\mathfrak{n}}(1+(\breve{\beta}-1)(1-\breve{\mathfrak{y}}_r^j))^{\ddot{\varpi}_r}+(\breve{\beta}-1)\prod_{r=1}^{\mathfrak{n}}(\breve{\mathfrak{y}}_r^j)^{\ddot{\varpi}_r}}, \frac{\breve{\beta}\prod_{r=1}^{\mathfrak{n}}(\breve{\mathfrak{F}}_r^j)^{\ddot{\varpi}_r}}{\prod_{r=1}^{\mathfrak{n}}(1+(\breve{\beta}-1)(1-\breve{\mathfrak{F}}_r^j))^{\ddot{\varpi}_r}+(\breve{\beta}-1)\prod_{r=1}^{\mathfrak{n}}(\breve{\mathfrak{F}}_r^j)^{\ddot{\varpi}_r}} \right\rangle$$

.....(5.2.1)

where $\ddot{\varpi} = (\ddot{\varpi}_1,\ \ddot{\varpi}_2,....,\ddot{\varpi}_{\mathfrak{n}})$ be a weighted vector of k_r^* such that $\ddot{\varpi}_r \in [0,1]$ and $\sum_{r=1}^{\mathfrak{n}}(\ddot{\varpi}_r) = 1$, $\breve{\beta} > 0$.

Proof. we can prove this theorem by mathematical induction.

For $\mathfrak{n}$=1,by Eq(5.2.1),we get

$\mathcal{QSVNRHWA}_{\ddot{\varpi}}(k_1^*) = (\ddot{\varpi}_1 k_1^*) = k_1^*.$

$$= \left\langle \frac{(1+(\breve{\beta}-1)\breve{\mathfrak{T}}_1^j)-(1-\breve{\mathfrak{T}}_1^j)}{(1+(\breve{\beta}-1)\breve{\mathfrak{T}}_1^j)+(\breve{\beta}-1)(1-\breve{\mathfrak{T}}_1^j)}, \frac{(1+(\breve{\beta}-1)\breve{\mathfrak{D}}_1^j)-(1-\breve{\mathfrak{D}}_1^j)}{(1+(\breve{\eta}-1)\breve{\mathfrak{D}}_1^j)+(\breve{\beta}-1)(1-\breve{\mathfrak{D}}_1^j)}, \right.$$

$$\left. \frac{\breve{\beta}(\breve{\mathfrak{y}}_1^j)}{(1+(\breve{\beta}-1)(1-\breve{\mathfrak{y}}_1^j))+(\breve{\beta}-1)(\breve{\mathfrak{y}}_1^j)}, \frac{\breve{\beta}(\breve{\mathfrak{F}}_1^j)}{(1+(\breve{\beta}-1)(1-\breve{\mathfrak{F}}_1^j))+(\breve{\beta}-1)(\breve{\mathfrak{F}}_1^j)} \right\rangle$$

Thus,satisfied for $\mathfrak{n}$=1.

put $\mathfrak{n} = \upsilon$ for Eq(5.2.1),then

$$\mathcal{QSVNRHWA}_{\ddot{\varpi}}(k_1^*,k_2^*,...,k_\upsilon^*) = \bigoplus_{r=1}^{\upsilon} {}_{\mathcal{H}}(\ddot{\varpi}_r k_r^*)$$

$$= \left\langle \frac{\prod_{r=1}^{\upsilon}(1+(\breve{\beta}-1)\breve{\mathfrak{T}}_r^j)^{\ddot{\varpi}_r}-\prod_{r=1}^{\upsilon}(1-\breve{\mathfrak{T}}_r^j)^{\ddot{\varpi}_r}}{\prod_{r=1}^{\upsilon}(1+(\breve{\beta}-1)\breve{\mathfrak{T}}_r^j)^{\ddot{\varpi}_r}+(\breve{\beta}-1)\prod_{r=1}^{\upsilon}(1-\breve{\mathfrak{T}}_1^j)^{\ddot{\varpi}_r}}, \frac{\prod_{r=1}^{\upsilon}(1+(\breve{\beta}-1)\breve{\mathfrak{D}}_r^j)^{\ddot{\varpi}_r}-\prod_{r=1}^{\upsilon}(1-\breve{\mathfrak{D}}_r^j)^{\ddot{\varpi}_r}}{\prod_{r=1}^{\upsilon}(1+(\breve{\beta}-1)\breve{\mathfrak{D}}_r^j)^{\ddot{\varpi}_r}+(\breve{\beta}-1)\prod_{r=1}^{\upsilon}(1-\breve{\mathfrak{D}}_r^j)^{\ddot{\varpi}_r}}, \right.$$

$$\left. \frac{\breve{\beta}\prod_{r=1}^{\upsilon}(\breve{\mathfrak{y}}_r^j)^{\ddot{\varpi}_r}}{\prod_{r=1}^{\upsilon}(1+(\breve{\beta}-1)(1-\breve{\mathfrak{y}}_r^j))^{\ddot{\varpi}_r}+(\breve{\beta}-1)\prod_{r=1}^{\upsilon}(\breve{\mathfrak{y}}_r^j)^{\ddot{\varpi}_r}}, \frac{\breve{\beta}\prod_{r=1}^{\upsilon}(\breve{\mathfrak{F}}_r^j)^{\ddot{\varpi}_r}}{\prod_{r=1}^{\upsilon}(1+(\breve{\beta}-1)(1-\breve{\mathfrak{F}}_r^j))^{\ddot{\varpi}_r}+(\breve{\beta}-1)\prod_{r=1}^{\upsilon}(\breve{\mathfrak{F}}_r^j)^{\ddot{\varpi}_r}} \right\rangle$$

If Eq(5.2.1) is true for $\mathfrak{n} = \upsilon$.

For $\mathfrak{n} = \upsilon$+1,then

$$\mathcal{QSVNRHWA}_{\ddot{\varpi}}(k_1^*,k_2^*,...,k_{\upsilon+1}^*) = \bigoplus_{r=1}^{\upsilon+1} {}_{\mathcal{H}}(\ddot{\varpi}_r k_r^*) \oplus_{\mathcal{H}} (\ddot{\varpi}_{\upsilon+1} k_{\upsilon+1}^*)$$

$$= \left\langle \frac{\prod_{r=1}^{\upsilon}(1+(\breve{\beta}-1)\breve{\mathfrak{T}}_r^j)^{\ddot{\varpi}_r}-\prod_{r=1}^{\upsilon}(1-\breve{\mathfrak{T}}_r^j)^{\ddot{\varpi}_r}}{\prod_{r=1}^{\upsilon}(1+(\breve{\beta}-1)\breve{\mathfrak{T}}_r^j)^{\ddot{\varpi}_r}+(\breve{\beta}-1)\prod_{r=1}^{\upsilon}(1-\breve{\mathfrak{T}}_1^j)^{\ddot{\varpi}_r}}, \frac{\prod_{r=1}^{\upsilon}(1+(\breve{\beta}-1)\breve{\mathfrak{D}}_r^j)^{\ddot{\varpi}_r}-\prod_{r=1}^{\upsilon}(1-\breve{\mathfrak{D}}_r^j)^{\ddot{\varpi}_r}}{\prod_{r=1}^{\upsilon}(1+(\breve{\beta}-1)\breve{\mathfrak{D}}_r^j)^{\ddot{\varpi}_r}+(\breve{\beta}-1)\prod_{r=1}^{\upsilon}(1-\breve{\mathfrak{D}}_r^j)^{\ddot{\varpi}_r}}, \right.$$

$$\left. \frac{\breve{\beta}\prod_{r=1}^{\upsilon}(\breve{\mathfrak{y}}_r^j)^{\ddot{\varpi}_r}}{\prod_{r=1}^{\upsilon}(1+(\breve{\beta}-1)(1-\breve{\mathfrak{y}}_r^j))^{\ddot{\varpi}_r}+(\breve{\beta}-1)\prod_{r=1}^{\upsilon}(\breve{\mathfrak{y}}_r^j)^{\ddot{\varpi}_r}}, \frac{\breve{\beta}\prod_{r=1}^{\upsilon}(\breve{\mathfrak{F}}_r^j)^{\ddot{\varpi}_r}}{\prod_{r=1}^{\upsilon}(1+(\breve{\beta}-1)(1-\breve{\mathfrak{F}}_r^j))^{\ddot{\varpi}_r}+(\breve{\beta}-1)\prod_{r=1}^{\upsilon}(\breve{\mathfrak{F}}_r^j)^{\ddot{\varpi}_r}} \right\rangle$$

$$\oplus_{\mathcal{H}} \left\langle \frac{(1+(\breve{\beta}-1)\breve{\mathfrak{T}}_{\upsilon+1}^j)-(1-\breve{\mathfrak{T}}_{\upsilon+1}^j)}{(1+(\breve{\beta}-1)\breve{\mathfrak{T}}_{\upsilon+1}^j)+(\breve{\beta}-1)(1-\breve{\mathfrak{T}}_{\upsilon+1}^j)}, \frac{(1+(\breve{\beta}-1)\breve{\mathfrak{D}}_{\upsilon+1}^j)-(1-\breve{\mathfrak{D}}_{\upsilon+1}^j)}{(1+(\breve{\eta}-1)\breve{\mathfrak{D}}_{\upsilon+1}^j)+(\breve{\beta}-1)(1-\breve{\mathfrak{D}}_{\upsilon+1}^j)}, \right.$$

$$\left. \frac{\breve{\beta}(\breve{\mathfrak{y}}_{\upsilon+1}^j)}{(1+(\breve{\beta}-1)(1-\breve{\mathfrak{y}}_{\upsilon+1}^j))+(\breve{\beta}-1)(\breve{\mathfrak{y}}_{\upsilon+1}^j)}, \frac{\breve{\beta}(\breve{\mathfrak{F}}_{\upsilon+1}^j)}{(1+(\breve{\beta}-1)(1-\breve{\mathfrak{F}}_{\upsilon+1}^j))+(\breve{\beta}-1)(\breve{\mathfrak{F}}_{\upsilon+1}^j)} \right\rangle$$

$$= \left\langle \frac{\prod_{r=1}^{\upsilon+1}(1+(\breve{\beta}-1)\breve{\mathfrak{T}}_r^j)^{\ddot{\varpi}_r}-\prod_{r=1}^{\upsilon+1}(1-\breve{\mathfrak{T}}_r^j)^{\ddot{\varpi}_r}}{\prod_{r=1}^{\upsilon+1}(1+(\breve{\beta}-1)\breve{\mathfrak{T}}_r^j)^{\ddot{\varpi}_r}+(\breve{\beta}-1)\prod_{r=1}^{\upsilon+1}(1-\breve{\mathcal{I}}_1^j)^{\ddot{\varpi}_r}}, \frac{\prod_{r=1}^{\upsilon+1}(1+(\breve{\beta}-1)\breve{\mathcal{D}}_r^j)^{\ddot{\varpi}_r}-\prod_{r=1}^{\upsilon+1}(1-\breve{\mathcal{D}}_r^j)^{\ddot{\varpi}_r}}{\prod_{r=1}^{\upsilon+1}(1+(\breve{\beta}-1)\breve{\mathcal{D}}_r^j)^{\ddot{\varpi}_r}+(\breve{\beta}-1)\prod_{r=1}^{\upsilon+1}(1-\breve{\mathcal{D}}_r^j)^{\ddot{\varpi}_r}}, \right.$$

$$\left. \frac{\breve{\beta}\prod_{r=1}^{\upsilon+1}(\breve{\mathcal{Y}}_r^j)^{\ddot{\varpi}_r}}{\prod_{r=1}^{\upsilon+1}(1+(\breve{\beta}-1)(1-\breve{\mathcal{Y}}_r^j))^{\ddot{\varpi}_r}+(\breve{\beta}-1)\prod_{r=1}^{\upsilon+1}(\breve{\mathcal{Y}}_r^j)^{\ddot{\varpi}_r}}, \frac{\breve{\beta}\prod_{r=1}^{\upsilon+1}(\breve{\mathcal{F}}_r^j)^{\ddot{\varpi}_r}}{\prod_{r=1}^{\upsilon+1}(1+(\breve{\beta}-1)(1-\breve{\mathcal{F}}_r^j))^{\ddot{\varpi}_r}+(\breve{\beta}-1)\prod_{r=1}^{\upsilon+1}(\breve{\mathcal{F}}_r^j)^{\ddot{\varpi}_r}} \right\rangle$$

Thus,Eq(5.2.1) is true for $\mathfrak{n} = \upsilon+1$.Therefore,Eq(5.2.1) is satisfied for every $\mathfrak{n}$. □

Theorem 3.6. (Idempotency)

Let k_r^* (r=1,2,...,$\mathfrak{n}$) is a collection of $\mathcal{QSVNRN}$s.If $k_r^* = k^*$ for all (r=1,2,...,$\mathfrak{n}$) then $\mathcal{QSVNRHWA}_{\ddot{\varpi}}(k_1^*,k_2^*,....,k_{\mathfrak{n}}^*) = k^*$.

Proof. Assume $k_r^* = k^*$ for all (r=1,2,...,$\mathfrak{n}$).

By Theorem (4.2) we obtain that

$\mathcal{QSVNRHWA}_{\ddot{\varpi}}(k_1^*,k_2^*,...,k_{\mathfrak{n}}^*) = \bigoplus_{r=1}^{\mathfrak{n}}{}_{\mathcal{H}}(\ddot{\varpi}_r k_r^*) = \bigoplus_{r=1}^{\mathfrak{n}}{}_{\mathcal{H}}(\ddot{\varpi}_r k^*) =$

$$\left\langle \frac{\prod_{r=1}^{\mathfrak{n}}(1+(\breve{\beta}-1)\breve{\mathfrak{T}}^j)^{\ddot{\varpi}_r}-\prod_{r=1}^{\mathfrak{n}}(1-\breve{\mathfrak{T}}^j)^{\ddot{\varpi}_r}}{\prod_{r=1}^{\mathfrak{n}}(1+(\breve{\beta}-1)\breve{\mathfrak{T}}^j)^{\ddot{\varpi}_r}+(\breve{\beta}-1)\prod_{r=1}^{\mathfrak{n}}(1-\breve{\mathcal{I}}_1^j)^{\ddot{\varpi}_r}}, \frac{\prod_{r=1}^{\mathfrak{n}}(1+(\breve{\beta}-1)\breve{\mathcal{D}}^j)^{\ddot{\varpi}_r}-\prod_{r=1}^{\mathfrak{n}}(1-\breve{\mathcal{D}}^j)^{\ddot{\varpi}_r}}{\prod_{r=1}^{\mathfrak{n}}(1+(\breve{\beta}-1)\breve{\mathcal{D}}^j)^{\ddot{\varpi}_r}+(\breve{\beta}-1)\prod_{r=1}^{\mathfrak{n}}(1-\breve{\mathcal{D}}^j)^{\ddot{\varpi}_r}}, \right.$$

$$\left. \frac{\breve{\beta}\prod_{r=1}^{\mathfrak{n}}(\breve{\mathcal{Y}}^j)^{\ddot{\varpi}_r}}{\prod_{r=1}^{\mathfrak{n}}(1+(\breve{\beta}-1)(1-\breve{\mathcal{Y}}^j))^{\ddot{\varpi}_r}+(\breve{\beta}-1)\prod_{r=1}^{\mathfrak{n}}(\breve{\mathcal{Y}}^j)^{\ddot{\varpi}_r}}, \frac{\breve{\beta}\prod_{r=1}^{\mathfrak{n}}(\breve{\mathcal{F}}^j)^{\ddot{\varpi}_r}}{\prod_{r=1}^{\mathfrak{n}}(1+(\breve{\beta}-1)(1-\breve{\mathcal{F}}^j))^{\ddot{\varpi}_r}+(\breve{\beta}-1)\prod_{r=1}^{\mathfrak{n}}(\breve{\mathcal{F}}^j)^{\ddot{\varpi}_r}} \right\rangle = k^* \ \square$$

Theorem 3.7. (Monotonicity)

Let k_r^* and $k_r^{*'}$ (r=1,2,...,$\mathfrak{n}$) be two collections of $\mathcal{QSVNRN}$s.If $k_r^* \leq k_r^{*'}$ for all r=1,2,...,$\mathfrak{n}$ then $\mathcal{QSVNRHWA}_{\ddot{\varpi}}(k_1^*,k_2^*,....,k_{\mathfrak{n}}^*) \leq \mathcal{QSVNRHWA}_{\ddot{\varpi}}(k_1^{*'},k_2^{*'},....,k_{\mathfrak{n}}^{*'})$.

Proof. If $k_r^* \leq k_r^{*'}$ then we have

$\breve{\mathfrak{T}}_r^j \leq \breve{\mathfrak{T}}_r^{j'}$, $\breve{\mathcal{D}}_r^j \leq \breve{\mathcal{D}}_r^{j'}$,$\breve{\mathcal{Y}}_r^j \leq \breve{\mathcal{Y}}_r^{j'}$,$\breve{\mathcal{F}}_r^j \leq \breve{\mathcal{F}}_r^{j'}$ for all r=1,2,...,$\mathfrak{n}$.

With these assumptions,we find that

$$\left\langle \frac{\prod_{r=1}^{\mathfrak{n}}(1+(\breve{\beta}-1)\breve{\mathfrak{T}}_r^j)^{\ddot{\varpi}_r}-\prod_{r=1}^{\mathfrak{n}}(1-\breve{\mathfrak{T}}_r^j)^{\ddot{\varpi}_r}}{\prod_{r=1}^{\mathfrak{n}}(1+(\breve{\beta}-1)\breve{\mathfrak{T}}_r^j)^{\ddot{\varpi}_r}+(\breve{\beta}-1)\prod_{r=1}^{\mathfrak{n}}(1-\breve{\mathcal{I}}_1^j)^{\ddot{\varpi}_r}}, \frac{\prod_{r=1}^{\mathfrak{n}}(1+(\breve{\beta}-1)\breve{\mathcal{D}}_r^j)^{\ddot{\varpi}_r}-\prod_{r=1}^{\mathfrak{n}}(1-\breve{\mathcal{D}}_r^j)^{\ddot{\varpi}_r}}{\prod_{r=1}^{\mathfrak{n}}(1+(\breve{\beta}-1)\breve{\mathcal{D}}_r^j)^{\ddot{\varpi}_r}+(\breve{\beta}-1)\prod_{r=1}^{\mathfrak{n}}(1-\breve{\mathcal{D}}_r^j)^{\ddot{\varpi}_r}}, \right.$$

$$\left. \frac{\breve{\beta}\prod_{r=1}^{\mathfrak{n}}(\breve{\mathcal{Y}}_r^j)^{\ddot{\varpi}_r}}{\prod_{r=1}^{\mathfrak{n}}(1+(\breve{\beta}-1)(1-\breve{\mathcal{Y}}_r^j))^{\ddot{\varpi}_r}+(\breve{\beta}-1)\prod_{r=1}^{\mathfrak{n}}(\breve{\mathcal{Y}}_r^j)^{\ddot{\varpi}_r}}, \frac{\breve{\beta}\prod_{r=1}^{\mathfrak{n}}(\breve{\mathcal{F}}_r^j)^{\ddot{\varpi}_r}}{\prod_{r=1}^{\mathfrak{n}}(1+(\breve{\beta}-1)(1-\breve{\mathcal{F}}_r^j))^{\ddot{\varpi}_r}+(\breve{\beta}-1)\prod_{r=1}^{\mathfrak{n}}(\breve{\mathcal{F}}_r^j)^{\ddot{\varpi}_r}} \right\rangle$$

$$\leq \left\langle \frac{\prod_{r=1}^{\mathfrak{n}}(1+(\breve{\beta}-1)\breve{\mathfrak{T}}_r^{j'})^{\ddot{\varpi}_r}-\prod_{r=1}^{\mathfrak{n}}(1-\breve{\mathfrak{T}}_r^{j'})^{\ddot{\varpi}_r}}{\prod_{r=1}^{\mathfrak{n}}(1+(\breve{\beta}-1)\breve{\mathfrak{T}}_r^{j'})^{\ddot{\varpi}_r}+(\breve{\beta}-1)\prod_{r=1}^{\mathfrak{n}}(1-\breve{\mathcal{I}}_1^j)^{\ddot{\varpi}_r}}, \frac{\prod_{r=1}^{\mathfrak{n}}(1+(\breve{\beta}-1)\breve{\mathcal{D}}_r^{j'})^{\ddot{\varpi}_r}-\prod_{r=1}^{\mathfrak{n}}(1-\breve{\mathcal{D}}_r^{j'})^{\ddot{\varpi}_r}}{\prod_{r=1}^{\mathfrak{n}}(1+(\breve{\beta}-1)\breve{\mathcal{D}}_r^{j'})^{\ddot{\varpi}_r}+(\breve{\beta}-1)\prod_{r=1}^{\mathfrak{n}}(1-\breve{\mathcal{D}}_r^{j'})^{\ddot{\varpi}_r}}, \right.$$

$$\left. \frac{\breve{\beta}\prod_{r=1}^{\mathfrak{n}}(\breve{\mathcal{Y}}_r^{j'})^{\ddot{\varpi}_r}}{\prod_{r=1}^{\mathfrak{n}}(1+(\breve{\beta}-1)(1-\breve{\mathcal{Y}}_r^{j'}))^{\ddot{\varpi}_r}+(\breve{\beta}-1)\prod_{r=1}^{\mathfrak{n}}(\breve{\mathcal{Y}}_r^{j'})^{\ddot{\varpi}_r}}, \frac{\breve{\beta}\prod_{r=1}^{\mathfrak{n}}(\breve{\mathcal{F}}_r^{j'})^{\ddot{\varpi}_r}}{\prod_{r=1}^{\mathfrak{n}}(1+(\breve{\beta}-1)(1-\breve{\mathcal{F}}_r^{j'}))^{\ddot{\varpi}_r}+(\breve{\beta}-1)\prod_{r=1}^{\mathfrak{n}}(\breve{\mathcal{F}}_r^{j'})^{\ddot{\varpi}_r}} \right\rangle$$

Then $\bigoplus_{r=1}^{\mathfrak{n}}{}_{\mathcal{H}}(\ddot{\varpi}_r k_r^*) = \bigoplus_{r=1}^{\mathfrak{n}}{}_{\mathcal{H}}(\ddot{\varpi}_r k^{*'})$,so $\mathcal{QSVNRHWA}_{\ddot{\varpi}}(k_1^*,k_2^*,....,k_{\mathfrak{n}}^*) \leq \mathcal{QSVNRHWA}_{\ddot{\varpi}}(k_1^{*'},k_2^{*'},....,k_{\mathfrak{n}}^{*'})$. □

Theorem 3.8. (Boundedness)
Let k_r^* (r=1,2,...,$\mathfrak{n}$) be a collection of $\mathcal{QSVNRN}$s.Then $k_{min}^* \leq \mathcal{QSVNRHWA}_{\ddot{\varpi}}(k_1^*,k_2^*,....,k_{\mathfrak{n}}^*) \leq k_{max}^*$.

Proof. $k_{min}^* = \langle \min(\breve{\mathfrak{T}}_r^j), \min(\breve{\mathcal{D}}_r^j), \max(\breve{\mathcal{Y}}_r^j), \max(\breve{\mathcal{F}}_r^j)$: r=1,2,...,$\mathfrak{n}$ $\rangle$
and $k_{max}^* = \langle \max(\breve{\mathfrak{T}}_r^j), \max(\breve{\mathcal{D}}_r^j), \min(\breve{\mathcal{Y}}_r^j), \min(\breve{\mathcal{F}}_r^j)$: r=1,2,...,$\mathfrak{n}$ $\rangle$
They can be proved using similar techniques, therefore omitted. □

4. TOPSIS method for MADM problems using $\mathcal{QSVNRHWA}$

One of the conventional methods for handling MADM problems is TOPSIS. It is used to rank the practical options in order of priority and select the best option based on the available data. Here, we will demonstrate how to solve the MADM problem using the TOPSIS technique, which is based on the $\mathcal{QSVNRHWA}$ operator.
Let us consider a collection of evaluation attributes $\hat{\Omega} = \{\hat{\Omega}_1,\hat{\Omega}_2,...,\hat{\Omega}_{\mathfrak{n}}\}$, $\hat{\zeta} = \{\hat{\zeta}_1,\hat{\zeta}_2,...,\hat{\zeta}_{\mathfrak{m}}\}$ be the set of feasible Preferences(Alternatives) and $\ddot{\kappa}_{\alpha\beta}$, α = 1,2,...,$\mathfrak{m}$; β = 1,2,...,$\mathfrak{n}$ is the score of preferencei $\hat{\zeta}_\alpha$ with respect to attributes $\hat{\Omega}_\beta$.Let $\ddot{\varpi} = (\ddot{\varpi}_1, \ddot{\varpi}_2,....,\ddot{\varpi}_{\mathfrak{q}})$ be a weighted vector satisfying $0 \leq \ddot{\varpi}_\beta \leq 1$ and $\sum_{\beta=1}^{\mathfrak{n}}(\ddot{\varpi}_\beta) = 1$.
TOPSIS approach is summarized as follows:
Step 1: Computation of normalized decision matrix.
Formulate the normalized value $\breve{\omega}_{\alpha\beta}$ is a follows:

For the profit matrix, $\breve{\omega}_{\alpha\beta}^N = \dfrac{\breve{\omega}_{\alpha\beta} - \breve{\omega}_\beta^-}{\breve{\omega}_\beta^+ - \breve{\omega}_\beta^-}$

where $\breve{\omega}_\beta^+ = \max_\beta(\breve{\omega}_{\alpha\beta})$ and $\breve{\omega}_\beta^- = \min_\beta(\breve{\omega}_{\alpha\beta})$or setting $\breve{\omega}_\beta^+$ is the best position and $\breve{\omega}_\beta^-$ is the worst position.

For the cost matrix, $\breve{\omega}_{\alpha\beta}^N = \dfrac{\breve{\omega}_\beta^- - \breve{\omega}_{\alpha\beta}}{\breve{\omega}_\beta^+ - \breve{\omega}_\beta^-}$.

Step 2: Compute the $\mathcal{QSVNRN}$ for each alternative $\hat{\zeta}_\alpha$ (α = 1,2,...,$\mathfrak{m}$) by using the opertor $\mathcal{QSVNRHWA}$.

$$\mathcal{QSVNRHWA}_{\ddot{\varpi}}(k_1^*,k_2^*,...,k_{\mathfrak{n}}^*) = \bigoplus_{r=1}^{\mathfrak{n}}(\ddot{\varpi}_r k_r^*)$$

$$= \left\langle \frac{\prod_{r=1}^{\mathfrak{n}}(1+(\breve{\beta}-1)\breve{\mathfrak{T}}_r^j)^{\ddot{\varpi}_r} - \prod_{r=1}^{\mathfrak{n}}(1-\breve{\mathfrak{T}}_r^j)^{\ddot{\varpi}_r}}{\prod_{r=1}^{\mathfrak{n}}(1+(\breve{\beta}-1)\breve{\mathfrak{T}}_r^j)^{\ddot{\varpi}_r} + (\breve{\beta}-1)\prod_{r=1}^{\mathfrak{n}}(1-\breve{\mathfrak{T}}_1^j)^{\ddot{\varpi}_r}}, \frac{\prod_{r=1}^{\mathfrak{n}}(1+(\breve{\beta}-1)\breve{\mathcal{D}}_r^j)^{\ddot{\varpi}_r} - \prod_{r=1}^{\mathfrak{n}}(1-\breve{\mathcal{D}}_r^j)^{\ddot{\varpi}_r}}{\prod_{r=1}^{\mathfrak{n}}(1+(\breve{\beta}-1)\breve{\mathcal{D}}_r^j)^{\ddot{\varpi}_r} + (\breve{\beta}-1)\prod_{r=1}^{\mathfrak{n}}(1-\breve{\mathcal{D}}_r^j)^{\ddot{\varpi}_r}}, \right.$$

$$\left. \frac{\breve{\beta}\prod_{r=1}^{\mathfrak{n}}(\breve{\mathcal{Y}}_r^j)^{\ddot{\varpi}_r}}{\prod_{r=1}^{\mathfrak{n}}(1+(\breve{\beta}-1)(1-\breve{\mathcal{Y}}_r^j))^{\ddot{\varpi}_r} + (\breve{\beta}-1)\prod_{r=1}^{\mathfrak{n}}(\breve{\mathcal{Y}}_r^j)^{\ddot{\varpi}_r}}, \frac{\breve{\beta}\prod_{r=1}^{\mathfrak{n}}(\breve{\mathcal{F}}_r^j)^{\ddot{\varpi}_r}}{\prod_{r=1}^{\mathfrak{n}}(1+(\breve{\beta}-1)(1-\breve{\mathcal{F}}_r^j))^{\ddot{\varpi}_r} + (\breve{\beta}-1)\prod_{r=1}^{\mathfrak{n}}(\breve{\mathcal{F}}_r^j)^{\ddot{\varpi}_r}} \right\rangle$$

where $\ddot{\varpi} = (\ddot{\varpi}_1, \ddot{\varpi}_2,....,\ddot{\varpi}_{\mathfrak{n}})$ be a weighted vector of k_r^* such that $\ddot{\varpi}_r \in [0,1]$ and $\sum_{r=1}^{\mathfrak{n}}(\ddot{\varpi}_r) = 1$, $\breve{\beta} > 0$.

Step 3: Determine the quadripartitioned single valued neutrosophic refined positive ideal solution ($\mathcal{QSVNRPIS}$) and quadripartitioned single valued neutrosophic refined negative ideal solutions ($\mathcal{QSVNRNIS}$).

$$\mathcal{QSVNRPIS} = \ddot{\mathfrak{D}}_\beta^+ = (\ddot{\varepsilon}_1^+, \ddot{\varepsilon}_2^+,....,\ddot{\varepsilon}_{\mathfrak{n}}^+)$$

$$= \left\langle \max_\beta(\breve{\mathcal{T}}_{\alpha\beta}), \max_\beta(\breve{\mathcal{D}}_{\alpha\beta}), \min_\beta(\breve{\mathcal{Y}}_{\alpha\beta}), \min_\beta(\breve{\mathcal{F}}_{\alpha\beta}) \right\rangle \ , \ \alpha = 1,2,...,\mathfrak{m} \ ; \ \beta = 1,2,...,\mathfrak{n}.$$

$$\mathcal{QSVNRNIS} = \ddot{\mathfrak{D}}_\beta^- = (\ddot{\varepsilon}_1^-, \ddot{\varepsilon}_2^-,....,\ddot{\varepsilon}_{\mathfrak{n}}^-)$$

$$= \left\langle \min_\beta(\breve{\mathcal{T}}_{\alpha\beta}), \min_\beta(\breve{\mathcal{D}}_{\alpha\beta}), \max_\beta(\breve{\mathcal{Y}}_{\alpha\beta}), \max_\beta(\breve{\mathcal{F}}_{\alpha\beta}) \right\rangle \ , \ \alpha = 1,2,...,\mathfrak{m} \ ; \ \beta = 1,2,...,\mathfrak{n}.$$

Step 4: Calculate the distance measures of a perferencei.

For the seperation measures, $\hat{\xi}^+$ and $\hat{\xi}^-$ can be calculated by using Normalized Hamming distance as follows:

$$\hat{\xi}_\alpha^+ = \sum_{\beta=1}^{\mathfrak{q}} \ddot{\varpi}_\beta d(\hat{\zeta}_{\alpha\beta}, \ddot{\mathfrak{D}}_\beta^+) \text{ and } \hat{\xi}_\alpha^- = \sum_{\beta=1}^{\mathfrak{q}} \ddot{\varpi}_\beta d(\hat{\zeta}_{\alpha\beta}, \ddot{\mathfrak{D}}_\beta^-) \ , \ \alpha = 1,2,...,\mathfrak{m}.$$

Step 5: Formulate the relative closeness coefficient($\mathcal{RCC}$) for the alternative $\hat{\zeta}_\alpha$,which is defined by

$$\breve{\Gamma}_\alpha = \frac{\hat{\xi}_\alpha^+}{(\hat{\xi}_\alpha^+ + \hat{\xi}_\alpha^-)}, \ \alpha = 1,2,...,\mathfrak{m} \ ; \ 0 \leq \breve{\Gamma}_\alpha \leq 1.$$

Step 6: Ranking the alternatives in decreasing or ascending order of their $\mathcal{RCC}$.The greater value $\breve{\Gamma}_\alpha$ indicates desirable alternative $\hat{\zeta}_\alpha$.

4.1. *Application*

In today's world, the amount of solid waste produced rises daily as a result of both population increase and the development of several technologies. While the government is employing various disposal techniques to cut down on garbage, they are still searching for the best way to solve this issue without negatively affecting society or the environment. Currently, solid waste management refers to the undesired or worthless solid materials produced by human activity

in commercial, industrial, or domestic settings. Therefore, it is essential to ethically reduce solid waste and convert it into energy.

Let us consider a set of solid waste-to-energy technologies: 1)Biological conversion($\hat{\zeta}_1$); 2)Bioelectrochemical conversion($\hat{\zeta}_2$);3)Thermal conversion($\hat{\zeta}_3$);4) Mechanical conversion ($\hat{\zeta}_4$).These four kinds of solid waste-to-energy technologies are evaluated based on five attributes: :1) Economy($\hat{\Omega}_1$);2) Technical aspects($\hat{\Omega}_2$) 3)Environment($\hat{\Omega}_3$) 4) Ecosystem ($\hat{\Omega}_4$) 5) Society ($\hat{\Omega}_5$).The weight vector assigned for the above five attributes is as follows $\ddot{\varpi}$ = (0.2,0.1,0.3,0.2,0.2).Hence the proposed operator $\mathcal{QSVNRHWA}$ and here to solve MADM problem under $\mathcal{QSVNRN}$ information.

Step 1: Compute the normalized decision matrix.

The decision matrix for the profit type can be expressed as

	$\hat{\Omega}_1$	$\hat{\Omega}_2$	$\hat{\Omega}_3$	$\hat{\Omega}_4$	$\hat{\Omega}_5$
$\hat{\zeta}_1$	⟨0.6,0.3,0.5,0.4⟩	⟨0.6,0.6,0.4,0.3⟩	⟨0.5,0.6,0.5,0.4⟩	⟨0.4,0.5,0.4,0.2⟩	⟨0.8,0.6,0.4,0.3⟩
	⟨0.5,0.6,0.2,0.3⟩	⟨0.9,0.7,0.3,0.1⟩	⟨0.6,0.4,0.3,0.6⟩	⟨0.6,0.7,0.5,0.6⟩	⟨0.6,0.7,0.2,0.5⟩
	⟨0.6,0.7,0.5,0.2⟩	⟨0.8,0.8,0.3,0.4⟩	⟨0.7,0.5,0.6,0.3⟩	⟨0.7,0.4,0.6,0.5⟩	⟨0.7,0.5,0.6,0.2⟩
$\hat{\zeta}_2$	⟨0.5,0.7,0.3,0.5⟩	⟨0.5,0.5,0.4,0.2⟩	⟨0.4,0.4,0.7,0.6⟩	⟨0.5,0.3,0.6,0.4⟩	⟨0.8,0.5,0.4,0.4⟩
	⟨0.6,0.4,0.3,0.4⟩	⟨0.8,0.7,0.3,0.4⟩	⟨0.7,0.7,0.3,0.2⟩	⟨0.9,0.6,0.4,0.5⟩	⟨0.4,0.4,0.5,0.6⟩
	⟨0.8,0.5,0.7,0.3⟩	⟨0.7,0.6,0.6,0.3⟩	⟨0.6,0.5,0.3,0.4⟩	⟨0.7,0.8,0.4,0.2⟩	⟨0.5,0.7,0.6,0.2⟩
$\hat{\zeta}_3$	⟨0.7,0.5,0.6,0.4⟩	⟨0.7,0.6,0.4,0.3⟩	⟨0.6,0.5,0.7,0.4⟩	⟨0.5,0.7,0.6,0.4⟩	⟨0.8,0.5,0.6,0.5⟩
	⟨0.5,0.8,0.4,0.6⟩	⟨0.8,0.8,0.2,0.2⟩	⟨0.5,0.6,0.4,0.2⟩	⟨0.7,0.6,0.5,0.3⟩	⟨0.7,0.6,0.3,0.3⟩
	⟨0.6,0.3,0.5,0.2⟩	⟨0.9,0.7,0.3,0.2⟩	⟨0.4,0.4,0.3,0.6⟩	⟨0.5,0.3,0.5,0.7⟩	⟨0.5,0.7,0.5,0.3⟩
$\hat{\zeta}_4$	⟨0.6,0.4,0.3,0.2⟩	⟨0.7,0.6,0.5,0.3⟩	⟨0.4,0.6,0.3,0.5⟩	⟨0.8,0.5,0.6,0.4⟩	⟨0.4,0.6,0.7,0.3⟩
	⟨0.7,0.5,0.6,0.8⟩	⟨0.5,0.5,0.6,0.7⟩	⟨0.8,0.5,0.6,0.3⟩	⟨0.5,0.4,0.3,0.5⟩	⟨0.6,0.3,0.4,0.7⟩
	⟨0.3,0.6,0.5,0.4⟩	⟨0.9,0.9,0.4,0.4⟩	⟨0.5,0.6,0.4,0.2⟩	⟨0.6,0.8,0.4,0.2⟩	⟨0.7,0.6,0.4,0.3⟩

Step 2: Computed $\mathcal{QSVNRHWA}$ operator with $\breve{\beta} = 2$,we obtain the collective $\mathcal{QSVNR}$ decision matrix.

	$\hat{\Omega}_1$	$\hat{\Omega}_2$	$\hat{\Omega}_3$
$\hat{\zeta}_1$	⟨0.281,0.243,0.667,0.560⟩	⟨0.294,0.232,0.794,0.730⟩	⟨0.462,0.302,0.639,0.587⟩
$\hat{\zeta}_2$	⟨0.369,0.247,0.715,0.679⟩	⟨0.212,0.165,0.855,0.768⟩	⟨0.413,0.361,0.587,0.522⟩
$\hat{\zeta}_3$	⟨0.321,0.256,0.795,0.667⟩	⟨0.314,0.232,0.768,0.723⟩	⟨0.302,0.241,0.644,0.522⟩
$\hat{\zeta}_4$	⟨0.243,0.205,0.753,0.757⟩	⟨0.251,0.230,0.896,0.876⟩	⟨0.429,0.408,0.587,0.430⟩

	$\hat{\Omega}_4$	$\hat{\Omega}_5$
$\hat{\zeta}_1$	⟨0.312,0.258,0.795,0.703⟩	⟨0.440,0.321,0.667,0.679⟩
$\hat{\zeta}_2$	⟨0.540,0.306,0.757,0.631⟩	⟨0.297,0.247,0.795,0.667⟩
$\hat{\zeta}_3$	⟨0.315,0.244,0.834,0.757⟩	⟨0.406,0.321,0.753,0.640⟩
$\hat{\zeta}_4$	⟨0.424,0.319,0.715,0.631⟩	⟨0.285,0.202,0.798,0.715⟩

Step 3: Calculate the $\mathcal{QSVNRPIS}$ and $\mathcal{QSVNRNIS}$ are defined as

$\ddot{\mathfrak{D}}_\beta^{+}$	$\mathcal{QSVNRPIS}$
$\ddot{\mathfrak{D}}_1^{+}$	⟨0.369,0.256,0.667,0.560⟩
$\ddot{\mathfrak{D}}_2^{+}$	⟨0.294,0.232,0.794,0.730⟩
$\ddot{\mathfrak{D}}_3^{+}$	⟨0.462,0.408,0.587,0.430⟩
$\ddot{\mathfrak{D}}_4^{+}$	⟨0.540,0.319,0.715,0.631⟩
$\ddot{\mathfrak{D}}_5^{+}$	⟨0.440,0.321,0.667,0.640⟩

$\ddot{\mathfrak{D}}_\beta^{-}$	$\mathcal{QSVNRNIS}$
$\ddot{\mathfrak{D}}_1^{-}$	⟨0.243,0.205,0.795,0.757⟩
$\ddot{\mathfrak{D}}_2^{-}$	⟨0.212,0.165,0.896,0.876⟩
$\ddot{\mathfrak{D}}_3^{-}$	⟨0.302,0.241,0.639,0.587⟩
$\ddot{\mathfrak{D}}_4^{-}$	⟨0.312,0.244,0.834,0.757⟩
$\ddot{\mathfrak{D}}_5^{-}$	⟨0.285,0.202,0.798,0.715⟩

Step 4: Calculate the distance measures of a perferencei.

For the seperation measures, $\hat{\xi}^{+}$ and $\hat{\xi}^{-}$ can be calculated by using Normalized Hamming distance of each perference.

$$\hat{\xi}_\alpha^{+} = \sum_{\beta=1}^{5} \ddot{\varpi}_\beta d(\ddot{\mathfrak{P}}_{\alpha\beta}, \ddot{\mathfrak{D}}_\beta^{+}) \text{ and } \hat{\xi}_\alpha^{-} = \sum_{\beta=1}^{4} \ddot{\varpi}_\beta d(\ddot{\mathfrak{P}}_{\alpha\beta}, \ddot{\mathfrak{D}}_\beta^{-}), \alpha = 1,2,3,4.$$

$$\hat{\xi}_1^{+} = 0.0106,\ \hat{\xi}_2^{+} = 0.0101,\ \hat{\xi}_3^{+} = 0.0169,\ \hat{\xi}_4^{+} = 0.0126.$$

$$\hat{\xi}_1^{-} = 0.0148,\ \hat{\xi}_2^{-} = 0.0152,\ \hat{\xi}_3^{-} = 0.0091,\ \hat{\xi}_4^{-} = 0.0128.$$

Step 5: Calculate($\mathcal{RCC}$) for each alternative $\hat{\zeta}_\alpha$,which is defined by

$$\breve{\Gamma}_\alpha = \frac{\hat{\xi}_\alpha^{+}}{(\hat{\xi}_\alpha^{+} + \hat{\xi}_\alpha^{-})},\ \alpha = 1,2,3,4.$$

$$\Rightarrow \breve{\Gamma}_1 = 0.5825,\ \breve{\Gamma}_2 = 0.6013,\ \breve{\Gamma}_3 = 0.3505,\ \breve{\Gamma}_4 = 0.5040.$$

Step 5: Select the desirable alterative $\hat{\zeta}_\alpha$.

The four alternatives are ranked based on the $\mathcal{RCC}$ values for each preference.

$$\hat{\zeta}_2 > \hat{\zeta}_1 > \hat{\zeta}_4 > \hat{\zeta}_3.$$

Therefore $\hat{\zeta}_2$ is the desirable.

Results:

The highest $\mathcal{RCC}$ indicates that Bioelectrochemical conversion is the optimal approach for solid waste-to-energy technology due to its environmentally friendly nature. This process generates more green energy, contributing to environmental cleanliness and offering significant employment opportunities to our society.

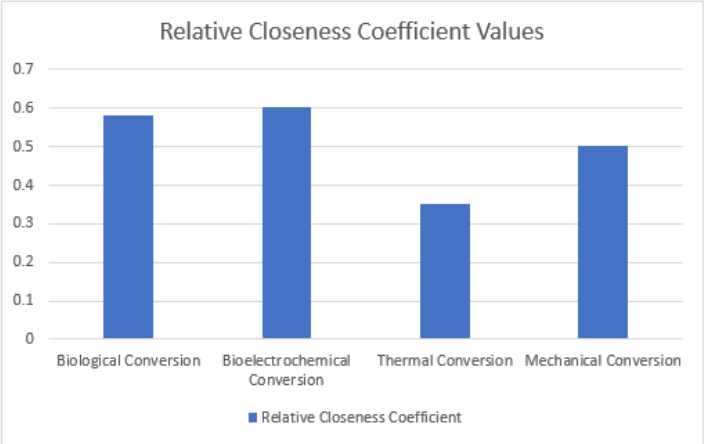

FIGURE 1. Rating values assigned to the alternatives

5. Conclusion

In this study, we introduced a comparative approach for two quadripartitioned single-valued neutrosophic refined numbers.We proposed new aggregation operators for these quadripartitioned single-valued neutrosophic refined sets, based on Hamacher t-norm and Hamacher t-conorm, and examined their fundamental properties. These operators were incorporated into the TOPSIS method, and the approach was extended to solve MADM problems involving quadripartitioned single-valued neutrosophic refined numbers. A practical application was provided, demonstrating the conversion of solid waste into energy, a solution relevant to current global issues. For future research, we aim to expand the proposed approach and applying it to a range of decision-making challenges, including information management, project selection, and other decision-making areas.

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Performance Evaluation of Venture Capital for Small and Medium-Sized Technology Startups under Uncertainty Environment

Zhihao Cao
School of Accounting, Chongqing College of Finance and Economics, Yongchuan, Chongqing, 402160, China
*Corresponding author, E-mail: caozhihao@126.com

Abstract
This study presents a novel approach to evaluating venture capital in an uncertain environment using a multi-criteria decision-making (MCDM) methodology integrated with the plithogenic framework. Venture capital decisions often involve numerous conflicting factors and vague information, making precise evaluations challenging. The MARCOS method was applied to rank eleven alternatives based on eight criteria, such as return on investment and market growth. The criteria weights were computed using normalized crisp values, and the plithogenic operator was utilized to handle uncertainty. Sensitivity analysis confirmed the stability of the results, showing that the proposed method effectively ranks alternatives under varying conditions. The findings highlight the robustness of the MARCOS approach when integrated with plithogenic sets in addressing complex decision-making scenarios.

Keywords: Venture Capital; Uncertainty Environment; Venture Capital; MCDM.

1. Introduction

Venture capital, frequently shortened to VC, is a sort of private equity and startup finance that enables a new firm to provide a significant portion of its business to one or a small group of investors in exchange for money or other advantages like talent or mentorship. For both investors and businesses, venture capital can have both high risks and high

returns. Through venture capital, startups can obtain funds without having to make monthly payments, but they might have to cede some degree of control over the company's administration and innovation. Although there is a significant chance that the firm will fail, investors also have the chance to profit if it succeeds[1], [2].

Depending on how new the company is, a startup may look for one of three primary forms of venture financing. For example, companies that are prepared to begin selling their goods or services may look for seed capital, whereas fresh startups that are still refining their concepts may seek pre-seed funding. Startups may attempt to obtain early-stage investment if they have already achieved some sales success and are prepared to increase production[3], [4].

Since venture capitalists usually seek long-term partnerships, the startup funding process for obtaining venture money might be drawn out. To thoroughly evaluate the startup and decide whether to invest, requires time[5], [6]. Although the exact time frame will differ from case to case, obtaining venture capital funding normally takes three to nine months from the time of initial contact to funding. After that, several years will pass between when the company or investor begins to provide funds and when they leave[7], [8].

Evaluating Venture Capital is difficult since there are many factors to consider, and these factors frequently contradict one another. Multi-criteria decision-making (MCDM), which is a component of Evaluating Venture Capital, requires decision-makers to evaluate both qualitative and quantitative considerations. The ability of MCDM to assess many options based on predetermined criteria makes it a highly useful tool for assessing complicated real-world issues. Using MCDM techniques, complicated issues are divided into manageable chunks that, upon analysis, can be combined to provide a comprehensive view of the issue[9], [10].

When it comes to solving difficulties in complex sectors, the decision-making process necessitates the prior definition and fulfillment of certain variables. In the scientific community, the notion of multi-criteria decision-making has a unique place. These approaches take advantage of the standards that characterize options for decision-making. Using predetermined criteria, a decision-maker uses MCDM techniques to analyze potential options and determine how well they meet the decision-making objective. The use of MCDM techniques to address a variety of issues has grown recently, and several new techniques have been created. All MCDM techniques fall into one of three categories: interactive, synthesis, or outranking techniques. Optimization problems are solved interactively, and a non-dominant solution is sought[11], [12].

Outranking techniques allow the ranking of options based on the preferences of the decision maker (DM). Making decisions when evaluating several, frequently opposing situations is made possible by synthesis procedures. By using the MCDM approaches, the decision-maker can establish a compromise between all potential conflicting factors by considering various criteria or goals[13], [14]. Finding the optimum MCDM method is nearly impossible, even though there are many different approaches available in the literature. However, researchers and practitioners typically choose the approach to be used based on the nature and complexity of the topic being studied[15], [16].

The Measurement of Alternatives and Ranking according to COmpromise Solution (MARCOS) is a novel approach that has been developed in this research. It consists of seven easy steps. The foundation of this approach is the ranking of options concerning a compromise solution once they have been measured. The compromise solution involves calculating utility functions based on the separation between ideal and anti-ideal solutions as well as their aggregations. This paper's primary contribution is to advance the field of decision-making by creating a novel approach that will assist decision-makers (DMs) in resolving complex issues[14], [17].

Uncertain, ambiguous, and incomplete information is a flaw that can result in a less-than-ideal choice, as is the case with many assessments and decision-making issues. Therefore, more accurate evaluation findings should be obtained by combining the plithogenic set with the triple components of the neutrosophic set (truth-membership, falsity-membership, and indeterminacy-membership). Plithogenic sets improve decision-making efficiency and accuracy. Plithogeny is a generalization of neutrosophy that was first proposed by Florentin Smarandache in 2017.

The plithogenic set is a collection of elements where each element x is distinguished by an appurtenance degree d(x, v) of element x to the plithogenic set and attribute values v that have a matching contradiction degree c(v, D) between them and a dominant attribute value D[18], [19].

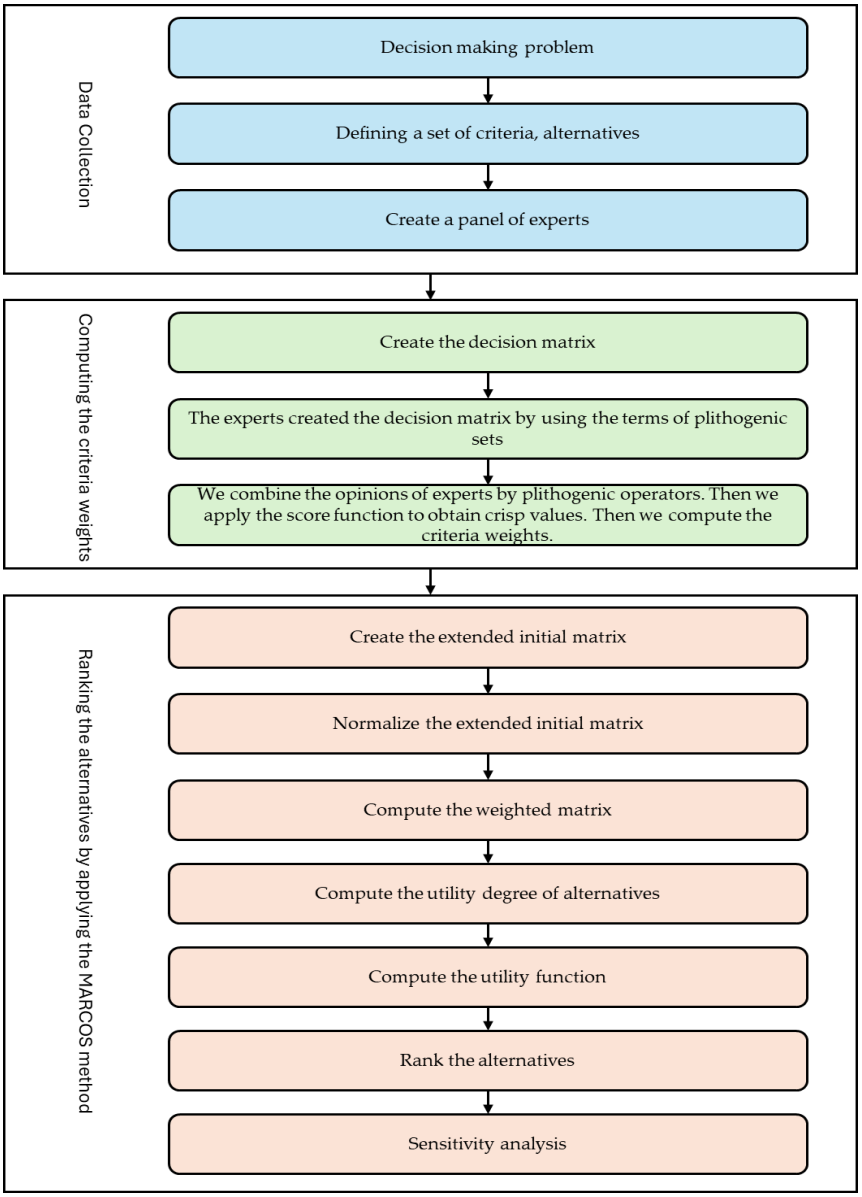
Data Collection
Decision making problem
Defining a set of criteria, alternatives
Create a panel of experts
Computing the criteria weights
Create the decision matrix
The experts created the decision matrix by using the terms of plithogenic sets
We combine the opinions of experts by plithogenic operators. Then we apply the score function to obtain crisp values. Then we compute the criteria weights.
Ranking the alternatives by applying the MARCOS method
Create the extended initial matrix
Normalize the extended initial matrix
Compute the weighted matrix
Compute the utility degree of alternatives
Compute the utility function
Rank the alternatives
Sensitivity analysis

Figure 1. Details of the MARCOS method under plithogenic.

2. Methods

This section of the work presents the steps of the method to rank the alternatives as shown in Figure 1. This section shows the criteria weights and rank of alternatives.

2.1. Evaluation of Plithogenic Set Integration

In the context of evaluating venture capital for small and medium-sized technology startups, handling vague and uncertain information is critical due to fluctuating market conditions and subjective expert opinions. The integration of plithogenic sets into the

decision-making process enhances the framework by accounting for these uncertainties. Unlike traditional fuzzy or neutrosophic sets, plithogenic sets introduce the concept of a contradiction degree, which quantifies inconsistencies between conflicting attributes.

2.1.1 Example Application

For instance, when evaluating the "Return on Investment" criterion, expert opinions might conflict due to differences in industry perspective. A plithogenic operator is used to combine these opinions into a comprehensive evaluation, as shown in the table below.

Criterion	Expert 1 Opinion	Expert 2 Opinion	Expert 3 Opinion	Plithogenic Value
Return on Investment (C1)	0.80	0.75	0.85	0.80 ± 0.05
Market Growth (C2)	0.70	0.65	0.75	0.70 ± 0.05
Innovation (C7)	0.90	0.85	0.88	0.88 ± 0.03

Figure 2 illustrates the variability in expert opinions for each criterion and the resulting plithogenic values. The red line represents the consolidated plithogenic values derived from the expert evaluations, showcasing how the framework incorporates varying opinions into a comprehensive assessment.

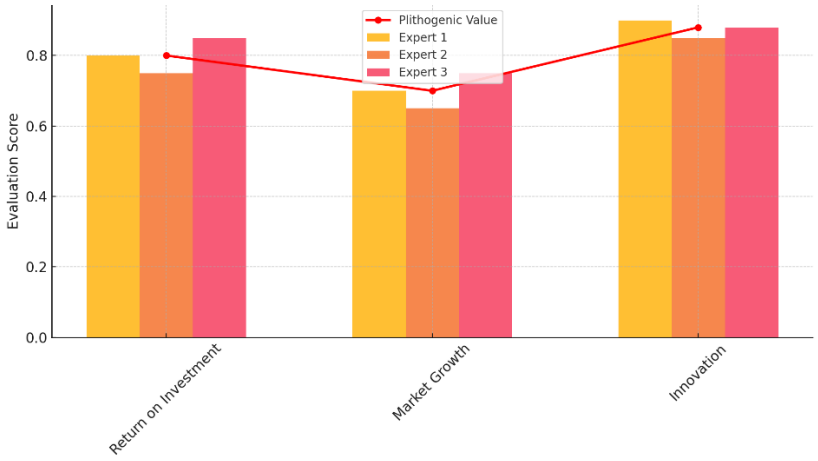

Figure 2.Variability in Expert Opinions and Resulting Plithogenic Values

2.2 Measurement of alternatives and ranking according to COmpromise solution (MARCOS)

In this part, the MARCOS method is described in detail (See Figuer 3). The MARCOS method is the MCDM method used to rank the alternatives. It is obtaining the relationship between the alternatives and reference values[20], [21], [22], [23]. The steps of MARCOS methodology as:

A) Create the decision matrix. Define a set of criteria and alternatives. The set of experts has evaluated the criteria and alternatives. The experts created the decision matrix by using the terms of plithogenic sets. We combine the opinions of experts by plithogenic operators. Then we apply the score function to obtain crisp values.

B) Create the extended initial matrix.

This step defines the ideal (AI) and anti-ideal (AAI) solution.

$$AAI = \min x_{ij} \ \ if\ j \in B\ and \max x_{ij} \ \ if\ j \in C \tag{1}$$

$$AI = \max x_{ij} \ \ if\ j \in B\ and \min x_{ij} \ \ if\ j \in C \tag{2}$$

Where C refers to the cost criteria and B refers to the positive criteria.

C) Normalize the extended initial matrix

$$q_{ij} = \frac{x_{ai}}{x_{ij}} \ \ if\ j \in C \tag{3}$$

$$q_{ij} = \frac{x_{ij}}{x_{ai}} \ \ if\ j \in B \tag{4}$$

D) Compute the weighted matrix.

$$y_{ij} = q_{ij} w_j \tag{5}$$

E) Compute the utility degree of alternatives

$$U_i^- = \frac{p_i}{p_{aai}} \tag{6}$$

$$U_i^+ = \frac{p_i}{p_{ai}} \tag{7}$$

$$p_i = \sum_{i=1}^{n} y_{ij} \tag{8}$$

F) Compute the utility function

$$f(U_i) = \frac{U_i^+ + U_i^-}{1 + \frac{1-f(U_i^+)}{f(U_i^+)} + \frac{1-f(U_i^-)}{f(U_i^-)}} \tag{9}$$

$$f(U_i^+) = \frac{U_i^-}{U_i^- + U_i^-} \tag{10}$$

$$f(U_i^-) = \frac{U_i^+}{U_i^- + U_i^-} \tag{11}$$

G) Rank the alternatives.

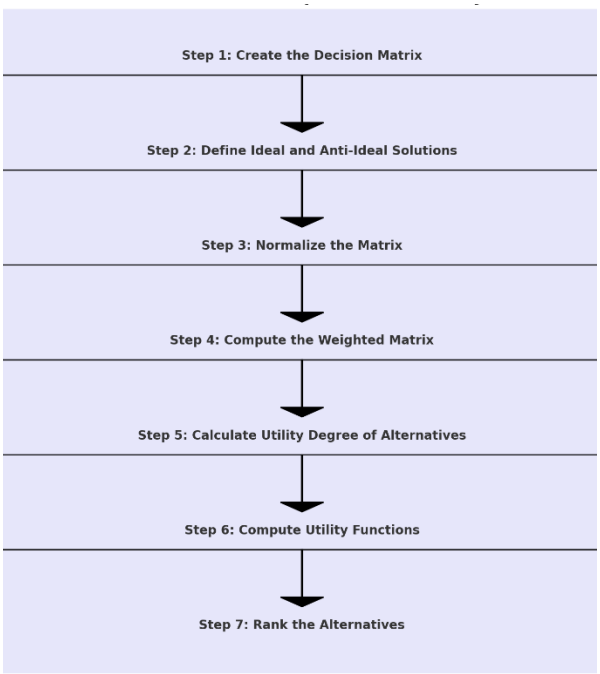
Step 1: Create the Decision Matrix
Step 2: Define Ideal and Anti-Ideal Solutions
Step 3: Normalize the Matrix
Step 4: Compute the Weighted Matrix
Step 5: Calculate Utility Degree of Alternatives
Step 6: Compute Utility Functions
Step 7: Rank the Alternatives

Figure 3. The main steps of MARCOS method

3. Case Study of Venture Capital

This study focuses on evaluating venture capital using a set of defined criteria and potential alternatives. To ensure accuracy and depth, three experts, each with over 20 years of experience in venture capital evaluation, were consulted. They assessed eight key criteria and eleven alternatives to determine the best investment options. The details of the criteria and alternatives are presented in Table 1.

Table 1. The criteria.

	Criteria	**Alternatives**
C_1	Exit Strategy Viability	A_1
C_2	Customer Base Growth and Retention	A_2
C_3	Quality of the Management Team	A_3
C_4	Market Growth and Scalability	A_4
C_5	Return on Investment	A_5
C_6	Risk Mitigation and Resilience	A_6
C_7	Innovation and Technological Advancement	A_7

C_8	Financial Stability and Liquidity	A_8
		A_9
		A_{10}
		A_{11}

4. Results

4.1 Overview of the Methodology

The evaluation of venture capital alternatives involved applying the MARCOS method to rank eleven alternatives based on eight criteria. Three decision-makers provided evaluations, which were aggregated using a plithogenic framework to account for vagueness and uncertainty. The results were systematically derived through a series of steps, including normalizing the decision matrix, calculating the weighted normalized matrix, determining utility degrees, and ranking the alternatives. The process and results are supported by detaile Tables 2–8 and Figure 4.

4.1 Expert Evaluations and Decision Matrix

The process began with gathering input from three decision-makers with expertise in venture capital evaluation. These experts evaluated the importance of eight criteria for each of the eleven alternatives. The criteria include Exit Strategy Viability (C1), Customer Base Growth and Retention (C2), Quality of the Management Team (C3), Market Growth and Scalability (C4), Return on Investment (C5), Risk Mitigation and Resilience (C6), Innovation and Technological Advancement (C7), and Financial Stability and Liquidity (C8).

The experts' evaluations were aggregated into a decision matrix using the plithogenic framework, which effectively manages vague and uncertain information. The evaluations from each expert are detailed in Tables 2–4.

Table 2 contains the evaluations provided by Decision-Maker 1. Table 3 Shows the assessments from Decision-Maker 2. Table 4 displays the input from Decision-Maker 3.

Table 2. The significance of the criteria of each alternative by decision maker 1.

	C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8
A_1	(0.80, 0.10, 0.30)	(0.25, 0.60, 0.80)	(0.40, 0.70, 0.50)	(0.50, 0.40, 0.60)	(0.65, 0.30, 0.45)	(0.95, 0.05, 0.05)	(0.95, 0.05, 0.05)	(0.95, 0.05, 0.05)
A_2	(0.95, 0.05, 0.05)	(0.80, 0.10, 0.30)	(0.65, 0.30, 0.45)	(0.80, 0.10, 0.30)	(0.95, 0.05, 0.05)	(0.10, 0.75, 0.85)	(0.25, 0.60, 0.80)	(0.80, 0.10, 0.30)
A_3	(0.10, 0.75, 0.85)	(0.95, 0.05, 0.05)	(0.80, 0.10, 0.30)	(0.65, 0.30, 0.45)	(0.80, 0.10, 0.30)	(0.25, 0.60, 0.80)	(0.40, 0.70, 0.50)	(0.95, 0.05, 0.05)
A_4	(0.25, 0.60, 0.80)	(0.10, 0.75, 0.85)	(0.95, 0.05, 0.05)	(0.80, 0.10, 0.30)	(0.95, 0.05, 0.05)	(0.80, 0.10, 0.30)	(0.50, 0.40, 0.60)	(0.10, 0.75, 0.85)
A_5	(0.40, 0.70, 0.50)	(0.25, 0.60, 0.80)	(0.10, 0.75, 0.85)	(0.95, 0.05, 0.05)	(0.10, 0.75, 0.85)	(0.95, 0.05, 0.05)	(0.80, 0.10, 0.30)	(0.25, 0.60, 0.80)
A_6	(0.95, 0.05, 0.05)	(0.40, 0.70, 0.50)	(0.25, 0.60, 0.80)	(0.10, 0.75, 0.85)	(0.25, 0.60, 0.80)	(0.10, 0.75, 0.85)	(0.95, 0.05, 0.05)	(0.40, 0.70, 0.50)
A_7	(0.95, 0.05, 0.05)	(0.95, 0.05, 0.05)	(0.40, 0.70, 0.50)	(0.25, 0.60, 0.80)	(0.40, 0.70, 0.50)	(0.25, 0.60, 0.80)	(0.10, 0.75, 0.85)	(0.95, 0.05, 0.05)
A_8	(0.10, 0.75, 0.85)	(0.10, 0.75, 0.85)	(0.95, 0.05, 0.05)	(0.40, 0.70, 0.50)	(0.95, 0.05, 0.05)	(0.40, 0.70, 0.50)	(0.25, 0.60, 0.80)	(0.10, 0.75, 0.85)
A_9	(0.25, 0.60, 0.80)	(0.25, 0.60, 0.80)	(0.25, 0.60, 0.80)	(0.95, 0.05, 0.05)	(0.95, 0.05, 0.05)	(0.95, 0.05, 0.05)	(0.40, 0.70, 0.50)	(0.25, 0.60, 0.80)
A_{10}	(0.40, 0.70, 0.50)	(0.40, 0.70, 0.50)	(0.40, 0.70, 0.50)	(0.10, 0.75, 0.85)	(0.10, 0.75, 0.85)	(0.25, 0.60, 0.80)	(0.95, 0.05, 0.05)	(0.40, 0.70, 0.50)
A_{11}	(0.10, 0.75, 0.85)	(0.25, 0.60, 0.80)	(0.10, 0.75, 0.85)	(0.25, 0.60, 0.80)	(0.25, 0.60, 0.80)	(0.40, 0.70, 0.50)	(0.50, 0.40, 0.60)	(0.50, 0.40, 0.60)

Table 3. The significance of the criteria of each alternative by decision maker 2.

	C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8
A_1	(0.50, 0.40, 0.60)	(0.25, 0.60, 0.80)	(0.40, 0.70, 0.50)	(0.25, 0.60, 0.80)	(0.65, 0.30, 0.45)	(0.80, 0.10, 0.30)	(0.95, 0.05, 0.05)	(0.10, 0.75, 0.85)
A_2	(0.65, 0.30, 0.45)	(0.25, 0.60, 0.80)	(0.65, 0.30, 0.45)	(0.50, 0.40, 0.60)	(0.95, 0.05, 0.05)	(0.10, 0.75, 0.85)	(0.25, 0.60, 0.80)	(0.25, 0.60, 0.80)
A_3	(0.25, 0.60, 0.80)	(0.50, 0.40, 0.60)	(0.25, 0.60, 0.80)	(0.65, 0.30, 0.45)	(0.25, 0.60, 0.80)	(0.95, 0.05, 0.05)	(0.50, 0.40, 0.60)	(0.25, 0.60, 0.80)
A_4	(0.50, 0.40, 0.60)	(0.65, 0.30, 0.45)	(0.50, 0.40, 0.60)	(0.25, 0.60, 0.80)	(0.50, 0.40, 0.60)	(0.10, 0.75, 0.85)	(0.65, 0.30, 0.45)	(0.50, 0.40, 0.60)
A_5	(0.65, 0.30, 0.45)	(0.80, 0.10, 0.30)	(0.65, 0.30, 0.45)	(0.50, 0.40, 0.60)	(0.25, 0.60, 0.80)	(0.25, 0.60, 0.80)	(0.80, 0.10, 0.30)	(0.65, 0.30, 0.45)
A_6	(0.80, 0.10, 0.30)	(0.95, 0.05, 0.05)	(0.80, 0.10, 0.30)	(0.65, 0.30, 0.45)	(0.50, 0.40, 0.60)	(0.25, 0.60, 0.80)	(0.95, 0.05, 0.05)	(0.80, 0.10, 0.30)
A_7	(0.95, 0.05, 0.05)	(0.10, 0.75, 0.85)	(0.95, 0.05, 0.05)	(0.80, 0.10, 0.30)	(0.65, 0.30, 0.45)	(0.50, 0.40, 0.60)	(0.25, 0.60, 0.80)	(0.95, 0.05, 0.05)
A_8	(0.65, 0.30, 0.45)	(0.80, 0.10, 0.30)	(0.10, 0.75, 0.85)	(0.95, 0.05, 0.05)	(0.80, 0.10, 0.30)	(0.65, 0.30, 0.45)	(0.50, 0.40, 0.60)	(0.65, 0.30, 0.45)
A_9	(0.80, 0.10, 0.30)	(0.95, 0.05, 0.05)	(0.80, 0.10, 0.30)	(0.80, 0.10, 0.30)	(0.95, 0.05, 0.05)	(0.80, 0.10, 0.30)	(0.65, 0.30, 0.45)	(0.80, 0.10, 0.30)
A_{10}	(0.95, 0.05, 0.05)	(0.10, 0.75, 0.85)	(0.95, 0.05, 0.05)	(0.95, 0.05, 0.05)	(0.95, 0.05, 0.05)	(0.95, 0.05, 0.05)	(0.80, 0.10, 0.30)	(0.95, 0.05, 0.05)
A_{11}	(0.10, 0.75, 0.85)	(0.25, 0.60, 0.80)	(0.10, 0.75, 0.85)	(0.10, 0.75, 0.85)	(0.10, 0.75, 0.85)	(0.65, 0.30, 0.45)	(0.95, 0.05, 0.05)	(0.10, 0.75, 0.85)

Table 4. The significance of the criteria of each alternative by decision maker 3.

	C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8
A_1	(0.10, 0.75, 0.85)	(0.25, 0.60, 0.80)	(0.40, 0.70, 0.50)	(0.50, 0.40, 0.60)	(0.65, 0.30, 0.45)	(0.80, 0.10, 0.30)	(0.95, 0.05, 0.05)	(0.10, 0.75, 0.85)
A_2	(0.40, 0.70, 0.50)	(0.10, 0.75, 0.85)	(0.65, 0.30, 0.45)	(0.80, 0.10, 0.30)	(0.95, 0.05, 0.05)	(0.10, 0.75, 0.85)	(0.25, 0.60, 0.80)	(0.25, 0.60, 0.80)
A_3	(0.10, 0.75, 0.85)	(0.25, 0.60, 0.80)	(0.10, 0.75, 0.85)	(0.65, 0.30, 0.45)	(0.80, 0.10, 0.30)	(0.10, 0.75, 0.85)	(0.40, 0.70, 0.50)	(0.40, 0.70, 0.50)
A_4	(0.25, 0.60, 0.80)	(0.40, 0.70, 0.50)	(0.25, 0.60, 0.80)	(0.10, 0.75, 0.85)	(0.95, 0.05, 0.05)	(0.25, 0.60, 0.80)	(0.10, 0.75, 0.85)	(0.50, 0.40, 0.60)
A_5	(0.40, 0.70, 0.50)	(0.50, 0.40, 0.60)	(0.40, 0.70, 0.50)	(0.25, 0.60, 0.80)	(0.10, 0.75, 0.85)	(0.40, 0.70, 0.50)	(0.25, 0.60, 0.80)	(0.65, 0.30, 0.45)
A_6	(0.50, 0.40, 0.60)	(0.65, 0.30, 0.45)	(0.50, 0.40, 0.60)	(0.40, 0.70, 0.50)	(0.25, 0.60, 0.80)	(0.50, 0.40, 0.60)	(0.40, 0.70, 0.50)	(0.80, 0.10, 0.30)
A_7	(0.65, 0.30, 0.45)	(0.80, 0.10, 0.30)	(0.65, 0.30, 0.45)	(0.50, 0.40, 0.60)	(0.40, 0.70, 0.50)	(0.10, 0.75, 0.85)	(0.50, 0.40, 0.60)	(0.95, 0.05, 0.05)
A_8	(0.50, 0.40, 0.60)	(0.65, 0.30, 0.45)	(0.10, 0.75, 0.85)	(0.65, 0.30, 0.45)	(0.50, 0.40, 0.60)	(0.25, 0.60, 0.80)	(0.65, 0.30, 0.45)	(0.10, 0.75, 0.85)
A_9	(0.40, 0.70, 0.50)	(0.50, 0.40, 0.60)	(0.95, 0.05, 0.05)	(0.80, 0.10, 0.30)	(0.65, 0.30, 0.45)	(0.40, 0.70, 0.50)	(0.25, 0.60, 0.80)	(0.25, 0.60, 0.80)
A_{10}	(0.25, 0.60, 0.80)	(0.40, 0.70, 0.50)	(0.50, 0.40, 0.60)	(0.25, 0.60, 0.80)	(0.80, 0.10, 0.30)	(0.50, 0.40, 0.60)	(0.40, 0.70, 0.50)	(0.40, 0.70, 0.50)
A_{11}	(0.10, 0.75, 0.85)	(0.25, 0.60, 0.80)	(0.10, 0.75, 0.85)	(0.95, 0.05, 0.05)	(0.80, 0.10, 0.30)	(0.65, 0.30, 0.45)	(0.50, 0.40, 0.60)	(0.50, 0.40, 0.60)

4.2 Normalization of the Decision Matrix (Table 5)

The decision matrix was normalized to scale all values between 0 and 1, allowing for comparability across different criteria. Normalization was performed using Equations (3) and (4):

For benefit criteria, each value was divided by the maximum value in its column.

For cost criteria, the minimum value in the column was divided by each value.

4.2.1 *Example from Table 5*

For C1 (a benefit criterion), where the maximum value is 0.95 and x_{11}=0.80, the normalized value for A1 is N_{11}=0.842

For C2 (a cost criterion), where the minimum value is 0.10 and x_{12}= 0.25, the normalized value for A1 is N_{12} = 0.400

The complete normalized decision matrix is shown in Table 5, where each entry represents the scaled value of an alternative for a specific criterion.

4.3 Computation of the Weighted Normalized Matrix (Table 6)

After normalization, the criteria weights were incorporated into the matrix to reflect their relative importance. This was done using Equation (5).

4.3.1 *Example from Table 6*

For C1 , with N_{11}=0.842 and w1=0.20, the weighted normalized value for A1 is: W11=0.1684

The resulting weighted normalized matrix, which aggregates the relative contributions of each criterion to the performance of alternatives, is presented in Table 6.

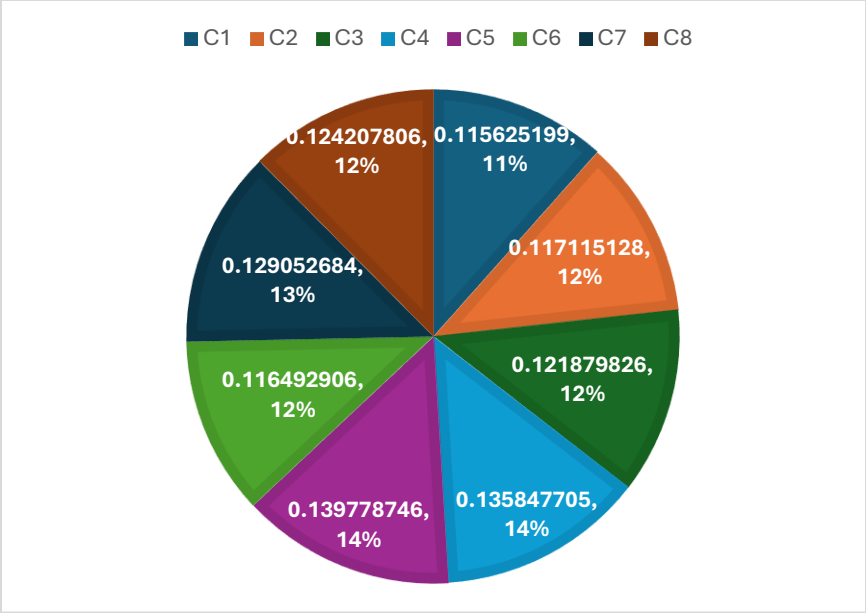

Figure 4. The criteria weights.

Table 5. The normalization matrix.

	C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8
A_1	0.686291	0.399358	0.572889	0.576918	0.800064	1	1	0.594608
A_2	0.855007	0.664704	0.915644	0.898623	1	0.146852	0.337556	0.614678
A_3	0.217822	0.915742	0.643003	0.822098	0.821442	0.66682	0.553925	0.717294
A_4	0.474384	0.621442	0.885846	0.577311	0.943778	0.577311	0.577664	0.508998
A_5	0.63353	0.832314	0.601154	0.795344	0.20919	0.737049	0.821442	0.684716
A_6	0.953153	0.971461	0.805142	0.539738	0.455584	0.407738	0.903063	0.829355
A_7	1	1	0.939746	0.722885	0.608424	0.407738	0.396809	1
A_8	0.601502	0.848094	0.680507	0.843738	0.91538	0.591475	0.626101	0.413114
A_9	0.677786	0.915742	1	1	0.96037	0.895344	0.575622	0.614678
A_{10}	0.746893	0.482144	0.896874	0.66682	0.845246	0.795344	0.871347	0.730185
A_{11}	0.148814	0.399358	0.163563	0.66682	0.561838	0.729115	0.834588	0.508998

Table 6. The weighted normalization matrix.

	C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8
A_1	0.079353	0.046771	0.069824	0.078373	0.111832	0.116493	0.129053	0.073855
A_2	0.09886	0.077847	0.111599	0.122076	0.139779	0.017107	0.043562	0.076348
A_3	0.025186	0.107247	0.078369	0.11168	0.11482	0.07768	0.071485	0.089094
A_4	0.054851	0.07278	0.107967	0.078426	0.13192	0.067253	0.074549	0.063222
A_5	0.073252	0.097477	0.073269	0.108046	0.02924	0.085861	0.106009	0.085047
A_6	0.110209	0.113773	0.098131	0.073322	0.063681	0.047499	0.116543	0.103012
A_7	0.115625	0.117115	0.114536	0.098202	0.085045	0.047499	0.051209	0.124208
A_8	0.069549	0.099325	0.08294	0.11462	0.127951	0.068903	0.0808	0.051312

A9	0.078369	0.107247	0.12188	0.135848	0.134239	0.104301	0.074286	0.076348
A10	0.08636	0.056466	0.109311	0.090586	0.118147	0.092652	0.11245	0.090695
A11	0.017207	0.046771	0.019935	0.090586	0.078533	0.084937	0.107706	0.063222

4.4 Utility Degree Calculation and Ranking of Alternatives (Table 7)

The utility degree of each alternative was calculated by comparing its weighted normalized values to the ideal solution (AI) and the anti-ideal solution (AAI). The utility degree measures the proximity of each alternative to the ideal solution, as determined by Equations (6), (7), and (8).

Ideal Solution: Represents the best possible performance for all criteria.

Anti-Ideal Solution: Represents the worst possible performance for all criteria.

4.4.1 *Example from Table 7*

Alternative 9 achieved the highest utility degree (U9=1.635931), indicating it is the closest to the ideal solution.

Alternative 11 had the lowest utility degree (U11=1.0), indicating it is the farthest from the ideal solution.

The computed utility degrees for all alternatives and their corresponding rankings are detailed in Table 7. The table shows that Alternative 9 consistently outperformed the others, while Alternative 11 performed the weakest.

Table 7. The MARCOS results.

	U_i^-	U_i^+	$f(U_i^-)$	$f(U_i^+)$	$f(U_i)$	**Rank**
A1	1.386439	0.847492	0.379373	0.620627	0.644244	5
A2	1.350332	0.825421	0.379373	0.620627	0.627466	7
A3	1.327505	0.811467	0.379373	0.620627	0.616859	8
A4	1.279177	0.781926	0.379373	0.620627	0.594402	10
A5	1.29339	0.790614	0.379373	0.620627	0.601007	9
A6	1.42695	0.872256	0.379373	0.620627	0.663069	4
A7	1.480538	0.905013	0.379373	0.620627	0.687969	3
A8	1.366486	0.835296	0.379373	0.620627	0.634972	6

A_9	1.635931	1	0.379373	0.620627	0.760177	1
A_{10}	1.48688	0.908889	0.379373	0.620627	0.690916	2
A_{11}	1	0.611273	0.379373	0.620627	0.464675	11

The results demonstrate that the MARCOS method, combined with the plithogenic framework, provides a robust and systematic approach for evaluating venture capital alternatives under uncertainty. By incorporating expert evaluations, normalization, weighted aggregation, and sensitivity analysis, the methodology ensures that the final rankings are comprehensive and reliable. This structured approach enables decision-makers to identify the most favorable alternatives with confidence.

5. Result Validation

We validated the results by conducting a sensitivity analysis to show the different ranks of alternatives. We performed nine cases in criteria weights as shown in Figure 5. We increased the criteria weights by 20% and reduced other criteria. Then we applied the MARCOS method to show different ranks of alternatives as shown in Table 8. We show that alternative 11 is the best and alternative 9 is the worst.

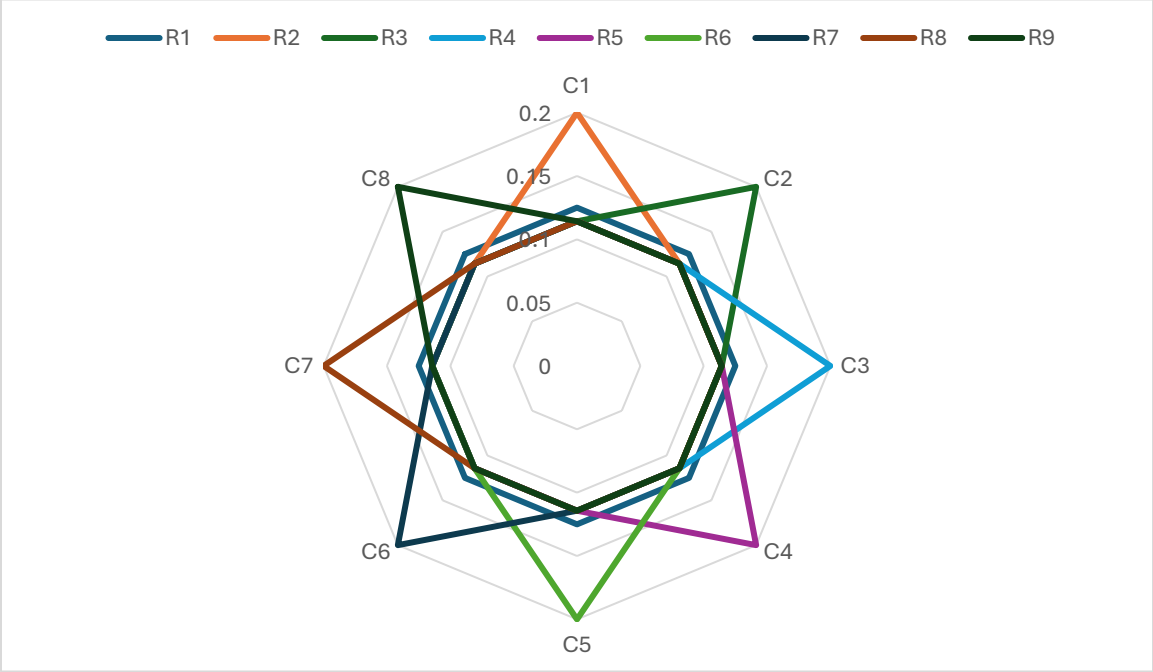

Figure 5. The change in criteria weights.

Table 8. The rank of alternatives under different criteria weights.

	R_1	R_2	R_3	R_4	R_5	R_6	R_7	R_8	R_9
A_1	5	5	9	6	7	4	4	4	5
A_2	7	6	8	5	6	7	10	9	7
A_3	8	10	6	8	8	8	8	8	6
A_4	10	9	10	9	10	9	9	10	10
A_5	9	8	7	10	9	10	7	7	9
A_6	4	4	3	4	4	5	5	3	4
A_7	2	2	2	2	2	3	3	5	2
A_8	6	7	5	7	5	6	6	6	8
A_9	1	1	1	1	1	1	1	1	1
A_{10}	3	3	4	3	3	2	2	2	3
A_{11}	11	11	11	11	11	11	11	11	11

The stability of the rankings was validated through sensitivity analysis. This involved varying the weights of the criteria by ±20% and observing the impact on the rankings. The results, summarized in Table 8, indicate that the rankings remained consistent under different weighting scenarios:

Alternative 9 consistently ranked first across all scenarios.

Alternative 11 consistently ranked last.

Figure 3 provides a visual representation of how changes in criteria weights influence the rankings. The figure highlights the robustness of the MARCOS methodology, as the rankings are largely unaffected by moderate changes in weight distributions.

6. Discussion and Advantage of the Proposed Method

The multi-criteria aspect of decisions is stressed at the managerial level when objectives are established and the best options are chosen. Decision-makers at the managerial level have the authority to approve or disapprove of the engineering-level solution. Sometimes political factors, rather than just technical ones, determine the structure of preferences. By using thorough analysis and highlighting key features of non-inferior and/or compromised solutions, the system analyst can support the decision-making process in these circumstances.

When there are competing criteria, a compromise solution might aid decision-makers in reaching a final choice. Zeleny laid the groundwork for the compromise solution. A compromise is an agreement reached via mutual concessions, and a compromise solution is a workable solution that is closest to the ideal.

7. Conclusions and Future Works

This research successfully developed an MCDM methodology using the MARCOS method integrated with the plithogenic framework to evaluate venture capital options for small and medium-sized technology startups. By employing eight criteria and analyzing eleven alternatives, the approach demonstrated its capability to handle uncertainty and vague information effectively. The results revealed that return on investment carried the highest weight among the criteria, while exit strategy viability had the lowest. The rankings of the alternatives remained stable under various sensitivity analyses, confirming the robustness of the proposed method.

Future research can expand on this study by exploring additional decision-making frameworks or hybrid approaches to further improve the accuracy of venture capital evaluations. Incorporating machine learning algorithms into the MCDM process could enhance the adaptability and predictive capabilities of the framework. Furthermore, applying this methodology to real-world case studies across diverse industries can validate its effectiveness and highlight sector-specific criteria.

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Different operators via weighted averaging and geometric approach using trigonometric neutrosophic interval-valued set and its extension

Raed Hatamleh[1], Ahmed Salem Heilat[2], M.Palanikumar[(3,*)] and Abdallah Al-Husban[4,5],

[1] Department of Mathematics, Faculty of Science, Jadara University, P.O. Box 733, Irbid 21110, Jordan; raed@jadara.edu.jo.

[2] Department of Mathematics, Faculty of Science, Jadara University, P.O. Box 733, Irbid 21110, Jordan; ahmed_heilat@yahoo.com

[3] Department of Mathematics, Saveetha School of Engineering, Saveetha Institute of Medical and Technical Sciences, Chennai-602105, India; palanimaths86@gmail.com.

[4] Department of Mathematics, Faculty of Science and Technology, Irbid National University, P.O. Box: 2600 Irbid, Jordan; dralhosban@inu.edu.jo.

[5] Jadara Research Center, Jadara University, Irbid 21110, Jordan.

*Correspondence: palanimaths86@gmail.com;

Abstract. This work presents a new method for generating tangent trigonometric neutrosophic interval-valued sets. This article will cover tangent trigonometric neutrosophic interval-valued weighted averaging, geometric, generalized notions. We used an aggregating model to get the weighted average and geometric. Several sets with substantial characteristics will be further studied using the algebraic techniques.

Keywords: weighted averaging, weighted geometric, generalized weighted averaging, generalized weighted geometric.

1. Introduction

To explain uncertainty, a number of theories have been put forth, including fuzzy sets (FS) [1], which have membership grades (MG) that range from 0. Atanassov [2] constructed an intuitionistic FS (IFS) for $\zeta, \omega \in [0,1]$ using two MGs: $0 \leq \zeta + \omega \leq 1$ and positive ζ and negative ω. Yager [3] developed the Pythagorean FSs (PFS) idea, which is distinguished by its MG and non-MG (NMG) with $\zeta + \omega \geq 1$ to $\zeta^2 + \omega^2 \leq 1$. Numerous studies have examined

the use of IFSs and PFSs in various fields. Their ability to communicate information is still restricted. Because of this, the experts were still having trouble interpreting the data in these sets and the associated data. Wang et al. [4] investigated the concept of complex IFS with DOMBI prioritized AOs and its application for trustworthy green supplier selection. Hatamleh et al. [5]- [10] discussed the concept of new algebraic structures such as neutrosophic sets, semigroups and its applications. According to Cuong et al. [11], the three primary ideas of the picture FS are positive MG (ζ), neutral MG ($\propto$), and negative MG (ω). Additionally, it offers greater benefits than PFS and IFS. Since $\zeta, \omega \in [0, 1]$, it has been noted that the picture FS is an upgrade of the IFS that may handle greater inconsistency and $0 \leq \zeta + \omega \leq 1$. According on the picture FS description, expert comments like "yes," "abstain," "no," and "refusal" will be supplied.

Shahzaib et al. [12] defined the SFS for certain AOs using MADM. SFS requires that $0 \leq \zeta^2 + \omega^2 \leq 1$ rather than $0 \leq \zeta + \omega \leq 1$. Hussain et al. [13] first proposed the concept of an intelligent decision support system for SFS. Hatamleh et al. [14]- [18] deals that the different algebraic concepts and its generalization. SFSs and their applications in DM were initially presented by Rafiq et al. [19]. For instance, $\zeta^2 + \omega^2 \geq 1$ is a DM problem with a property. Senapati et al. [20] invented Fermatean FS (FFS) in 2019 with the condition that $0 \leq \zeta^3 + \omega^3 \leq 1$. The concept of generalized orthopair FSs was initially proposed by Yager [21]. In the $\propto$-rung orthogonal pair FS ($\propto$-ROFS), both the MG and the NMG have power $\propto$; however, their sum can never be more than one. Palanikumar et al. [22] introduced the MADM approach for Pythagorean neutrosophic normal interval-valued fuzzy AOs. Aggregation operators are essential to solving MADM problems (AOs). According to Xu et al. [23], there are IFS averaging operators that may be used to average IFS data. Recently many authors discussed the new research and its aggregating operators [24]- [28].

Based on IFSs, Xu et al. [29] developed geometric operators, such as weighted, ordered weighted, and hybrid operators. Li et al. [30] proposed generalized ordered weighted averaging operators (GOWs) in 2002. Using AOs and distance measurements, Zeng et al. [31] described how to calculate ordered weighted distances. Peng et al. [32] examined a basic PFS based on the characteristics of AOs. Fuzzy sphere Dombi AOs were developed by Ashraf et al. [33]. Additionally, Ullah et al. [34, 35] offer more details on SFSs and T-SFSs. Al-Husban [36] introduced the concept of multi-fuzzy rings and its extension. Palanikumar et al. [37–39] investigated a variety of algebraic structures and aggregation methods with applications. The rest of this work will be completed in the manner described below. For an introduction, see section 1. In Section 2, PFS and NS were covered. Section 3 describes a number of techniques on $\propto$ IVNNs. Section 4 discusses the AOs based on IVT $\propto$ NN. The conclusion is covered in section 5.

2. Background

This section has several crucial definitions that we should examine for future learning.

Definition 2.1. Let $\mathcal{A}$ be a universal. The PFS $\Gamma = \left\{\flat, \left\langle \mathcal{L}^T(\flat), \mathcal{L}^F(\flat)\right\rangle \middle| \flat \in \mathcal{A}\right\}$, $\mathcal{L}^T : \mathcal{A} \to (0,1)$ and $\mathcal{L}^F : \mathcal{A} \to (0,1)$ called the MG and NMG of $\flat \in \mathcal{A}$ to Γ, respectively and $0 \preceq (\mathcal{L}^T(\flat))^2 + (\mathcal{L}^F(\flat))^2 \preceq 1$. For $\Gamma = \left\langle \mathcal{L}^T, \mathcal{L}^F\right\rangle$ is called a Pythagorean fuzzy number (PFN).

Definition 2.2. The NS $\Gamma = \left\{\flat, \left\langle \mathcal{L}^T(\flat), \mathcal{L}^I(\flat), \mathcal{L}^F(\flat)\right\rangle \middle| \flat \in \mathcal{A}\right\}$, where $\mathcal{L}^T, \mathcal{L}^I, \mathcal{L}^F : \mathcal{A} \to (0,1)$ is denote the MG, IMG and NMG of $\flat \in \mathcal{A}$, respectively and $0 \leq (\mathcal{L}^T(\flat)) + (\mathcal{L}^I(\flat)) + (\mathcal{L}^F(\flat)) \leq 2$. For $M = \left\langle \mathcal{L}^T, \mathcal{L}^I, \mathcal{L}^F\right\rangle$ is called a neutrosophic number (NN).

Definition 2.3. The Pythagorean NS $\Gamma = \left\{\flat, \left\langle \mathcal{L}^T(\flat), \mathcal{L}^I(\flat), \mathcal{L}^F(\flat)\right\rangle \middle| \flat \in \mathcal{A}\right\}$, where $\mathcal{L}^T, \mathcal{L}^I, \mathcal{L}^F : \mathcal{A} \to (0,1)$ is called the MG, IMG and NMG of $\flat \in \mathcal{A}$, respectively and $0 \leq (\mathcal{L}^T(\flat))^2 + (\mathcal{L}^I(\flat))^2 + (\mathcal{L}^F(\flat))^2 \leq 2$. For $M = \left\langle \mathcal{L}^T, \mathcal{L}^I, \mathcal{L}^F\right\rangle$ is called a Pythagorean neutrosophic number (PyNN).

Definition 2.4. Let $\Gamma_1 = (a_1, b_1) \in N$and $\Gamma_2 = (a_2, b_2) \in N$. Then the distance between Γ_1 and Γ_2 is defined as $\Lambda(\Gamma_1, \Gamma_2) = \sqrt{(a_1 - a_2)^2 + \frac{1}{2}(b_1 - b_2)^2}$, where N is a natural number.

3. Operations for IVT $\propto$ NN

We present the concept of the $\propto$ IVNN, which is a tangent trigonometric. As a consequence, the IVT $\propto$ NN and its operations were established and $\tan \pi/2 = \beth$

Definition 3.1. The $\propto$ NS $\Gamma = \left\{\flat, \left\langle \left([(\beth \circ \mathcal{O})(\flat), (\beth \circ \mathcal{V})(\flat)], [(\beth \circ \mathcal{P})(\flat), (\beth \circ \mathcal{X})(\flat)], [(\beth \circ \mathcal{Q})(\flat), (\beth \circ \mathcal{Y})(\flat)] \right) \right\rangle \middle| \flat \in \mathcal{A}\right\}$, where $(\beth \circ \mathcal{O}), (\beth \circ \mathcal{P}), (\beth \circ \mathcal{Q}) : \mathcal{A} \to (0,1)$ denote the MG, IMG and NMG of $\flat \in \mathcal{A}$ to Γ, respectively and $0 \leq ((\beth \circ \mathcal{V})(\flat))^{\propto} + ((\beth \circ \mathcal{X})(\flat))^{\propto} + ((\beth \circ \mathcal{Y})(\flat))^{\propto} \leq 1$. For convenience, $\Gamma = \left\langle \left([(\beth \circ \mathcal{O}), (\beth \circ \mathcal{V})], [(\beth \circ \mathcal{P}), (\beth \circ \mathcal{X})], [(\beth \circ \mathcal{Q}), (\beth \circ \mathcal{Y})] \right) \right\rangle$ is represent a IVT $\propto$ NN.

Definition 3.2. Let $\Gamma = \left\langle \left([(\beth \circ \mathcal{O}), (\beth \circ \mathcal{V})], [(\beth \circ \mathcal{P}), (\beth \circ \mathcal{X})], [(\beth \circ \mathcal{Q}), (\beth \circ \mathcal{Y})] \right) \right\rangle, \Gamma_1 = \langle ([(\beth \circ \mathcal{O}_1), (\beth \circ \mathcal{V}_1)], [(\beth \circ \mathcal{P}_1), (\beth \circ \mathcal{X}_1)], [(\beth \circ \mathcal{Q}_1), (\beth \circ \mathcal{Y}_1)]) \rangle, \Gamma_2 = \langle ([(\beth \circ \mathcal{O}_2), (\beth \circ \mathcal{V}_2)], [(\beth \circ \mathcal{P}_2), (\beth \circ \mathcal{X}_2)], [(\beth \circ \mathcal{Q}_2), (\beth \circ \mathcal{Y}_2)]) \rangle$ be any three IVT $\propto$ NNs, and $q > 0$. Then

$$(1)\ \Gamma_1 \curlyvee \Gamma_2 = \begin{bmatrix} \overset{\propto}{\sqrt{\begin{matrix} ((\beth \circ \mathcal{O}_1))^{\propto} + ((\beth \circ \mathcal{O}_2))^{\propto} \\ -((\beth \circ \mathcal{O}_1))^{\propto} \cdot ((\beth \circ \mathcal{O}_2))^{\propto} \end{matrix}}}, & \overset{\propto}{\sqrt{\begin{matrix} ((\beth \circ \mathcal{V}_1))^{\propto} + ((\beth \circ \mathcal{V}_2))^{\propto} \\ -((\beth \circ \mathcal{V}_1))^{\propto} \cdot ((\beth \circ \mathcal{V}_2))^{\propto} \end{matrix}}} \\ \overset{\propto}{\sqrt{\begin{matrix} ((\beth \circ \mathcal{P}_1))^{\propto} + ((\beth \circ \mathcal{P}_2))^{\propto} \\ -((\beth \circ \mathcal{P}_1))^{\propto} \cdot ((\beth \circ \mathcal{P}_2))^{\propto} \end{matrix}}}, & \overset{\propto}{\sqrt{\begin{matrix} ((\beth \circ \mathcal{X}_1))^{\propto} + ((\beth \circ \mathcal{X}_2))^{\propto} \\ -((\beth \circ \mathcal{X}_1))^{\propto} \cdot ((\beth \circ \mathcal{X}_2))^{\propto} \end{matrix}}} \\ ((\beth \circ \mathcal{Q}_1))^{\propto}((\beth \circ \mathcal{Q}_2))^{\propto}, & ((\beth \circ \mathcal{Y}_1))^{\propto}((\beth \circ \mathcal{Y}_2))^{\propto} \end{bmatrix},$$

$$(2)\ \Gamma_1 \oslash \Gamma_2 = \begin{bmatrix} ((\beth \circ \mathcal{O}_1))^{\propto}((\beth \circ \mathcal{O}_2))^{\propto}, ((\beth \circ \mathcal{V}_1))^{\propto}((\beth \circ \mathcal{V}_2))^{\propto}, \\ \sqrt[\propto]{\begin{matrix}((\beth \circ \mathcal{P}_1))^{\propto} + ((\beth \circ \mathcal{P}_2))^{\propto} \\ -((\beth \circ \mathcal{P}_1))^{\propto} \cdot ((\beth \circ \mathcal{P}_2))^{\propto}\end{matrix}}, \sqrt[\propto]{\begin{matrix}((\beth \circ \mathcal{X}_1))^{\propto} + ((\beth \circ \mathcal{X}_2))^{\propto} \\ -((\beth \circ \mathcal{X}_1))^{\propto} \cdot ((\beth \circ \mathcal{X}_2))^{\propto}\end{matrix}} \\ \sqrt[\propto]{\begin{matrix}((\beth \circ \mathcal{Q}_1))^{\propto} + ((\beth \circ \mathcal{Q}_2))^{\propto} \\ -((\beth \circ \mathcal{Q}_1))^{\propto} \cdot ((\beth \circ \mathcal{Q}_2))^{\propto}\end{matrix}}, \sqrt[\propto]{\begin{matrix}((\beth \circ \mathcal{Y}_1))^{\propto} + ((\beth \circ \mathcal{Y}_2))^{\propto} \\ -((\beth \circ \mathcal{Y}_1))^{\propto} \cdot ((\beth \circ \mathcal{Y}_2))^{\propto}\end{matrix}} \end{bmatrix}$$

$$(3)\ \partial \cdot \Gamma = \begin{bmatrix} \sqrt[\propto]{1-\left(1-(\beth \circ (\mathcal{O})^{\propto}\right)^{\partial}}, \sqrt[\propto]{1-\left(1-(\beth \circ (\mathcal{V})^{\propto}\right)^{\partial}}, \\ \sqrt[\propto]{1-\left(1-(\beth \circ (\mathcal{P})^{\propto}\right)^{\partial}}, \sqrt[\propto]{1-\left(1-(\beth \circ (\mathcal{X})^{\propto}\right)^{\partial}}, \\ ((\beth \circ (\mathcal{Q})^{\propto})^{\partial}, ((\beth \circ (\mathcal{Y})^{\propto})^{\partial} \end{bmatrix}$$

$$(4)\ \Gamma^{\partial} = \begin{bmatrix} ((\beth \circ (\mathcal{O})^{\propto})^{\partial}, ((\beth \circ (\mathcal{V})^{\propto})^{\partial}, \\ \sqrt[\propto]{1-\left(1-(\beth \circ (\mathcal{P})^{\propto}\right)^{\partial}}, \sqrt[\propto]{1-\left(1-(\beth \circ (\mathcal{X})^{\propto}\right)^{\partial}}, \\ \sqrt[\propto]{1-\left(1-(\beth \circ (\mathcal{Q})^{\propto}\right)^{\partial}}, \sqrt[\propto]{1-\left(1-(\beth \circ (\mathcal{Y})^{\propto}\right)^{\partial}} \end{bmatrix}.$$

We present ED and HD measures for IVT $\propto$ NNs and investigate their mathematical characteristics.

Definition 3.3. For any two IVT $\propto$ NNs $\Gamma_1 = \langle([(\beth \circ \mathcal{O}_1), (\beth \circ \mathcal{V}_1)], [(\beth \circ \mathcal{P}_1), (\beth \circ \mathcal{X}_1)], [(\beth \circ \mathcal{Q}_1), (\beth \circ \mathcal{Y}_1)])\rangle, \Gamma_2 = \langle([(\beth \circ \mathcal{O}_2), (\beth \circ \mathcal{V}_2)], [(\beth \circ \mathcal{P}_2), (\beth \circ \mathcal{X}_2)], [(\beth \circ \mathcal{Q}_2), (\beth \circ \mathcal{Y}_2)])\rangle$. Then

$$\Lambda_E(\Gamma_1, \Gamma_2) = \sqrt{\frac{1}{2}\left[\begin{bmatrix} 1 + ((\beth \circ \mathcal{O}_1))^2 - ((\beth \circ \mathcal{P}_1))^2 - ((\beth \circ \mathcal{Q}_1))^2 \\ -\left(1 + ((\beth \circ \mathcal{O}_2))^2 - ((\beth \circ \mathcal{P}_2))^2 - ((\beth \circ \mathcal{Q}_2))^2\right) \end{bmatrix}^2 + \begin{bmatrix} ((\beth \circ \mathcal{V}_1))^2 - ((\beth \circ \mathcal{X}_1))^2 - ((\beth \circ \mathcal{Y}_1))^2 \\ -\left(((\beth \circ \mathcal{V}_2))^2 - ((\beth \circ \mathcal{X}_2))^2 - ((\beth \circ \mathcal{Y}_2))^2\right) \end{bmatrix}^2\right]}$$

where $\Lambda_E(\Gamma_1, \Gamma_2)$ is called the ED between Γ_1 and Γ_2.

$$\Lambda_H(\Gamma_1, \Gamma_2) = \frac{1}{2}\left[\begin{vmatrix} 1 + ((\beth \circ \mathcal{O}_1))^2 - ((\beth \circ \mathcal{P}_1))^2 - ((\beth \circ \mathcal{Q}_1))^2 \\ -\left(1 + ((\beth \circ \mathcal{O}_2))^2 - ((\beth \circ \mathcal{P}_2))^2 - ((\beth \circ \mathcal{Q}_2))^2\right) \end{vmatrix} + \begin{vmatrix} ((\beth \circ \mathcal{V}_1))^2 - ((\beth \circ \mathcal{X}_1))^2 - ((\beth \circ \mathcal{Y}_1))^2 \\ \left(-((\beth \circ \mathcal{V}_2))^2 - ((\beth \circ \mathcal{X}_2))^2 - ((\beth \circ \mathcal{Y}_2))^2\right) \end{vmatrix}\right]$$

where $\Lambda_H(\Gamma_1, \Gamma_2)$ is called the HD between Γ_1 and Γ_2.

4. **Aggregating operators**

We use IVT $\propto$ NWA, IVT $\propto$ NWG, GIVT $\propto$ NWA, and GIVT $\propto$ NWG to describe the AOs.

4.1. *IVT qNWA*

Definition 4.1. Let $\Gamma_i = \langle(([(\beth \circ \mathcal{O}_i), (\beth \circ \mathcal{V}_i)], [(\beth \circ \mathcal{P}_i), (\beth \circ \mathcal{X}_i)], [(\beth \circ \mathcal{Q}_i), (\beth \circ \mathcal{Y}_i)]))\rangle$ be the IVT $\propto$ NNs, $W = (\eth_1, \eth_2, ..., \eth_n)$ be the weight of Γ_i, $\eth_i \geq 0$ and $\curlyvee_{i=1}^{n} \eth_i = 1$. Then IVT $\propto$ NWA $(\Gamma_1, \Gamma_2, ..., \Gamma_n) = \curlyvee_{i=1}^{n} \eth_i \Gamma_i$.

Theorem 4.2. *Let* $\Gamma_i = \langle(([(\beth \circ \mathcal{O}_i), (\beth \circ \mathcal{V}_i)], [(\beth \circ \mathcal{P}_i), (\beth \circ \mathcal{X}_i)], [(\beth \circ \mathcal{Q}_i), (\beth \circ \mathcal{Y}_i)]))\rangle$ *be the IVT* $\propto$ *NNs. Then IVT qNWA*$(\Gamma_1, \Gamma_2, ..., \Gamma_n)$

$$= \begin{bmatrix} \sqrt[\propto]{1 - \curlywedge_{i=1}^{n}\left(1 - ((\beth \circ \mathcal{O}_i))^{\propto}\right)^{\eth_i}}, \sqrt[\propto]{1 - \curlywedge_{i=1}^{n}\left(1 - ((\beth \circ \mathcal{V}_i))^{\propto}\right)^{\eth_i}}, \\ \sqrt[\propto]{1 - \curlywedge_{i=1}^{n}\left(1 - ((\beth \circ \mathcal{P}_i))^{\propto}\right)^{\eth_i}}, \sqrt[\propto]{1 - \curlywedge_{i=1}^{n}\left(1 - ((\beth \circ \mathcal{X}_i))^{\propto}\right)^{\eth_i}}, \\ \curlywedge_{i=1}^{n}(((\beth \circ \mathcal{Q}_i))^{\propto})^{\eth_i}, \curlywedge_{i=1}^{n}(((\beth \circ \mathcal{Y}_i))^{\propto})^{\eth_i} \end{bmatrix}.$$

Proof If $n = 2$, then IVT $\propto$ NWA$(\Gamma_1, \Gamma_2) = \eth_1\Gamma_1 \curlyvee \eth_2\Gamma_2$, where

$$\eth_1\Gamma_1 = \begin{bmatrix} \sqrt[\propto]{1 - \left(1 - ((\beth \circ \mathcal{O}_1))^{\propto}\right)^{\eth_1}}, \sqrt[\propto]{1 - \left(1 - ((\beth \circ \mathcal{V}_1))^{\propto}\right)^{\eth_1}}, \\ \sqrt[\propto]{1 - \left(1 - ((\beth \circ \mathcal{P}_1))^{\propto}\right)^{\eth_1}}, \sqrt[\propto]{1 - \left(1 - ((\beth \circ \mathcal{X}_1))^{\propto}\right)^{\eth_1}}, \\ (((\beth \circ \mathcal{Q}_1))^{\propto})^{\eth_1}, (((\beth \circ \mathcal{Y}_1))^{\propto})^{\eth_1} \end{bmatrix}$$

$$\eth_2\Gamma_2 = \begin{bmatrix} \sqrt[\propto]{1 - \left(1 - ((\beth \circ \mathcal{O}_2))^{\propto}\right)^{\eth_2}}, \sqrt[\propto]{1 - \left(1 - ((\beth \circ \mathcal{V}_2))^{\propto}\right)^{\eth_2}}, \\ \sqrt[\propto]{1 - \left(1 - ((\beth \circ \mathcal{P}_2))^{\propto}\right)^{\eth_2}}, \sqrt[\propto]{1 - \left(1 - ((\beth \circ \mathcal{X}_2))^{\propto}\right)^{\eth_2}} \\ (((\beth \circ \mathcal{Q}_2))^{\propto})^{\eth_2}, (((\beth \circ \mathcal{Y}_2))^{\propto})^{\eth_2} \end{bmatrix}.$$

Now, $\eth_1\Gamma_1 \curlyvee \eth_2\Gamma_2$

$$= \begin{bmatrix} \propto\sqrt{\begin{matrix}\left(1 - \left(1 - ((\beth \circ \mathcal{O}_1))^{\propto}\right)^{\eth_1}\right) + \\ \left(1 - \left(1 - ((\beth \circ \mathcal{O}_2))^{\propto}\right)^{\eth_2}\right) \\ -\left(1 - \left(1 - ((\beth \circ \mathcal{O}_1))^{\propto}\right)^{\eth_1}\right)\cdot \\ \left(1 - \left(1 - ((\beth \circ \mathcal{O}_2))^{\propto}\right)^{\eth_2}\right)\end{matrix}}, \propto\sqrt{\begin{matrix}\left(1 - \left(1 - ((\beth \circ \mathcal{V}_1))^{\propto}\right)^{\eth_1}\right) + \\ \left(1 - \left(1 - ((\beth \circ \mathcal{V}_2))^{\propto}\right)^{\eth_2}\right) \\ -\left(1 - \left(1 - ((\beth \circ \mathcal{V}_1))^{\propto}\right)^{\eth_1}\right)\cdot \\ \left(1 - \left(1 - ((\beth \circ \mathcal{V}_2))^{\propto}\right)^{\eth_2}\right)\end{matrix}}, \\ \propto\sqrt{\begin{matrix}\left(1 - \left(1 - ((\beth \circ \mathcal{P}_1))^{\propto}\right)^{\eth_1}\right) + \\ \left(1 - \left(1 - ((\beth \circ \mathcal{P}_2))^{\propto}\right)^{\eth_2}\right) \\ -\left(1 - \left(1 - ((\beth \circ \mathcal{P}_1))^{\propto}\right)^{\eth_1}\right)\cdot \\ \left(1 - \left(1 - ((\beth \circ \mathcal{P}_2))^{\propto}\right)^{\eth_2}\right)\end{matrix}}, \propto\sqrt{\begin{matrix}\left(1 - \left(1 - ((\beth \circ \mathcal{X}_1))^{\propto}\right)^{\eth_1}\right) + \\ \left(1 - \left(1 - ((\beth \circ \mathcal{X}_2))^{\propto}\right)^{\eth_2}\right) \\ -\left(1 - \left(1 - ((\beth \circ \mathcal{X}_1))^{\propto}\right)^{\eth_1}\right)\cdot \\ \left(1 - \left(1 - ((\beth \circ \mathcal{X}_2))^{\propto}\right)^{\eth_2}\right),\end{matrix}} \\ (((\beth \circ \mathcal{Q}_1))^{\propto})^{\eth_1}, (((\beth \circ \mathcal{Q}_2))^{\propto})^{\eth_2}, (((\beth \circ \mathcal{Y}_1))^{\propto})^{\eth_1}, (((\beth \circ \mathcal{Y}_2))^{\propto})^{\eth_2} \end{bmatrix}$$

$$= \begin{bmatrix} \sqrt[\propto]{1-\left(1-((\lrcorner\circ\mathcal{O}_1))^{\propto}\right)^{\eth_1}\left(1-((\lrcorner\circ\mathcal{O}_2))^{\propto}\right)^{\eth_2}}, \\ \sqrt[\propto]{1-\left(1-((\lrcorner\circ\mathcal{V}_1))^{\propto}\right)^{\eth_1}\left(1-((\lrcorner\circ\mathcal{V}_2))^{\propto}\right)^{\eth_2}}, \\ \sqrt[\propto]{1-\left(1-((\lrcorner\circ\mathcal{P}_1))^{\propto}\right)^{\eth_1}\left(1-((\lrcorner\circ\mathcal{P}_2))^{\propto}\right)^{\eth_2}}, \\ \sqrt[\propto]{1-\left(1-((\lrcorner\circ\mathcal{X}_1))^{\propto}\right)^{\eth_1}\left(1-((\lrcorner\circ\mathcal{X}_2))^{\propto}\right)^{\eth_2}}, \\ (((\lrcorner\circ\mathcal{Q}_1))^{\propto})^{\eth_1}\cdot(((\lrcorner\circ\mathcal{Q}_2))^{\propto})^{\eth_2}, \\ (((\lrcorner\circ\mathcal{Y}_1))^{\propto})^{\eth_1}\cdot(((\lrcorner\circ\mathcal{Y}_2))^{\propto})^{\eth_2} \end{bmatrix}$$

Hence, IVT $qNWA(\Gamma_1,\Gamma_2)$

$$= \begin{bmatrix} \sqrt[\propto]{1-\curlywedge_{i=1}^{2}\left(1-((\lrcorner\circ\mathcal{O}_i))^{\propto}\right)^{\eth_i}}, \sqrt[\propto]{1-\curlywedge_{i=1}^{2}\left(1-((\lrcorner\circ\mathcal{V}_i))^{\propto}\right)^{\eth_i}}, \\ \sqrt[\propto]{1-\curlywedge_{i=1}^{2}\left(1-((\lrcorner\circ\mathcal{P}_i))^{\propto}\right)^{\eth_i}}, \sqrt[\propto]{1-\curlywedge_{i=1}^{2}\left(1-((\lrcorner\circ\mathcal{X}_i))^{\propto}\right)^{\eth_i}}, \\ \curlywedge_{i=1}^{2}(((\lrcorner\circ\mathcal{Q}_i))^{\propto})^{\eth_i}, \curlywedge_{i=1}^{2}(((\lrcorner\circ\mathcal{Y}_i))^{\propto})^{\eth_i} \end{bmatrix}.$$

It valid for $n \geq 3$. Thus, IVT $qNWA(\Gamma_1,\Gamma_2,...,\Gamma_l)$

$$= \begin{bmatrix} \sqrt[\propto]{1-\curlywedge_{i=1}^{l}\left(1-((\lrcorner\circ\mathcal{O}_i))^{\propto}\right)^{\eth_i}}, \sqrt[\propto]{1-\curlywedge_{i=1}^{l}\left(1-((\lrcorner\circ\mathcal{V}_i))^{\propto}\right)^{\eth_i}}, \\ \sqrt[\propto]{1-\curlywedge_{i=1}^{l}\left(1-((\lrcorner\circ\mathcal{P}_i))^{\propto}\right)^{\eth_i}}, \sqrt[\propto]{1-\curlywedge_{i=1}^{l}\left(1-((\lrcorner\circ\mathcal{X}_i))^{\propto}\right)^{\eth_i}}, \\ \curlywedge_{i=1}^{l}(((\lrcorner\circ\mathcal{Q}_i))^{\propto})^{\eth_i}, \curlywedge_{i=1}^{l}(((\lrcorner\circ\mathcal{Y}_i))^{\propto})^{\eth_i} \end{bmatrix}.$$

If $n = l+1$, then IVT $\propto$ NWA $(\Gamma_1,\Gamma_2,...,\Gamma_l,\Gamma_{l+1})$

$$= \begin{bmatrix} \sqrt[\propto]{\begin{array}{c}\curlyvee_{i=1}^{l}\left(1-\left(1-((\lrcorner\circ\mathcal{O}_i))^{\propto}\right)^{\eth_i}\right)+\left(1-\left(1-(\mathcal{O}_{l+1})^{\propto}\right)^{\eth_{l+1}}\right) \\ -\curlywedge_{i=1}^{l}\left(1-\left(1-((\lrcorner\circ\mathcal{O}_i))^{\propto}\right)^{\eth_i}\right)\cdot\left(1-\left(1-(\mathcal{O}_{l+1})^{\propto}\right)^{\eth_{l+1}}\right)\end{array}}, \\ \sqrt[\propto]{\begin{array}{c}\curlyvee_{i=1}^{l}\left(1-\left(1-((\lrcorner\circ\mathcal{V}_i))^{\propto}\right)^{\eth_i}\right)+\left(1-\left(1-(\mathcal{V}_{l+1})^{\propto}\right)^{\eth_{l+1}}\right) \\ -\curlywedge_{i=1}^{l}\left(1-\left(1-((\lrcorner\circ\mathcal{V}_i))^{\propto}\right)^{\eth_i}\right)\cdot\left(1-\left(1-(\mathcal{V}_{l+1})^{\propto}\right)^{\eth_{l+1}}\right),\end{array}} \\ \sqrt[\propto]{\begin{array}{c}\curlyvee_{i=1}^{l}\left(1-\left(1-((\lrcorner\circ\mathcal{P}_i))^{\propto}\right)^{\eth_i}\right)+\left(1-\left(1-(\mathcal{P}_{l+1})^{\propto}\right)^{\eth_{l+1}}\right) \\ -\curlywedge_{i=1}^{l}\left(1-\left(1-((\lrcorner\circ\mathcal{P}_i))^{\propto}\right)^{\eth_i}\right)\cdot\left(1-\left(1-(\mathcal{P}_{l+1})^{\propto}\right)^{\eth_{l+1}}\right),\end{array}} \\ \sqrt[\propto]{\begin{array}{c}\curlyvee_{i=1}^{l}\left(1-\left(1-((\lrcorner\circ\mathcal{X}_i))^{\propto}\right)^{\eth_i}\right)+\left(1-\left(1-(\mathcal{X}_{l+1})^{\propto}\right)^{\eth_{l+1}}\right) \\ -\curlywedge_{i=1}^{l}\left(1-\left(1-((\lrcorner\circ\mathcal{X}_i))^{\propto}\right)^{\eth_i}\right)\cdot\left(1-\left(1-(\mathcal{X}_{l+1})^{\propto}\right)^{\eth_{l+1}}\right),\end{array}} \\ \curlywedge_{i=1}^{l}(((\lrcorner\circ\mathcal{Q}_i))^{\propto})^{\eth_i}, ((\mathcal{Q}_{l+1})^{\propto})^{\eth_{l+1}}, \curlywedge_{i=1}^{l}(((\lrcorner\circ\mathcal{Y}_i))^{\propto})^{\eth_i}, ((\mathcal{Y}_{l+1})^{\propto})^{\eth_{l+1}} \end{bmatrix}$$

$$= \begin{bmatrix} \sqrt[\propto]{1-\curlywedge_{i=1}^{l+1}\left(1-((\lrcorner\circ\mathcal{O}_i))^{\propto}\right)^{\eth_i}}, \sqrt[\propto]{1-\curlywedge_{i=1}^{l+1}\left(1-((\lrcorner\circ\mathcal{V}_i))^{\propto}\right)^{\eth_i}}, \\ \sqrt[\propto]{1-\curlywedge_{i=1}^{l+1}\left(1-((\lrcorner\circ\mathcal{P}_i))^{\propto}\right)^{\eth_i}}, \sqrt[\propto]{1-\curlywedge_{i=1}^{l+1}\left(1-((\lrcorner\circ\mathcal{X}_i))^{\propto}\right)^{\eth_i}}, \\ \curlywedge_{i=1}^{l+1}(((\lrcorner\circ\mathcal{Q}_i))^{\propto})^{\eth_i}, \curlywedge_{i=1}^{l+1}(((\lrcorner\circ\mathcal{Y}_i))^{\propto})^{\eth_i} \end{bmatrix}$$

Theorem 4.3. *Let* $\Gamma_i = \langle(([(\lrcorner\circ\mathcal{O}_i),(\lrcorner\circ\mathcal{V}_i)],[(\lrcorner\circ\mathcal{P}_i),(\lrcorner\circ\mathcal{X}_i)],\ [(\lrcorner\circ\mathcal{Q}_i),(\lrcorner\circ\mathcal{Y}_i)]))\rangle$ *be the IVT* $\propto$ *NNs. Then IVT* $\propto$ *NWA* $(\Gamma_1,\Gamma_2,...,\Gamma_n)=\Gamma$ *(idempotency property).*

Proof Since $(\lrcorner\circ\mathcal{O}_i)=(\lrcorner\circ\mathcal{O})$, $(\lrcorner\circ\mathcal{P}_i)=(\lrcorner\circ\mathcal{P})$ and $(\lrcorner\circ\mathcal{Q}_i)=(\lrcorner\circ\mathcal{Q})$ and $(\lrcorner\circ\mathcal{V}_i)=(\lrcorner\circ\mathcal{V})$, $(\lrcorner\circ\mathcal{X}_i)=(\lrcorner\circ\mathcal{X})$ and $(\lrcorner\circ\mathcal{Y}_i)=(\lrcorner\circ\mathcal{Y})$ and $\curlyvee_{i=1}^{n}\eth_i=1$. Now, IVT $qNWA(\Gamma_1,\Gamma_2,...,\Gamma_n)$

$$= \begin{bmatrix} \sqrt[\propto]{1-\curlywedge_{i=1}^{n}\left(1-((\lrcorner\circ\mathcal{O}_i))^{\propto}\right)^{\eth_i}}, \sqrt[\propto]{1-\curlywedge_{i=1}^{n}\left(1-((\lrcorner\circ\mathcal{V}_i))^{\propto}\right)^{\eth_i}}, \\ \sqrt[\propto]{1-\curlywedge_{i=1}^{n}\left(1-((\lrcorner\circ\mathcal{P}_i))^{\propto}\right)^{\eth_i}}, \sqrt[\propto]{1-\curlywedge_{i=1}^{n}\left(1-((\lrcorner\circ\mathcal{X}_i))^{\propto}\right)^{\eth_i}}, \\ \curlywedge_{i=1}^{n}(((\lrcorner\circ\mathcal{Q}_i))^{\propto})^{\eth_i}, \curlywedge_{i=1}^{n}(((\lrcorner\circ\mathcal{Y}_i))^{\propto})^{\eth_i} \end{bmatrix}$$

$$= \begin{bmatrix} \sqrt[\propto]{1-\left(1-(\lrcorner\circ(\mathcal{O})^{\propto}\right)^{\curlyvee_{i=1}^{n}\eth_i}}, \sqrt[\propto]{1-\left(1-(\lrcorner\circ(\mathcal{V})^{\propto}\right)^{\curlyvee_{i=1}^{n}\eth_i}}, \\ \sqrt[\propto]{1-\left(1-(\lrcorner\circ(\mathcal{P})^{\propto}\right)^{\curlyvee_{i=1}^{n}\eth_i}}, \sqrt[\propto]{1-\left(1-(\lrcorner\circ(\mathcal{X})^{\propto}\right)^{\curlyvee_{i=1}^{n}\eth_i}}, \\ ((\lrcorner\circ(\mathcal{Q})^{\propto})^{\curlyvee_{i=1}^{n}\eth_i}, ((\lrcorner\circ(\mathcal{Y})^{\propto})^{\curlyvee_{i=1}^{n}\eth_i} \end{bmatrix}$$

$$= \begin{bmatrix} \sqrt[\propto]{1-\left(1-(\lrcorner\circ(\mathcal{O})^{\propto}\right)}, \sqrt[\propto]{1-\left(1-(\lrcorner\circ(\mathcal{V})^{\propto}\right)}, \\ \sqrt[\propto]{1-\left(1-(\lrcorner\circ(\mathcal{P})^{\propto}\right)}, \sqrt[\propto]{1-\left(1-(\lrcorner\circ(\mathcal{X})^{\propto}\right)}, \\ (\lrcorner\circ(\mathcal{Q})^{\propto}, (\lrcorner\circ(\mathcal{Y})^{\propto} \end{bmatrix}$$

$$= \Gamma.$$

Theorem 4.4. *Let* $\Gamma_i = \langle(([(\lrcorner\circ\mathcal{O}_i),(\lrcorner\circ\mathcal{V}_i)],[(\lrcorner\circ\mathcal{P}_i),(\lrcorner\circ\mathcal{X}_i)],\ [(\lrcorner\circ\mathcal{Q}_i),(\lrcorner\circ\mathcal{Y}_i)]))\rangle$ *be the IVT* $\propto$ *NNs. Then IVT* $\propto$ *NWA*$(\Gamma_1,\Gamma_2,...,\Gamma_n)$
where $\underline{(\lrcorner\circ\mathcal{O})}=\min(\lrcorner\circ\mathcal{O}_{ij})$, $\overline{(\lrcorner\circ\mathcal{O})}=\max(\lrcorner\circ\mathcal{O}_{ij})$, $\underline{(\lrcorner\circ\mathcal{P})}=\min(\lrcorner\circ\mathcal{P}_{ij})$, $\overline{(\lrcorner\circ\mathcal{P})}=\max(\lrcorner\circ\mathcal{P}_{ij})$, $\underline{(\lrcorner\circ\mathcal{Q})}=\min(\lrcorner\circ\mathcal{Q}_{ij})$, $\overline{(\lrcorner\circ\mathcal{Q})}=\max(\lrcorner\circ\mathcal{Q}_{ij})$
and $\underline{(\lrcorner\circ\mathcal{V})}=\min(\lrcorner\circ\mathcal{V}_{ij})$, $\overline{(\lrcorner\circ\mathcal{V})}=\max(\lrcorner\circ\mathcal{V}_{ij})$, $\underline{(\lrcorner\circ\mathcal{X})}=\min(\lrcorner\circ\mathcal{X}_{ij})$, $\overline{(\lrcorner\circ\mathcal{X})}=\max(\lrcorner\circ\mathcal{X}_{ij})$, $\underline{(\lrcorner\circ\mathcal{Y})}=\min(\lrcorner\circ\mathcal{Y}_{ij})$, $\overline{(\lrcorner\circ\mathcal{Y})}=\max(\lrcorner\circ\mathcal{Y}_{ij})$ *and where* $1\le i\le n,\ j=1,2,...,i_j$. *Then,*
$\left\langle\underline{(\lrcorner\circ\mathcal{O}),(\lrcorner\circ\mathcal{V})},\underline{(\lrcorner\circ\mathcal{P}),(\lrcorner\circ\mathcal{X})},\overline{(\lrcorner\circ\mathcal{Q}),(\lrcorner\circ\mathcal{Y})}\right\rangle$

$$\le IVTqNWA(\Gamma_1,\Gamma_2,...,\Gamma_n)$$
$$\le \left\langle\overline{(\lrcorner\circ\mathcal{O}),(\lrcorner\circ\mathcal{V})},\overline{(\lrcorner\circ\mathcal{P}),(\lrcorner\circ\mathcal{X})},\underline{(\lrcorner\circ\mathcal{Q}),(\lrcorner\circ\mathcal{Y})}\right\rangle.$$

(Boundedness property).

Proof Since, $\underline{(\beth \circ \mathcal{O})} = \min(\beth \circ \mathcal{O}_{ij})$, $\overline{(\beth \circ \mathcal{O})} = \max(\beth \circ \mathcal{O}_{ij})$ and $\underline{(\beth \circ \mathcal{O})} \leq (\beth \circ \mathcal{O}_{ij}) \leq \overline{(\beth \circ \mathcal{O})}$ and $\underline{(\beth \circ \mathcal{V})} = \min(\beth \circ \mathcal{V}_{ij})$, $\overline{(\beth \circ \mathcal{V})} = \max(\beth \circ \mathcal{V}_{ij})$ and $\underline{(\beth \circ \mathcal{V})} \leq (\beth \circ \mathcal{V}_{ij}) \leq \overline{(\beth \circ \mathcal{V})}$.

Now $\underline{(\beth \circ \mathcal{O})}, \underline{(\beth \circ \mathcal{V})}$

$$\begin{aligned}
&= \sqrt[\propto]{1 - \curlywedge_{i=1}^{n}\left(1 - (\underline{(\beth \circ \mathcal{O})})^{\propto}\right)^{\eth_i}}, \sqrt[\propto]{1 - \curlywedge_{i=1}^{n}\left(1 - (\underline{(\beth \circ \mathcal{V})})^{\propto}\right)^{\eth_i}} \\
&\leq \sqrt[\propto]{1 - \curlywedge_{i=1}^{n}\left(1 - (((\beth \circ \mathcal{O}_{ij})))^{\propto}\right)^{\eth_i}}, \sqrt[\propto]{1 - \curlywedge_{i=1}^{n}\left(1 - (((\beth \circ \mathcal{V}_{ij})))^{\propto}\right)^{\eth_i}} \\
&\leq \sqrt[\propto]{1 - \curlywedge_{i=1}^{n}\left(1 - (\overline{(\beth \circ \mathcal{O})})^{\propto}\right)^{\eth_i}}, \sqrt[\propto]{1 - \curlywedge_{i=1}^{n}\left(1 - (\overline{(\beth \circ \mathcal{V})})^{\propto}\right)^{\eth_i}} \\
&= \overline{(\beth \circ \mathcal{O})}.
\end{aligned}$$

Since, $\underline{(\beth \circ \mathcal{P})} = \min(\beth \circ \mathcal{P}_{ij})$, $\overline{(\beth \circ \mathcal{P})} = \max(\beth \circ \mathcal{P}_{ij})$ and $\underline{(\beth \circ \mathcal{P})} \leq (\beth \circ \mathcal{P}_{ij}) \leq \overline{(\beth \circ \mathcal{P})}$ and $\underline{(\beth \circ \mathcal{X})} = \min(\beth \circ \mathcal{X}_{ij})$, $\overline{(\beth \circ \mathcal{X})} = \max(\beth \circ \mathcal{X}_{ij})$ and $\underline{(\beth \circ \mathcal{X})} \leq (\beth \circ \mathcal{X}_{ij}) \leq \overline{(\beth \circ \mathcal{X})}$.

Now, $\underline{(\beth \circ \mathcal{P})}, \underline{(\beth \circ \mathcal{X})}$

$$\begin{aligned}
&= \sqrt[\propto]{1 - \curlywedge_{i=1}^{n}\left(1 - (\underline{(\beth \circ \mathcal{P})})^{\propto}\right)^{\eth_i}}, \sqrt[\propto]{1 - \curlywedge_{i=1}^{n}\left(1 - (\underline{(\beth \circ \mathcal{X})})^{\propto}\right)^{\eth_i}} \\
&\leq \sqrt[\propto]{1 - \curlywedge_{i=1}^{n}\left(1 - ((\beth \circ \mathcal{P}_{ij}))^{\propto}\right)^{\eth_i}}, \sqrt[\propto]{1 - \curlywedge_{i=1}^{n}\left(1 - ((\beth \circ \mathcal{X}_{ij}))^{\propto}\right)^{\eth_i}} \\
&\leq \sqrt[\propto]{1 - \curlywedge_{i=1}^{n}\left(1 - (\overline{(\beth \circ \mathcal{P})})^{\propto}\right)^{\eth_i}}, \sqrt[\propto]{1 - \curlywedge_{i=1}^{n}\left(1 - (\overline{(\beth \circ \mathcal{X})})^{\propto}\right)^{\eth_i}} \\
&= \overline{(\beth \circ \mathcal{P})}, \overline{(\beth \circ \mathcal{X})}.
\end{aligned}$$

Since, $\underline{(\beth \circ (\mathcal{Q})^{\propto})} = \min((\beth \circ \mathcal{Q}_{ij}))^{\propto}$, $\overline{(\beth \circ (\mathcal{Q})^{\propto})} = \max((\beth \circ \mathcal{Q}_{ij}))^{\propto}$ and $\underline{(\beth \circ (\mathcal{Q})^{\propto})} \leq ((\beth \circ \mathcal{Q}_{ij}))^{\propto} \leq \overline{(\beth \circ (\mathcal{Q})^{\propto})}$ and $\underline{(\beth \circ (\mathcal{Y})^{\propto})} = \min((\beth \circ \mathcal{Y}_{ij}))^{\propto}$, $\overline{(\beth \circ (\mathcal{Y})^{\propto})} = \max((\beth \circ \mathcal{Y}_{ij}))^{\propto}$ and $\underline{(\beth \circ (\mathcal{Y})^{\propto})} \leq ((\beth \circ \mathcal{Y}_{ij}))^{\propto} \leq \overline{(\beth \circ (\mathcal{Y})^{\propto})}$.

We have,

$$\begin{aligned}
\underline{(\beth \circ (\mathcal{Q})^{\propto}} &= \curlywedge_{i=1}^{n}\underline{(\beth \circ (\mathcal{Q})^{\propto})}^{\eth_i}, \curlywedge_{i=1}^{n}\underline{(\beth \circ (\mathcal{Y})^{\propto})}^{\eth_i} \\
&\leq \curlywedge_{i=1}^{n}(((\beth \circ \mathcal{Q}_{ij}))^{\propto})^{\eth_i}, \curlywedge_{i=1}^{n}(((\beth \circ \mathcal{Y}_{ij}))^{\propto})^{\eth_i} \\
&\leq \curlywedge_{i=1}^{n}\overline{(\beth \circ (\mathcal{Q})^{\propto})}^{\eth_i}, \curlywedge_{i=1}^{n}\overline{(\beth \circ (\mathcal{Y})^{\propto})}^{\eth_i} \\
&= \overline{(\beth \circ (\mathcal{Q})^{\propto})}, \overline{(\beth \circ (\mathcal{Y})^{\propto})}.
\end{aligned}$$

Therefore,

$$\frac{1}{2}\times\left[\begin{array}{c}\left[\begin{array}{c}\left(\sqrt[\propto]{1-\curlywedge_{i=1}^{n}\left(1-(\underline{(\beth\circ\mathcal{O})})^{\propto}\right)^{\eth_i}}\right)^2-\left(\sqrt[\propto]{1-\curlywedge_{i=1}^{n}\left(1-(\underline{(\beth\circ\mathcal{P})})^{\propto}\right)^{\eth_i}}\right)^2\\ +1-\left(\curlywedge_{i=1}^{n}((\overline{(\beth\circ\mathcal{Q})})^{\propto})^{\eth_i}\right)^2\end{array}\right]\\ +\left[\begin{array}{c}\left(\sqrt[\propto]{1-\curlywedge_{i=1}^{n}\left(1-(\underline{(\beth\circ\mathcal{V})})^{\propto}\right)^{\eth_i}}\right)^2-\left(\sqrt[\propto]{1-\curlywedge_{i=1}^{n}\left(1-(\underline{(\beth\circ\mathcal{X})})^{\propto}\right)^{\eth_i}}\right)^2\\ -\left(\curlywedge_{i=1}^{n}((\overline{(\beth\circ\mathcal{Y})})^{\propto})^{\eth_i}\right)^2\end{array}\right]\end{array}\right]$$

$$\leq\ \frac{1}{2}\times\left[\begin{array}{c}\left[\begin{array}{c}\left(\sqrt[\propto]{1-\curlywedge_{i=1}^{n}\left(1-((\beth\circ(\beth\circ\mathcal{O}_{ij})))^{\propto}\right)^{\eth_i}}\right)^2-\left(\sqrt[\propto]{1-\curlywedge_{i=1}^{n}\left(1-((\beth\circ\mathcal{P}_{ij}))^{\propto}\right)^{\eth_i}}\right)^2+\\ +1-\left(\curlywedge_{i=1}^{n}(((\beth\circ\mathcal{Q}_{ij}))^{\propto})^{\eth_i}\right)^2\end{array}\right]\\ +\left[\begin{array}{c}\left(\sqrt[\propto]{1-\curlywedge_{i=1}^{n}\left(1-((\beth\circ(\beth\circ\mathcal{V}_{ij})))^{\propto}\right)^{\eth_i}}\right)^2-\left(\sqrt[\propto]{1-\curlywedge_{i=1}^{n}\left(1-((\beth\circ\mathcal{X}_{ij}))^{\propto}\right)^{\eth_i}}\right)^2+\\ -\left(\curlywedge_{i=1}^{n}(((\beth\circ\mathcal{Y}_{ij}))^{\propto})^{\eth_i}\right)^2\end{array}\right]\end{array}\right]$$

$$\leq\ \frac{1}{2}\times\left[\begin{array}{c}\left[\begin{array}{c}\left(\sqrt[\propto]{1-\curlywedge_{i=1}^{n}\left(1-(\overline{(\beth\circ\mathcal{O})})^{\propto}\right)^{\eth_i}}\right)^2-\left(\sqrt[\propto]{1-\curlywedge_{i=1}^{n}\left(1-(\overline{(\beth\circ\mathcal{P})})^{\propto}\right)^{\eth_i}}\right)^2+\\ +1-\left(\curlywedge_{i=1}^{n}(\underline{(\beth\circ\mathcal{Q})})^{\propto})^{\eth_i}\right)^2\end{array}\right]\\ +\left[\begin{array}{c}\left(\sqrt[\propto]{1-\curlywedge_{i=1}^{n}\left(1-(\overline{(\beth\circ\mathcal{V})})^{\propto}\right)^{\eth_i}}\right)^2-\left(\sqrt[\propto]{1-\curlywedge_{i=1}^{n}\left(1-(\overline{(\beth\circ\mathcal{X})})^{\propto}\right)^{\eth_i}}\right)^2+\\ -\left(\curlywedge_{i=1}^{n}(\underline{(\beth\circ\mathcal{Y})})^{\propto})^{\eth_i}\right)^2\end{array}\right]\end{array}\right].$$

Hence,

$$\left\langle\underline{(\beth\circ\mathcal{O}),(\beth\circ\mathcal{V})},\underline{(\beth\circ\mathcal{P}),(\beth\circ\mathcal{X})},\overline{(\beth\circ\mathcal{Q}),(\beth\circ\mathcal{Y})}\right\rangle$$
$$\leq IVTqNWA(\Gamma_1,\Gamma_2,...,\Gamma_n)$$
$$\leq\langle\overline{(\beth\circ\mathcal{O}),(\beth\circ\mathcal{V})},\overline{(\beth\circ\mathcal{P}),(\beth\circ\mathcal{X})},\underline{(\beth\circ\mathcal{Q}),(\beth\circ\mathcal{Y})}\rangle.$$

Theorem 4.5. *Let* $\Gamma_i=\langle([(\beth\circ\mathcal{O}_{t_{ij}}),(\beth\circ\mathcal{V}_{t_{ij}})],[(\beth\circ\mathcal{P}_{t_{ij}}),(\beth\circ\mathcal{X}_{t_{ij}})],[(\beth\circ\mathcal{Q}_{t_{ij}}),(\beth\circ\mathcal{Y}_{t_{ij}})])\rangle$ *and* $W_i=\langle([(\beth\circ\mathcal{O}_{h_{ij}}),(\beth\circ\mathcal{V}_{h_{ij}})],[(\beth\circ\mathcal{P}_{h_{ij}}),(\beth\circ\mathcal{X}_{h_{ij}})],[(\beth\circ\mathcal{Q}_{h_{ij}}),(\beth\circ\mathcal{Y}_{h_{ij}})])\rangle$, *be the IVT* $\propto$ *NWAs. For any* i, *if there is* $(\beth\circ\mathcal{O}_{t_{ij}})^2\leq(\beth\circ\mathcal{O}_{h_{ij}})^2$ *and* $(\beth\circ\mathcal{P}_{t_{ij}})^2\leq(\beth\circ\mathcal{P}_{h_{ij}})^2$ *and* $(\beth\circ\mathcal{Q}_{t_{ij}})^2\geq(\beth\circ\mathcal{Q}_{h_{ij}})^2$ *and* $(\beth\circ\mathcal{V}_{t_{ij}})^2\leq(\beth\circ\mathcal{V}_{h_{ij}})^2$ *and* $(\beth\circ\mathcal{X}_{t_{ij}})^2\leq(\beth\circ\mathcal{X}_{h_{ij}})^2$ *and* $(\beth\circ\mathcal{Y}_{t_{ij}})^2\geq(\beth\circ\mathcal{Y}_{h_{ij}})^2$ *or* $\Gamma_i\leq W_i$. *Prove that IVT* $qNWA(\Gamma_1,\Gamma_2,...,\Gamma_n)\leq IVTqNWA(W_1,W_2,...,W_n)$, *where* $(i=1,2,...,n);(j=1,2,...,i_j)$ *(monotonicity property).*

Proof For any i, $(\beth\circ\mathcal{O}_{t_{ij}})^2\leq(\beth\circ\mathcal{O}_{h_{ij}})^2$.
Therefore, $1-((\beth\circ\mathcal{O}_{t_i}))^2\geq1-((\beth\circ\mathcal{O}_{h_i}))^2$.
Hence, $\curlywedge_{i=1}^{n}\left(1-((\beth\circ\mathcal{O}_{t_i}))^2\right)^{\eth_i}\geq\curlywedge_{i=1}^{n}\left(1-((\beth\circ\mathcal{O}_{h_i}))^2\right)^{\eth_i}$
and $\sqrt[\propto]{1-\curlywedge_{i=1}^{n}\left(1-((\beth\circ\mathcal{O}_{t_i}))^{\propto}\right)^{\eth_i}}\leq\sqrt[\propto]{1-\curlywedge_{i=1}^{n}\left(1-((\beth\circ\mathcal{O}_{h_i}))^{\propto}\right)^{\eth_i}}$.
Similarly, $(\beth\circ\mathcal{V}_{t_{ij}})^2\leq(\beth\circ\mathcal{V}_{h_{ij}})^2$.
Therefore, $1-((\beth\circ\mathcal{V}_{t_i}))^2\geq1-((\beth\circ\mathcal{V}_{h_i}))^2$.
Hence, $\curlywedge_{i=1}^{n}\left(1-((\beth\circ\mathcal{V}_{t_i}))^2\right)^{\eth_i}\geq\curlywedge_{i=1}^{n}\left(1-((\beth\circ\mathcal{V}_{h_i}))^2\right)^{\eth_i}$

and $\sqrt[\propto]{1-\curlywedge_{i=1}^{n}\left(1-((\beth\circ\mathcal{V}_{t_i}))^{\propto}\right)^{\eth_i}} \leq \sqrt[\propto]{1-\curlywedge_{i=1}^{n}\left(1-((\beth\circ\mathcal{V}_{h_i}))^{\propto}\right)^{\eth_i}}$.

For any i, $(\beth\circ\mathcal{P}_{t_{ij}})^{\propto} \leq \left((\beth\circ\mathcal{P}_{h_{ij}})\right)^{\propto}$.

Therefore, $1-((\beth\circ\mathcal{P}_{t_i}))^{\propto} \geq 1-((\beth\circ\mathcal{P}_{h_i}))^{\propto}$.

Hence, $\curlywedge_{i=1}^{n}\left(1-((\beth\circ\mathcal{P}_{t_i}))^{\propto}\right)^{\eth_i} \geq \curlywedge_{i=1}^{n}\left(1-((\beth\circ\mathcal{P}_{h_i}))^{\propto}\right)^{\eth_i}$.

This implies that $\sqrt[\propto]{1-\curlywedge_{i=1}^{n}\left(1-((\beth\circ\mathcal{P}_{t_i}))^{\propto}\right)^{\eth_i}} \leq \sqrt[\propto]{1-\curlywedge_{i=1}^{n}\left(1-((\beth\circ\mathcal{P}_{h_i}))^{\propto}\right)^{\eth_i}}$.

Similarly, for any i, $(\beth\circ\mathcal{X}_{t_{ij}})^{\propto} \leq \left((\beth\circ\mathcal{X}_{h_{ij}})\right)^{\propto}$.

Therefore, $1-((\beth\circ\mathcal{X}_{t_i}))^{\propto} \geq 1-((\beth\circ\mathcal{X}_{h_i}))^{\propto}$.

Hence, $\curlywedge_{i=1}^{n}\left(1-((\beth\circ\mathcal{X}_{t_i}))^{\propto}\right)^{\eth_i} \geq \curlywedge_{i=1}^{n}\left(1-((\beth\circ\mathcal{X}_{h_i}))^{\propto}\right)^{\eth_i}$.

This implies that $\sqrt[\propto]{1-\curlywedge_{i=1}^{n}\left(1-((\beth\circ\mathcal{X}_{t_i}))^{\propto}\right)^{\eth_i}} \leq \sqrt[\propto]{1-\curlywedge_{i=1}^{n}\left(1-((\beth\circ\mathcal{X}_{h_i}))^{\propto}\right)^{\eth_i}}$.

For any i, $\left((\beth\circ\mathcal{Q}_{t_{ij}})\right)^2 \geq \left((\beth\circ\mathcal{Q}_{h_{ij}})\right)^2$ and $\left((\beth\circ\mathcal{Q}_{t_{ij}})\right)^{\propto} \geq \left((\beth\circ\mathcal{Q}_{h_{ij}})\right)^{\propto}$.

Therefore, $1-\frac{\left(\curlywedge_{i=1}^{n}(\beth\circ\mathcal{Q}_{t_{ij}})\right)^{\propto}}{2} \leq 1-\frac{\left(\curlywedge_{i=1}^{n}(\beth\circ\mathcal{Q}_{h_{ij}})\right)^{\propto}}{2}$.

Similarly, for any i,

$\left((\beth\circ\mathcal{Y}_{t_{ij}})\right)^2 \geq \left((\beth\circ\mathcal{Y}_{h_{ij}})\right)^2$ and $\left((\beth\circ\mathcal{Y}_{t_{ij}})\right)^{\propto} \geq \left((\beth\circ\mathcal{Y}_{h_{ij}})\right)^{\propto}$.

Therefore, $-\left(\curlywedge_{i=1}^{n}(\beth\circ\mathcal{Y}_{t_{ij}})\right)^{\propto} \leq -\left(\curlywedge_{i=1}^{n}(\beth\circ\mathcal{Y}_{h_{ij}})\right)^{\propto}$.

Hence,

$$\frac{1}{2}\times\left[\begin{array}{l}\left[\begin{array}{c}\left(\sqrt[\propto]{1-\curlywedge_{i=1}^{n}\left(1-((\beth\circ\mathcal{O}_{ti}))^{\propto}\right)^{\eth_i}}\right)^2-\left(\sqrt[\propto]{1-\curlywedge_{i=1}^{n}\left(1-((\beth\circ\mathcal{P}_{t_i}))^{\propto}\right)^{\eth_i}}\right)^2\\ +1-\left(\curlywedge_{i=1}^{n}((\beth\circ\mathcal{Q}_{t_i}))^{\propto}\right)^2\end{array}\right]\\ +\left[\begin{array}{c}\left(\sqrt[\propto]{1-\curlywedge_{i=1}^{n}\left(1-((\beth\circ\mathcal{V}_{ti}))^{\propto}\right)^{\eth_i}}\right)^2-\left(\sqrt[\propto]{1-\curlywedge_{i=1}^{n}\left(1-((\beth\circ\mathcal{X}_{t_i}))^{\propto}\right)^{\eth_i}}\right)^2\\ -\left(\curlywedge_{i=1}^{n}((\beth\circ\mathcal{Y}_{t_i}))^{\propto}\right)^2\end{array}\right]\end{array}\right]$$

$$\leq\ \frac{1}{2}\times\left[\begin{array}{l}\left[\begin{array}{c}\left(\sqrt[\propto]{1-\curlywedge_{i=1}^{n}\left(1-((\beth\circ\mathcal{O}_{hi}))^{\propto}\right)^{\eth_i}}\right)^2-\left(\sqrt[\propto]{1-\curlywedge_{i=1}^{n}\left(1-((\beth\circ\mathcal{P}_{h_i}))^{\propto}\right)^{\eth_i}}\right)^2\\ +1-\left(\curlywedge_{i=1}^{n}((\beth\circ\mathcal{Q}_{h_i}))^{\propto}\right)^2\end{array}\right]\\ +\left[\begin{array}{c}\left(\sqrt[\propto]{1-\curlywedge_{i=1}^{n}\left(1-((\beth\circ\mathcal{V}_{hi}))^{\propto}\right)^{\eth_i}}\right)^2-\left(\sqrt[\propto]{1-\curlywedge_{i=1}^{n}\left(1-((\beth\circ\mathcal{X}_{h_i}))^{\propto}\right)^{\eth_i}}\right)^2\\ -\left(\curlywedge_{i=1}^{n}((\beth\circ\mathcal{Y}_{h_i}))^{\propto}\right)^2\end{array}\right]\end{array}\right].$$

Hence, IVT $qNWA\,(\Gamma_1,\Gamma_2,...,\Gamma_n) \leq IVTqNWA\,(W_1,W_2,...,W_n)$.

4.2. *IVT* $\propto$ *NWG*

Definition 4.6. Let $\Gamma_i = \langle(([(\beth\circ\mathcal{O}_i),(\beth\circ\mathcal{V}_i)],[(\beth\circ\mathcal{P}_i),(\beth\circ\mathcal{X}_i)],\ [(\beth\circ\mathcal{Q}_i),(\beth\circ\mathcal{Y}_i)]))\rangle$ be the IVT $\propto$ NNs. Then $\propto$NWG $(\Gamma_1,\Gamma_2,...,\Gamma_n) = \curlywedge_{i=1}^{n}\Gamma_i^{\eth_i}$.

Corollary 4.7. *Let* $\Gamma_i = \langle(([(\beth \circ \mathcal{O}_i), (\beth \circ \mathcal{V}_i)], [(\beth \circ \mathcal{P}_i), (\beth \circ \mathcal{X}_i)], [(\beth \circ \mathcal{Q}_i), (\beth \circ \mathcal{Y}_i)]))\rangle$ *be the IVT* $\propto$ *NNs. Then IVT* $\propto$ *NWG* $(\Gamma_1, \Gamma_2, ..., \Gamma_n)$

$$= \begin{bmatrix} \curlywedge_{i=1}^{n}(((\beth \circ \mathcal{O}_i))^{\propto})^{\eth_i}, \curlywedge_{i=1}^{n}(((\beth \circ \mathcal{V}_i))^{\propto})^{\eth_i} \\ \sqrt[\propto]{1 - \curlywedge_{i=1}^{n}\left(1 - ((\beth \circ \mathcal{P}_i))^{\propto}\right)^{\eth_i}}, \sqrt[\propto]{1 - \curlywedge_{i=1}^{n}\left(1 - ((\beth \circ \mathcal{X}_i))^{\propto}\right)^{\eth_i}} \\ \sqrt[\propto]{1 - \curlywedge_{i=1}^{n}\left(1 - ((\beth \circ \mathcal{Q}_i))^{\propto}\right)^{\eth_i}}, \sqrt[\propto]{1 - \curlywedge_{i=1}^{n}\left(1 - ((\beth \circ \mathcal{Y}_i))^{\propto}\right)^{\eth_i}} \end{bmatrix}.$$

Corollary 4.8. *Let* $\Gamma_i = \langle(([(\beth \circ \mathcal{O}_i), (\beth \circ \mathcal{V}_i)], [(\beth \circ \mathcal{P}_i), (\beth \circ \mathcal{X}_i)], [(\beth \circ \mathcal{Q}_i), (\beth \circ \mathcal{Y}_i)]))\rangle$ *be the IVT* $\propto$ *NNs and all are equal. Then* $\propto NWG(\Gamma_1, \Gamma_2, ..., \Gamma_n) = \Gamma$.

Corollary 4.9. *It has other properties, including boundedness and monotonicity, as well as having* $\propto NWG$.

4.3. *Generalized IVT* $\propto$ *NWA (GIVT*$\propto$*NWA)*

Definition 4.10. Let $\Gamma_i = \langle(([(\beth \circ \mathcal{O}_i), (\beth \circ \mathcal{V}_i)], [(\beth \circ \mathcal{P}_i), (\beth \circ \mathcal{X}_i)], [(\beth \circ \mathcal{Q}_i), (\beth \circ \mathcal{Y}_i)]))\rangle$ be the IVT $\propto$ NN. Then GIVT $\propto$ NWA $(\Gamma_1, \Gamma_2, ..., \Gamma_n) = \left(\curlyvee_{i=1}^{n} \eth_i \Gamma_i^{\partial}\right)^{1/\partial}$.

Theorem 4.11. *Let* $\Gamma_i = \langle(([(\beth \circ \mathcal{O}_i), (\beth \circ \mathcal{V}_i)], [(\beth \circ \mathcal{P}_i), (\beth \circ \mathcal{X}_i)], [(\beth \circ \mathcal{Q}_i), (\beth \circ \mathcal{Y}_i)]))\rangle$ *be the IVT* $\propto$ *NNs. Then GIVT* $\propto$ *NWA* $(\Gamma_1, \Gamma_2, ..., \Gamma_n)$

$$= \begin{bmatrix} \left(\sqrt[\propto]{1 - \curlywedge_{i=1}^{n}\left(1 - \left((\beth \circ \mathcal{O}_i))^{\propto}\right)^{\propto}\right)^{\eth_i}}\right)^{1/\propto}, \left(\sqrt[\propto]{1 - \curlywedge_{i=1}^{n}\left(1 - \left((\beth \circ \mathcal{V}_i))^{\propto}\right)^{\propto}\right)^{\eth_i}}\right)^{1/\propto}, \\ \left(\sqrt[\propto]{1 - \curlywedge_{i=1}^{n}\left(1 - \left((\beth \circ \mathcal{P}_i))^{\propto}\right)^{\propto}\right)^{\eth_i}}\right)^{1/\propto}, \left(\sqrt[\propto]{1 - \curlywedge_{i=1}^{n}\left(1 - \left((\beth \circ \mathcal{X}_i))^{\propto}\right)^{\propto}\right)^{\eth_i}}\right)^{1/\propto}, \\ \sqrt[\propto]{1 - \left(1 - \left(\curlywedge_{i=1}^{n}\left(\sqrt[\propto]{1 - \left(1 - ((\beth \circ \mathcal{Q}_i))^{\propto}\right)^{\propto}}\right)^{\eth_i}\right)^{\propto}\right)^{1/\propto}}, \\ \sqrt[\propto]{1 - \left(1 - \left(\curlywedge_{i=1}^{n}\left(\sqrt[\propto]{1 - \left(1 - ((\beth \circ \mathcal{Y}_i))^{\propto}\right)^{\propto}}\right)^{\eth_i}\right)^{\propto}\right)^{1/\propto}} \end{bmatrix}.$$

Proof To illustrate this, we may first show that,

$$\curlyvee_{i=1}^{n}\eth_i\Gamma_i^{\propto} = \begin{bmatrix} \sqrt[\propto]{1 - \curlywedge_{i=1}^{n}\left(1 - \left((\beth \circ \mathcal{O}_i))^{\propto}\right)^{\propto}\right)^{\eth_i}}, \sqrt[\propto]{1 - \curlywedge_{i=1}^{n}\left(1 - \left((\beth \circ \mathcal{V}_i))^{\propto}\right)^{\propto}\right)^{\eth_i}} \\ \sqrt[\propto]{1 - \curlywedge_{i=1}^{n}\left(1 - \left((\beth \circ \mathcal{P}_i))^{\propto}\right)^{\propto}\right)^{\eth_i}}, \sqrt[\propto]{1 - \curlywedge_{i=1}^{n}\left(1 - \left((\beth \circ \mathcal{X}_i))^{\propto}\right)^{\propto}\right)^{\eth_i}} \\ \curlywedge_{i=1}^{n}\left(\sqrt[\propto]{1 - \left(1 - ((\beth \circ \mathcal{Q}_i))^{\propto}\right)^{\propto}}\right)^{\eth_i}, \curlywedge_{i=1}^{n}\left(\sqrt[\propto]{1 - \left(1 - ((\beth \circ \mathcal{Y}_i))^{\propto}\right)^{\propto}}\right)^{\eth_i} \end{bmatrix}.$$

Put $n = 2$, $\eth_1\Gamma_1 \curlyvee \eth_2\Gamma_2$

$$= \left[\begin{array}{c} \sqrt[\propto]{\left(\sqrt[\propto]{1-\left(1-\left(((\lrcorner\circ\mathcal{O}_1))^{\propto}\right)^{\propto}\right)^{\eth_1}}\right)^{\propto}+\left(\sqrt[\propto]{1-\left(1-\left(((\lrcorner\circ\mathcal{O}_2))^{\propto}\right)^{\propto}\right)^{\eth_1}}\right)^{\propto}} \\ -\left(\sqrt[\propto]{1-\left(1-\left(((\lrcorner\circ\mathcal{O}_1))^{\propto}\right)^{\propto}\right)^{\eth_1}}\right)^{\propto}\cdot\left(\sqrt[\propto]{1-\left(1-\left(((\lrcorner\circ\mathcal{O}_2))^{\propto}\right)^{\propto}\right)^{\eth_1}}\right)^{\propto} \\ , \sqrt[\propto]{\left(\sqrt[\propto]{1-\left(1-\left(((\lrcorner\circ\mathcal{V}_1))^{\propto}\right)^{\propto}\right)^{\eth_1}}\right)^{\propto}+\left(\sqrt[\propto]{1-\left(1-\left(((\lrcorner\circ\mathcal{V}_2))^{\propto}\right)^{\propto}\right)^{\eth_1}}\right)^{\propto}} \\ -\left(\sqrt[\propto]{1-\left(1-\left(((\lrcorner\circ\mathcal{V}_1))^{\propto}\right)^{\propto}\right)^{\eth_1}}\right)^{\propto}\cdot\left(\sqrt[\propto]{1-\left(1-\left(((\lrcorner\circ\mathcal{V}_2))^{\propto}\right)^{\propto}\right)^{\eth_1}}\right)^{\propto}, \\ \sqrt[\propto]{\left(\sqrt[\propto]{1-\left(1-\left(((\lrcorner\circ\mathcal{P}_1))^{\propto}\right)^{\propto}\right)^{\eth_1}}\right)^{\propto}+\left(\sqrt[\propto]{1-\left(1-\left(((\lrcorner\circ\mathcal{P}_2))^{\propto}\right)^{\propto}\right)^{\eth_1}}\right)^{\propto}} \\ -\left(\sqrt[\propto]{1-\left(1-\left(((\lrcorner\circ\mathcal{P}_1))^{\propto}\right)^{\propto}\right)^{\eth_1}}\right)^{\propto}\cdot\left(\sqrt[\propto]{1-\left(1-\left(((\lrcorner\circ\mathcal{P}_2))^{\propto}\right)^{\propto}\right)^{\eth_1}}\right)^{\propto} \\ , \sqrt[\propto]{\left(\sqrt[\propto]{1-\left(1-\left(((\lrcorner\circ\mathcal{X}_1))^{\propto}\right)^{\propto}\right)^{\eth_1}}\right)^{\propto}+\left(\sqrt[\propto]{1-\left(1-\left(((\lrcorner\circ\mathcal{X}_2))^{\propto}\right)^{\propto}\right)^{\eth_1}}\right)^{\propto}} \\ -\left(\sqrt[\propto]{1-\left(1-\left(((\lrcorner\circ\mathcal{X}_1))^{\propto}\right)^{\propto}\right)^{\eth_1}}\right)^{\propto}\cdot\left(\sqrt[\propto]{1-\left(1-\left(((\lrcorner\circ\mathcal{X}_2))^{\propto}\right)^{\propto}\right)^{\eth_1}}\right)^{\propto} \\ \left(\sqrt[\propto]{1-\left(1-((\lrcorner\circ\mathcal{Q}_1))^{\propto}\right)^{\propto}}\right)^{\eth_1}, \left(\sqrt[\propto]{1-\left(1-((\lrcorner\circ\mathcal{Q}_2))^{\propto}\right)^{\propto}}\right)^{\eth_1} \\ , \left(\sqrt[\propto]{1-\left(1-((\lrcorner\circ\mathcal{Y}_1))^{\propto}\right)^{\propto}}\right)^{\eth_1}, \left(\sqrt[\propto]{1-\left(1-((\lrcorner\circ\mathcal{Y}_2))^{\propto}\right)^{\propto}}\right)^{\eth_1} \end{array} \right]$$

$$= \left[\begin{array}{c} \sqrt[\propto]{1-\curlywedge_{i=1}^{2}\left(1-\left(((\lrcorner\circ\mathcal{O}_1))^{\propto}\right)^{\propto}\right)^{\eth_i}}, \sqrt[\propto]{1-\curlywedge_{i=1}^{2}\left(1-\left(((\lrcorner\circ\mathcal{V}_1))^{\propto}\right)^{\propto}\right)^{\eth_i}} \\ \sqrt[\propto]{1-\curlywedge_{i=1}^{2}\left(1-\left(((\lrcorner\circ\mathcal{P}_1))^{\propto}\right)^{\propto}\right)^{\eth_i}}, \sqrt[\propto]{1-\curlywedge_{i=1}^{2}\left(1-\left(((\lrcorner\circ\mathcal{X}_1))^{\propto}\right)^{\propto}\right)^{\eth_i}} \\ \curlywedge_{i=1}^{2}\left(\sqrt[\propto]{1-\left(1-((\lrcorner\circ\mathcal{Q}_i))^{\propto}\right)^{\propto}}\right)^{\eth_i}, \curlywedge_{i=1}^{2}\left(\sqrt[\propto]{1-\left(1-((\lrcorner\circ\mathcal{Y}_i))^{\propto}\right)^{\propto}}\right)^{\eth_i} \end{array} \right].$$

Hence,

$$\Upsilon_{i=1}^{l}\eth_i\Gamma_i^{\partial} = \begin{bmatrix} \sqrt[\propto]{1-\curlywedge_{i=1}^{l}\left(1-\left(((\lrcorner\circ\mathcal{O}_1))^{\propto}\right)^{\propto}\right)^{\eth_i}}, \sqrt[\propto]{1-\curlywedge_{i=1}^{l}\left(1-\left(((\lrcorner\circ\mathcal{V}_1))^{\propto}\right)^{\propto}\right)^{\eth_i}} \\ \sqrt[\propto]{1-\curlywedge_{i=1}^{l}\left(1-\left(((\lrcorner\circ\mathcal{P}_1))^{\propto}\right)^{\propto}\right)^{\eth_i}}, \sqrt[\propto]{1-\curlywedge_{i=1}^{l}\left(1-\left(((\lrcorner\circ\mathcal{X}_1))^{\propto}\right)^{\propto}\right)^{\eth_i}} \\ \curlywedge_{i=1}^{l}\left(\sqrt[\propto]{1-\left(1-((\lrcorner\circ\mathcal{Q}_i))^{\propto}\right)^{\propto}}\right)^{\eth_i}, \curlywedge_{i=1}^{l}\left(\sqrt[\propto]{1-\left(1-((\lrcorner\circ\mathcal{Y}_i))^{\propto}\right)^{\propto}}\right)^{\eth_i} \end{bmatrix}.$$

If $n = l+1$, then $\Upsilon_{i=1}^{l}\eth_i\Gamma_i^{\partial} + \eth_{l+1}\Gamma_{l+1}^{\partial} = \Upsilon_{i=1}^{l+1}\eth_i\Gamma_i^{\partial}$.
Now, $\Upsilon_{i=1}^{l}\eth_i\Gamma_i^{\partial} + \eth_{l+1}\Gamma_{l+1}^{\partial} = \eth_1\Gamma_1^{\partial} \curlyvee \eth_2\Gamma_2^{\partial} \curlyvee ... \curlyvee \eth_l\Gamma_l^{\partial} \curlyvee \eth_{l+1}\Gamma_{l+1}^{\partial}$

$$= \begin{bmatrix} \sqrt[\propto]{\left(\sqrt[\propto]{1-\curlywedge_{i=1}^{l}\left(1-\left(((\lrcorner\circ\mathcal{O}_i))^{\propto}\right)^{\propto}\right)^{\eth_i}}\right)^{\propto}+\left(\sqrt[\propto]{1-\left(1-\left((\mathcal{O}_{l+1})^{\propto}\right)^{\propto}\right)^{\eth_1}}\right)^{\propto} - \left(\sqrt[\propto]{1-\curlywedge_{i=1}^{l}\left(1-\left(((\lrcorner\circ\mathcal{O}_i))^{\propto}\right)^{\propto}\right)^{\eth_i}}\right)^{\propto}\cdot\left(\sqrt[\propto]{1-\left(1-\left((\mathcal{O}_{l+1})^{\propto}\right)^{\propto}\right)^{\eth_1}}\right)^{\propto}}, \\ \sqrt[\propto]{\left(\sqrt[\propto]{1-\curlywedge_{i=1}^{l}\left(1-\left(((\lrcorner\circ\mathcal{V}_i))^{\propto}\right)^{\propto}\right)^{\eth_i}}\right)^{\propto}+\left(\sqrt[\propto]{1-\left(1-\left((\mathcal{V}_{l+1})^{\propto}\right)^{\propto}\right)^{\eth_1}}\right)^{\propto} - \left(\sqrt[\propto]{1-\curlywedge_{i=1}^{l}\left(1-\left(((\lrcorner\circ\mathcal{V}_i))^{\propto}\right)^{\propto}\right)^{\eth_i}}\right)^{\propto}\cdot\left(\sqrt[\propto]{1-\left(1-\left((\mathcal{V}_{l+1})^{\propto}\right)^{\propto}\right)^{\eth_1}}\right)^{\propto}}, \\ \sqrt[\propto]{\left(\sqrt[\propto]{1-\curlywedge_{i=1}^{l}\left(1-\left(((\lrcorner\circ\mathcal{P}_i))^{\propto}\right)^{\propto}\right)^{\eth_i}}\right)^{\propto}+\left(\sqrt[\propto]{1-\left(1-\left((\mathcal{P}_{l+1})^{\propto}\right)^{\propto}\right)^{\eth_1}}\right)^{\propto} - \left(\sqrt[\propto]{1-\curlywedge_{i=1}^{l}\left(1-\left(((\lrcorner\circ\mathcal{P}_i))^{\propto}\right)^{\propto}\right)^{\eth_i}}\right)^{\propto}\cdot\left(\sqrt[\propto]{1-\left(1-\left((\mathcal{P}_{l+1})^{\propto}\right)^{\propto}\right)^{\eth_1}}\right)^{\propto}}, \\ \sqrt[\propto]{\left(\sqrt[\propto]{1-\curlywedge_{i=1}^{l}\left(1-\left(((\lrcorner\circ\mathcal{X}_i))^{\propto}\right)^{\propto}\right)^{\eth_i}}\right)^{\propto}+\left(\sqrt[\propto]{1-\left(1-\left((\mathcal{X}_{l+1})^{\propto}\right)^{\propto}\right)^{\eth_1}}\right)^{\propto} - \left(\sqrt[\propto]{1-\curlywedge_{i=1}^{l}\left(1-\left(((\lrcorner\circ\mathcal{X}_i))^{\propto}\right)^{\propto}\right)^{\eth_i}}\right)^{\propto}\cdot\left(\sqrt[\propto]{1-\left(1-\left((\mathcal{X}_{l+1})^{\propto}\right)^{\propto}\right)^{\eth_1}}\right)^{\propto}}, \\ \curlywedge_{i=1}^{l}\left(\sqrt[\propto]{1-\left(1-((\lrcorner\circ\mathcal{Q}_i))^{\propto}\right)^{\propto}}\right)^{\eth_i}, \left(\sqrt[\propto]{1-\left(1-(\mathcal{Q}_{l+1})^{\propto}\right)^{\propto}}\right)^{\eth_1} \\ , \curlywedge_{i=1}^{l}\left(\sqrt[\propto]{1-\left(1-((\lrcorner\circ\mathcal{Y}_i))^{\propto}\right)^{\propto}}\right)^{\eth_i}, \left(\sqrt[\propto]{1-\left(1-(\mathcal{Y}_{l+1})^{\propto}\right)^{\propto}}\right)^{\eth_1} \end{bmatrix}$$

$$\Upsilon_{i=1}^{l+1}\eth_i\Gamma_i^{\propto} = \begin{bmatrix} \sqrt[\propto]{1-\curlywedge_{i=1}^{l+1}\left(1-\left(((\lrcorner\circ\mathcal{O}_1))^{\propto}\right)^{\propto}\right)^{\eth_i}}, \sqrt[\propto]{1-\curlywedge_{i=1}^{l+1}\left(1-\left(((\lrcorner\circ\mathcal{V}_1))^{\propto}\right)^{\propto}\right)^{\eth_i}} \\ \sqrt[\propto]{1-\curlywedge_{i=1}^{l+1}\left(1-\left(((\lrcorner\circ\mathcal{P}_1))^{\propto}\right)^{\propto}\right)^{\eth_i}}, \sqrt[\propto]{1-\curlywedge_{i=1}^{l+1}\left(1-\left(((\lrcorner\circ\mathcal{X}_1))^{\propto}\right)^{\propto}\right)^{\eth_i}} \\ \curlywedge_{i=1}^{l+1}\left(\sqrt[\propto]{1-\left(1-((\lrcorner\circ\mathcal{Q}_i))^{\propto}\right)^{\propto}}\right)^{\eth_i}, \curlywedge_{i=1}^{l+1}\left(\sqrt[\propto]{1-\left(1-((\lrcorner\circ\mathcal{Y}_i))^{\propto}\right)^{\propto}}\right)^{\eth_i} \end{bmatrix}.$$

$$\left(\Upsilon_{i=1}^{l+1}\eth_i\Gamma_i^{\partial}\right)^{1/\partial} = \left[\begin{array}{c} \left(\sqrt[\propto]{1-\curlywedge_{i=1}^{l+1}\left(1-\left(((\lrcorner\circ\mathcal{O}_i))^{\propto}\right)^{\eth_i}\right)}\right)^{1/\propto}, \left(\sqrt[\propto]{1-\curlywedge_{i=1}^{l+1}\left(1-\left(((\lrcorner\circ\mathcal{V}_i))^{\propto}\right)^{\eth_i}\right)}\right)^{1/\propto} \\ \left(\sqrt[\propto]{1-\curlywedge_{i=1}^{l+1}\left(1-\left(((\lrcorner\circ\mathcal{P}_i))^{\propto}\right)^{\eth_i}\right)}\right)^{1/\propto}, \left(\sqrt[\propto]{1-\curlywedge_{i=1}^{l+1}\left(1-\left(((\lrcorner\circ\mathcal{X}_i))^{\propto}\right)^{\eth_i}\right)}\right)^{1/\propto} \\ \sqrt[\propto]{1-\left(1-\left(\curlywedge_{i=1}^{l+1}\left(\sqrt[\propto]{1-\left(1-((\lrcorner\circ\mathcal{Q}_i))^{\propto}\right)}\right)^{\eth_i}\right)^2\right)^{1/\propto}}, \\ \sqrt[\propto]{1-\left(1-\left(\curlywedge_{i=1}^{l+1}\left(\sqrt[\propto]{1-\left(1-((\lrcorner\circ\mathcal{Y}_i))^{\propto}\right)}\right)^{\eth_i}\right)^2\right)^{1/\propto}} \end{array}\right]$$

Corollary 4.12. *If $q=1$, then IVT $\propto$ NWA operator is used instead of the GIVT $\propto$ NWA operator.*

Theorem 4.13. *If all $\Gamma_i = \langle(([(\lrcorner\circ\mathcal{O}_i),(\lrcorner\circ\mathcal{V}_i)],[(\lrcorner\circ\mathcal{P}_i),(\lrcorner\circ\mathcal{X}_i)],\ [(\lrcorner\circ\mathcal{Q}_i),(\lrcorner\circ\mathcal{Y}_i)]))\rangle$ and all are equal. Then GIVT $\propto$ NWA$(\Gamma_1,\Gamma_2,...,\Gamma_n)=\Gamma$.*

Corollary 4.14. *The GIVT $\propto$ NWA operator meets both boundedness and monotonicity constraints.*

4.4. *Generalized IVT $\propto$NWG (GIVT $\propto$NWG)*

Definition 4.15. Let $\Gamma_i = \langle(([(\lrcorner\circ\mathcal{O}_i),(\lrcorner\circ\mathcal{V}_i)],[(\lrcorner\circ\mathcal{P}_i),(\lrcorner\circ\mathcal{X}_i)],\ [(\lrcorner\circ\mathcal{Q}_i),(\lrcorner\circ\mathcal{Y}_i)]))\rangle$ be the IVT $\propto$ NNs. Then GIVT$\propto$NWG $(\Gamma_1,\Gamma_2,...,\Gamma_n) = \frac{1}{\partial}\Big(\curlywedge_{i=1}^{n}(\partial\Gamma_i)^{\eth_i}\Big)$.

Corollary 4.16. *Let $\Gamma_i = \langle(([(\lrcorner\circ\mathcal{O}_i),(\lrcorner\circ\mathcal{V}_i)],[(\lrcorner\circ\mathcal{P}_i),(\lrcorner\circ\mathcal{X}_i)],\ [(\lrcorner\circ\mathcal{Q}_i),(\lrcorner\circ\mathcal{Y}_i)]))\rangle$ be the IVT $\propto$ NNs. Then GIVT$\propto$NWG$(\Gamma_1,\Gamma_2,...,\Gamma_n)$*

$$= \left[\begin{array}{c} \sqrt[\propto]{1-\left(1-\left(\curlywedge_{i=1}^{n}\left(\sqrt[\propto]{1-\left(1-((\lrcorner\circ\mathcal{O}_i))^{\propto}\right)}\right)^{\eth_i}\right)^{\propto}\right)^{1/\propto}}, \\ \sqrt[\propto]{1-\left(1-\left(\curlywedge_{i=1}^{n}\left(\sqrt[\propto]{1-\left(1-((\lrcorner\circ\mathcal{V}_i))^{\propto}\right)}\right)^{\eth_i}\right)^{\propto}\right)^{1/\propto}} \\ \left(\sqrt[\propto]{1-\curlywedge_{i=1}^{n}\left(1-\left(((\lrcorner\circ\mathcal{P}_i))^{\propto}\right)^{\eth_i}\right)}\right)^{1/\propto}, \left(\sqrt[\propto]{1-\curlywedge_{i=1}^{n}\left(1-\left(((\lrcorner\circ\mathcal{X}_i))^{\propto}\right)^{\eth_i}\right)}\right)^{1/\propto} \\ \left(\sqrt[\propto]{1-\curlywedge_{i=1}^{n}\left(1-\left(((\lrcorner\circ\mathcal{Q}_i))^{\propto}\right)^{\eth_i}\right)}\right)^{1/\propto}, \left(\sqrt[\propto]{1-\curlywedge_{i=1}^{n}\left(1-\left(((\lrcorner\circ\mathcal{Y}_i))^{\propto}\right)^{\eth_i}\right)}\right)^{1/\propto} \end{array}\right].$$

Corollary 4.17. *When $\partial=1$, the GIVT $\propto$ NWG is converted to the $\propto$NWG.*

Corollary 4.18. *GIVT∝NWG operators satisfy the boundness and monotonicity characteristics.*

Corollary 4.19. *If all* $\Gamma_i = \langle(([(\beth \circ \mathcal{O}_i), (\beth \circ \mathcal{V}_i)], [(\beth \circ \mathcal{P}_i), (\beth \circ \mathcal{X}_i)],\ [(\beth \circ \mathcal{Q}_i), (\beth \circ \mathcal{Y}_i)]))\rangle$ *are equal. Then* $GIVT{\propto}NWG(\Gamma_1, \Gamma_2, ..., \Gamma_n) = \Gamma$.

5. Conclusion:

This work presents novel weighted operators, such as geometric and averaging operators. Boundedness, idempotency, commutativity, associativity, and monotonicity are some of the characteristics of these operators. To describe the weighted vector, we looked at a number of common metrics. Many aggregation operator criteria have been studied. Some findings have been made after a few aggregating techniques for these IVT $\propto$ NNs have been examined.

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Predicting Customer Satisfaction based on Neutrosophic sets: Applied on mobile food ordering application.

Walaa Elsanabary[1], Nouran M. Radwan [2]

[1] Computer and Information Systems Dept.; Sadat Academy for Management Sciences; walaaams@gmail.com
[2] Computer and Information Systems Dept.; Sadat Academy for Management Sciences; radwannouran@yahoo.com

* Correspondence: e-mail@e-mail.com; Tel.: (optional; include country code)

Abstract: Customers' satisfaction prediction is a vital process for all business organizations to draw new customers and maintain existing customers. One of the most efficient methods to predict customers' satisfaction is classifying customers' feedback. In the real world, customer's feedback is ambiguous, confusing and inconsistent. This option can be stated in neutrosophic logic as indeterminacy membership, associated with truth and falsity membership. In this study, a classification model based on neutrosophic sets to handle the inconsistency of customer responses is presented. Also, significant factors that impact customers' satisfaction are defined. In order to show the procedures and the application of the proposed method, a case study to determine the customers' satisfaction while using food order application (Talabat) in Egypt is presented. A comparison between the classical classification models and the proposed model based on neutrosophic sets is presented. The experimental results indicate that the proposed classifying model achieved accuracy results around 95.36% to 99.95%, higher than the classical one that achieve around 90.3% to 93.99%. Next, a sensitivity analysis is performed for reliability validation as to determine the most factors that affect customers' satisfaction.

Keywords: Food Ordering Applications, Neutrosophic Logic, Neutrosophic set, Classification, Neutrosophic Classifier.

1. Introduction

Nowadays, customer satisfaction plays an important role in improving and developing any organization. Customer satisfaction is a crucial issue that can be considered one of the organization's assets, as customer satisfaction, in turn, affects the formulation of the organization's strategies and promotional campaigns and, thus, the success or failure of the organization [1, 2]. Moreover, obtaining and analyzing customer reviews in the restaurant sector is an extremely difficult issue. Valuable customer feedback can be challenging to establish as it includes variation in customer interpretation of their experiences, and reliability of survey results. On the other hand, the progress that has occurred in e-commerce and then in online applications has helped influence how to collect customer reviews in a more impartial manner [3]. Hence, online food ordering applications were able to obtain customer reviews about different restaurants in an unbiased manner [2, 4].

Recently, online food ordering applications have been widely used in different restaurant sectors to provide their services to customers. However, there are significant issues regarding the effect of applying food ordering applications on customer satisfaction [4]. Egyptian restaurants and stores started to consider mobile applications as a new mechanism to attract new customers and keep current customers' satisfaction. For example, more than 3800 Egyptian restaurants and and stores joined the Talabat application to take orders from customers in 2023 [5].

Customers, by using food ordering applications, can order their food at times and locations desired by to them. These applications provide customers with up-to-date information about the restaurant and store options [6, 7]. Also, it gives customers the ability to see their order progress through all its stages. Food ordering applications help customers and restaurants overcome difficulties such as misunderstandings, late deliveries, or handling complaints [8, 9]. Customers' opinions towards food ordering applications have been a significant issue recognized by a number of researchers. It was found that customers' satisfaction with food ordering applications is largely affected by the level of trust, product quality, and significant differences in the customers' points of view of such applications [10–12].

On the other hand, evaluating customer reviews about restaurants is considered one of the most complicated issues because it is vague, imprecise, and indeterminate. Generally, the customer expresses his opinion by selecting the most likely choice from: very good, good, normal, bad, or very bad. Many studies presented various theories to handle human thinking uncertainty [13]. Professor Zadeh introduced fuzzy logic, which considers the degree of the membership of elements in a set, in 1965 [14]. In 1975, Zadeh introduced type 2 fuzzy logic, in which membership degrees themselves are fuzzy [15]. Attanssov presented intuitionistic fuzzy logic as an extension of the standard fuzzy sets in 1983 [16]. Fuzzy set theory, intuitionistic fuzzy sets, and neutrosophy theory have been proposed to deal with uncertain, indeterminate, inaccurate, and vague data in everyday life problems such as decision-making, engineering, computer science, and finance [17]. Fuzzy logic can only express vagueness when information is naturally graded. However, type-2 and intuitionistic fuzzy logic, proposed by Attanssov as an extension of normal fuzzy sets, may describe vagueness and imprecision using a range of membership values. Previous study suggests that dealing with ambiguity and contradictory data is crucial for improving customer feedback accuracy [18].

The four main categories of uncertainty data are vagueness, impreciseness, inconsistency, and ambiguity, an example of vague information would be the following sentence: "The food quality is very good"; the Fuzzy set can handle vagueness as a type of uncertainty. An example of imprecise information would be the following sentence: "The food is neither good nor bad"; the Intuitionistic fuzzy set can represent imprecision. A statement of ambiguous information could be: "The food quality is typically about 75%." Inconsistent information is represented by the following sentence: "The probability that food quality is good is 60%" and that doesn't mean that the probability of the food quality is bad is 40% given that there may be unknown hidden reasons; All of the previously mentioned uncertainties may be handled by a neutrosophic set. [19, 20].

Smarandache introduced neutrosophic logic to handle inconsistent and imprecise information to express the percentage of unknown parameters [19]. Neutrosophic logic handles the indeterminacy of customers' responses better and simulates human thinking [20]. Therefore, neutrosophic logic has the potential to simulate human thought processes, resolve paradoxes that are both true contrasting to each other, and be applied to real-world issues. Hence, the researchers suppose that the outcome of evaluating the customer reviews may enhance the classification results after using neutrosophic logic. As a result of the above discussion, there are no previous studies that handle impreciseness and ambiguity in customers' responses for better classification accuracy. An example of vagueness, impreciseness, and ambiguity in customers' response choices is excellent, very good, good, normal, bad, or very bad [13]. The undecided response is a choice by the customer who is unsure. A neutral set can handle all the above uncertainties. Consequently, neutrosophic logic is a proposed approach for customer satisfaction classification to deal with uncertainty [19, 20]. Moreover, no research has investigated the relationship between customer satisfaction and the combination of order packaging, value for money, delivery time, and quality of the food in Egypt.

A research gap exists in related work, since no studies have been conducted to investigate the association between customer satisafaction and the combination of order packing, value for money, delivery time, and quality of food. In addition, no previous study addressed impreciseness and ambiguity in customer replies for better classification accuracy. Excellent, very good, good, normal, awful, or extremely bad are examples of vagueness, impreciseness, and ambiguity in customer responses [11]. The customer who is unsure may choose the undecided response. A neutral set can deal with all of the aforementioned uncertainties. As a result, neutrosophic logic is a recommended way to cope with uncertainty in customer satisfaction classification [1, 16].

In this study, a proposed model based on a neutrosophic set is presented. The main novelties of this research are shown in the following points:

- Obtaining and handling vagueness, impreciseness, and ambiguity customers' responses from Talabat ordering application by utilizing the neutrosophic set.
- Presenting a classification model to determine the customer satisfaction class as satisfied, neutral, and unsatisfied based on neutrosophic sets.
- Discovering the importance of neutrosophic sets by introducing a comparison between the classical classification models and the proposed model based on neutrosophic sets.
- Defining which attributes have the most effect on customer satisfaction in mobile food ordering applications.
- Following that, a sensitivity analysis is performed to validate the reliability.

The paper is organized as follows: section 2 discusses the related work; Section 3 explains the research methodology; Section 4 presents the applied experiment; Section 5 presents the discussion and results; Section 6 presents the conclusion and future work.

2. Related work

Customer satisfaction has become an important factor as the market has changed from a vendor-driven market to a customer-driven one [21]. Customer satisfaction answers questions related to

present customers and potential customers which contributes to higher profitability [20]. Customers' reviews play a critical role in the selection process to evaluate customers' satisfaction and dissatisfaction [22]. Analyzing the customer's reviews is essential for improving service quality to gain a competitive advantage [23,24].

Food delivery has become a rising business that has extended in many regions. Previous studies concern the factors affecting food ordering application customers' satisfaction while ordering food online. [4, 5]. A study was conducted to investigate the interest to use food ordering applications through the evaluation of customer satisfaction, food quality, and service quality [25]. Saad AT presented a study to explore customer behavior, the study concerns on the services that bring food from the restaurants, not the product. The survey results were analyzed using MS–Excel and SPSS, it was concluded that delivery time, service quality, price, and condition of food delivered are the most important factors affecting the success of online food delivery [26]. Dirsehan, T., & Cankat, E. found that food ordering application plays a decisive function in developing restaurants' satisfaction and loyalty. Theuiy conducted a survey including filtering, customers' attitudes and customers' behavior questions [27].

Another study collected customer opinions using survey to examine the effect of ease of use. The study concluded that customer trust was affected by perceived usefulness and perceived ease of use positively. It was shown that application's ease of use enhances continuous usage [28]. Tandon investigated factors that contribute to forming customers' loyalty in the context of food delivery application. The results indicated that food ordering application has changed the food delivery industry [29]. The customers' confidence in food ordering application was examined including the role of technology acceptance model, mobile service quality, and privacy in promoting customers' confidence. The study helped owners of food ordering applications by providing product design and service strategy to enhance their competitive advantage [30].

From previous studies, it can be concluded that many researchers have conducted custom surveys to investigate customer satisfaction for food delivery applications. The survey method has limitations due to human discourage, difference, and unawareness [31]. Survey results may be inaccurate as they may be a collection of overall customer opinions on the most important factors affecting their satisfaction [32].

To address this gap, order review mining takes place to gain more precise customer information through the application. There are only limited studies on customers' reviews that use mining to explore the factors reflecting customer satisfaction [33-34]. Therefore, there is a need to conclude meaningful implications by analyzing customer reviews and exploring which attributes affected customer satisfaction in mobile food ordering application. Collecting reviews using mobile food ordering application is more accurate than the general survey because the customer gives feedback immediately after the purchase. On the other hand, not all previous research has been applied in Egypt.

Customers' reviews are mostly collected using a scaling responses choice such as very good, good, normal, bad, very bad, or totally disagree, disagree, neutral, agree and totally agree. The neutral or undecided choice is an option by the customer who is unsure [11]. In real world, customers' responses are not strict, but it is vague, imprecise, ambiguous and inconsistent information. Neutrosophic logic can express this option by indeterminacy membership with truth and falsity membership [35].

Neutrosophic logic is an extension of fuzzy logic which is a better choice to handle indeterminacy that expresses the degree of unknown parameters. It is described by triple values t, i, and f where t is the grade of truth, f is the grade of false and i is the grade of indeterminacy [36]. Neutrosophic logic expresses inconsistency and contradictions found in human thinking. Also, it helps in the decision-making process in the case of information conflict due to human observation from the surrounding world [37].

Neutrosophic set can express customers' responses with three membership functions and then be analyzed based on the weights appointed to each response in the questionnaire. From this, the customer segment can be determined, and this can be used by the decision-makers to modify its strategy [38]. The indeterminacy degree gives the possibility to define the unrecognized information in the model. Therefore, decision-makers by handling uncertain information can make more reasonable decisions [39, 40]. Lately, neutrosophic set has been widely studied to solve different problems such as crowd management [41], obstacles analysis [42], energy resources management [43], and inventory control model [44] as ambiguous and partial data may be handled by fuzzy sets and intuitionistic fuzzy set theories, not neutral data, which is typically present in real-world world. Smarandache first provided a concept of an NS from a philosophical point of view that is a component of neutrosophic to manage all sorts of ambiguity, including neutrality [45, 46]. The researchers note that a limited number of previous research used neutrosophic logic in determining customer satisfaction. Moreover, the previous research did not use the same methodology or dataset as the researchers used in this research [31-34, 47-50].

Classification is an important technique in data mining to put out data into similar groups depending on the data features. Most machine learning techniques are used to classify all possible inputs into defined classes. How to classify ambiguous data is a major research topic, which has been getting serious concern from researchers recently [51, 52].

Classification performed on imprecise and ambiguous is a question in many research [53]. For example, a study presented fuzzy logic techniques for infectious diseases identification such as hepatitis and tuberculosis. The evaluation indicators such as sensitivity and accuracy are employed [54]. Furthermore, intuitionistic fuzzy set has a powerful ability to address ambiguous data by assigning a lower degree to the outliers. Intuitionistic fuzzy classification is more powerful as compared to fuzzy classification as it deals with membership, hesitant level, and non-membership [55].

However, previous research that applied classification techniques based on fuzzy logic and intuitionistic fuzzy logic cannot handle vagueness, ambiguous, inconsistent, and imprecise data which found in nature [56-58]. In this paper, classification model based on neutrosophic set is presented to enhance the results of customers' feedback analysis. A comparison between the famous classification techniques such as naive Bayes, c4.5, random tree, and random forest, and the combined one with neutrosophic is presented. The proposed classification model has been implemented and experimental results showed that this system did quite better than classical techniques.

3. Materials and Methods

The proposed method is divided into three parts; the first part is to convert linguistic terms to neutrosophic sets. The second part is to apply classical and neutrosophic classification techniques to dataset, and the third part is to compare between results.

Neutrosophic set consists of three values true degree (t), indeterminate degree (i) and false degree (f) with no restriction on t, i, f, nor on their sum. For example, assume that (0.6, 0.3, and 0.5) is a neutrosophic set. This means that t=0.6 degrees, i=0.3 and f=0.5 [19, 20]. The linguistic terms excellent, good, fair, and poor are expressed by neutrosophic values. Linguistic terms and assigned neutrosophic set presented in Table 1 are defined as:

Table 1. Linguistic terms and the assigned neutrosophic set

Linguistic term	Neutrosophic set
Excellent	(1,0,0)
Very Good	(0.8, 0.15, 0.2)
Good	(0.7,0.25, 0.3)
Fair	(0.5,0.5,0.5)
Poor	(0.2, 0.7, 0.7)

Definition [38]: If A = (t, i, f) is a single-valued neutrosophic set, a score function is mapped into the single crisp output as S(A) follows: $S(A) = \frac{2+t-i-f}{3}$.

In this section the proposed classification model based on neutrosophic set is presented, Figure 1 describes the proposed model steps. In which, neutrosophication unit accepts linguistic term input which assigns the appropriate membership functions of inputs in true, false, and indeterminacy degrees. Then, the triple neutrosophic set is converted into a single crisp value to take the decision which is called de-neutrosophication. Let x= (TA, IA, FA) be a single value of neutrosophic set, then the proposed score function is defined as the following function: $S(A) = \frac{2+t-i-f}{3}$ [59].

Then classical classification techniques and the neutrosophic combined into one is applied on the dataset, the last step is validating the results by measuring the accuracies as Accuracy = True Positive +True Negative /True Positive + True Negative + False Positive + False Negative [60][61] and determine the critical factors that affects customers' satisfaction about restaurants.

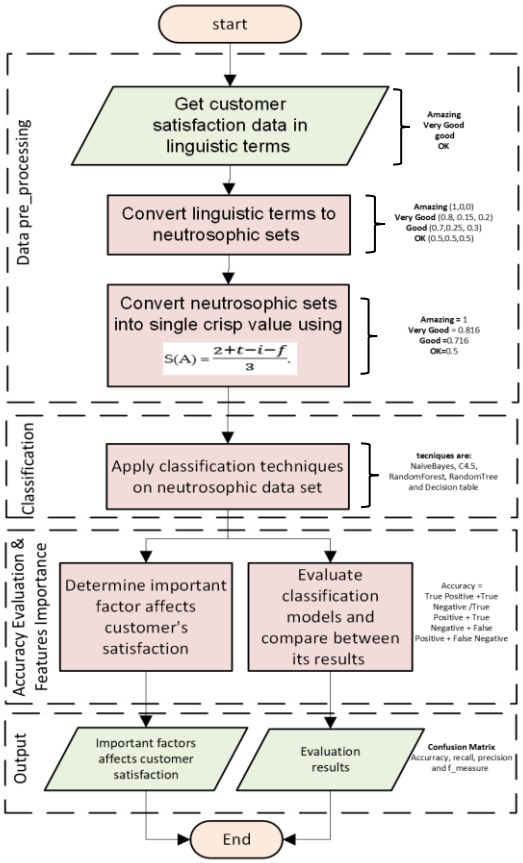

Figure 1. Classification Proposed Model based on Neutrosophic set Flowchart

4. Application

In this part, the proposed model is applied on customers scaling responses of food ordering applications called Talabat in Egypt.

4.1 The Dataset

The dataset characteristics are shown in Table 2. The dataset consists of 2330 reviews of different restaurant and stores that joined Talabat ordering food application in Egypt [62]. The dataset features are order packaging, value for money, delivery time and quality of food.

Table 2. Talabat dataset characteristics

Dataset characteristics	**Attribute characteristics**	**Associated tasks**
Multivariate	Categorical, integer, real	Classification
Number of instances	Number of attributes	Missing values?
2330	7	Yes

4.2 Data Preprocessing

There are many steps to prepare the data for use in a model. Data preprocessing includes removing the noise and handling missing data to obtain optimal results. Categorical variables needed to be encoded as satisfied, medium, and unsatisfied.

Table 3. Talabat dataset after preprocessing

ID	Reviews	OrderPackaging	ValueForMoney	DeliveryTime	QualityOfFood	Class
1	177	Good	Good	Good	Good	medium
6	564	Very Good	Very Good	Very Good	Very Good	Satisfied
10	40	Very Good	Very Good	Very Good	Very Good	Satisfied
12	334	Good	Good	Good	Good	medium
15	238	Very Good	Very Good	Very Good	Very Good	Satisfied
17	3117	Very Good	Very Good	Very Good	Very Good	Satisfied
20	79	Fair	Fair	Fair	Fair	Unsatisfied
22	3	Poor	Fair	Fair	Fair	Unsatisfied
25	7	Excellent	Excellent	Very good	Very good	Satisfied

4.3 Neutrosophic Set

In this phase; converting linguistic terms (dataset attributes: order packaging, value for money, delivery time and quality of food) into neutrosophic values with truth, falsity, and indeterminacy membership. Table 4 describes Talabat dataset after assigning neutrosophic set, while Table 5 shows Talabat dataset after applying de-neutrosophication process.

Table 4. A sample of Talabat dataset after assigning neutrosophic set

ID	Reviews	OrderPackaging	ValueForMoney	DeliveryTime	QualityOfFood	Class
1	177	(0.7,0.25, 0.3)	(0.7,0.25, 0.3)	(0.7,0.25, 0.3)	(0.7,0.25, 0.3)	medium
6	564	(0.8, 0.15, 0.2)	(0.8, 0.15, 0.2)	(0.8, 0.15, 0.2)	(0.8, 0.15, 0.2)	Satisfied
10	40	(0.8, 0.15, 0.2)	(0.8, 0.15, 0.2)	(0.8, 0.15, 0.2)	(0.8, 0.15, 0.2)	Satisfied
12	334	(0.7,0.25, 0.3)	(0.7,0.25, 0.3)	(0.7,0.25, 0.3)	(0.7,0.25, 0.3)	medium
15	238	(0.8, 0.15, 0.2)	(0.8, 0.15, 0.2)	(0.8, 0.15, 0.2)	(0.8, 0.15, 0.2)	Satisfied
17	3117	(0.8, 0.15, 0.2)	(0.8, 0.15, 0.2)	(0.8, 0.15, 0.2)	(0.8, 0.15, 0.2)	Satisfied
20	79	(0.5,0.5,0.5)	(0.5,0.5,0.5)	(0.5,0.5,0.5)	(0.5,0.5,0.5)	Unsatisfied
22	3	(0.2, 0.7, 0.7)	(0.5,0.5,0.5)	(0.5,0.5,0.5)	(0.5,0.5,0.5)	Unsatisfied
25	7	(1,0,0)	(1,0,0)	(0.8, 0.15, 0.2)	(0.8, 0.15, 0.2)	Satisfied

Table 5. A sample of Talabat dataset after applying deneutrosophication process

D	Reviews	OrderPackaging	ValueForMoney	DeliveryTime	QualityOfFood	Class
1	177	0.716	0.716	0.716	0.716	medium
6	564	0.816	0.816	0.816	0.816	Satisfied
10	40	0.816	0.816	0.816	0.816	Satisfied
12	334	0.716	0.716	0.716	0.716	medium
15	238	0.816	0.816	0.816	0.816	Satisfied
17	3117	0.816	0.816	0.816	0.816	Satisfied
20	79	0.5	0.5	0.5	0.5	Unsatisfied
22	3	0.266	0.5	0.5	0.5	Unsatisfied

25	7	1	1	0.816	0.816	Satisfied

4.5 Classification

Weka (open-source software for knowledge discovery tasks) is used for dataset classifying after applying the proposed model based on neutrosophic sets using famous algorithms such as naive Bayes, c4.5, random tree, random forest, and decision table.*4.4 Accuracy Measurement*

Accuracy = True Positive +True Negative /True Positive + True Negative + False Positive + False Negative [56].

5. Discussion and Results

Classification results for each algorithm: naive Bayes, c4.5, random tree, random forest and decision table are showed in Table 5. All algorithms achieve accepted results 93.99%, 93.95%, 93.65%, 90.3% respectively but the neutrosophic classifying based models achieve higher accuracy results 95.36%, 99.95%, 99.95%, 99.78%. Then the feature selection is applied to the dataset to define important factors that affects customers' satisfaction while using food ordering application in Egypt which are, value for money, and quality of food. Table 6 shows the classification results using classical models and combined one with neutrosophic set.

Table 6. Classification Results for Different Classification Algorithm

	NaiveBayes		C4.5		RandomForest		RandomTree		Decision table	
	Classic Model	Proposed Model	Classic Model	Proposed Model	Classic Model	Proposed Model	Classic Model	Proposed Model	Classic Model	Proposed Model
Precision	0.940	0.954	0.938	1.000	0.940	1.000	0.928	0.998	0.936	1.000
Recall	0.940	0.954	0.975	1.000	0.968	1.000	0.929	0.999	0.979	1.000
F-measure	0.940	0.954	0.956	1.000	0.954	1.000	0.928	0.999	0.957	1.000

The classified algorithms naive bayes, c4.5, random tree, random forest and decision table were applied to measure confusion matrix TP rate, FP rate, precision, recall, F-measure and ROC area). Table 7 and 8 show the confusion matrix for Naive Bayes performance and proposed model based on neutrosophic sets respectively. Figures 2 and 3 show the comparison between correctly and incorrectly classified results in classical model and neutrosophic based model.

Table 7. Confusion matrix for Naive Bayes performance using classical model

Confusion Matrix for Naive Bayes Classifier performance using Classical Model			
classified as	**a**	**b**	**c**
a= medium	**660**	**98**	**0**
b=satisfied	**41**	**1531**	**0**
c=unsatisfied	**1**	**0**	**0**

Table 8. Confusion matrix for Naive Bayes performance after applying the neutrosophic based model

Confusion Matrix for Naive Bayes Classifier performance after applying the Neutrosophic based model			
classified as	**a**	**b**	**c**
a= medium	**692**	**66**	**0**
b=satisfied	**41**	**1531**	**0**

c=unsatisfied	**1**	**0**	**0**

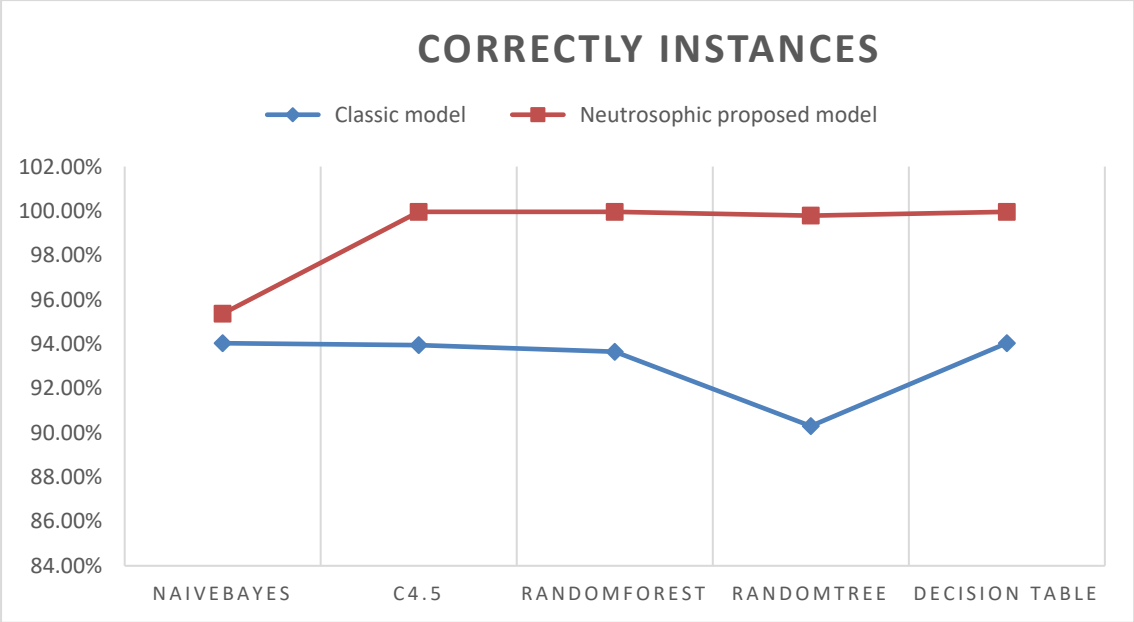

Figure 2. Correctly classified results for different classification algorithms.

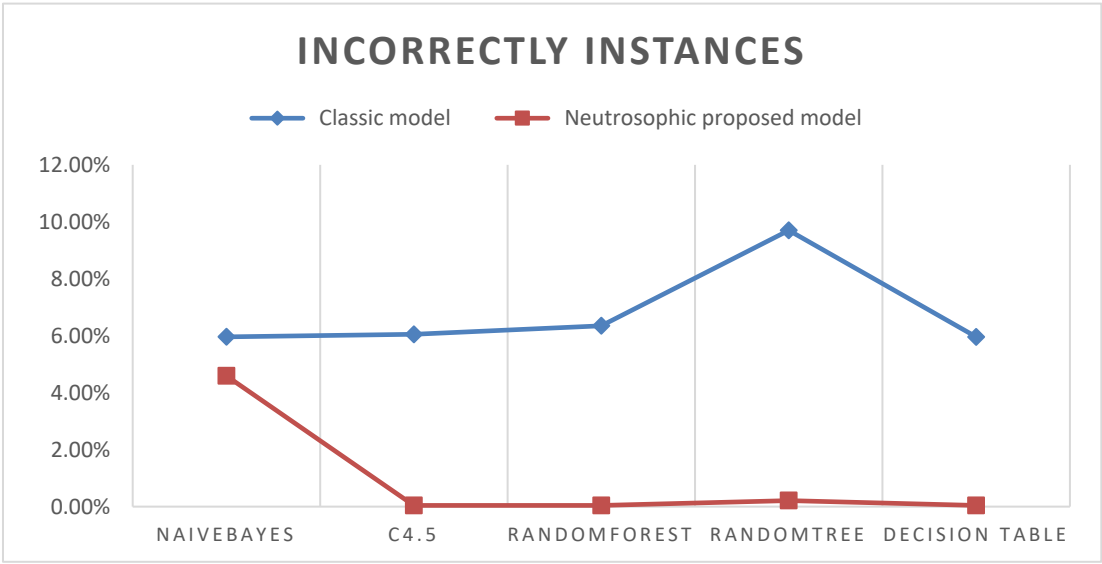

Figure 3. Incorrectly instances for different classification

6. Conclusion

Recently, studying the level of customer satisfaction has become one of the most critical issues facing researchers due to its importance in formulating corporate strategies, designing promotional campaigns, and identifying the most important factors that affect customer satisfaction. The emergence of online applications for ordering food and services has helped researchers facilitate the study of the level of customer satisfaction because these applications allow customers to evaluate food and services immediately after receiving them. Analyzing customers' reviews helps organizations understand overall customer satisfaction. Customers' reviews are mostly collected using scaling responses that have drawbacks like indeterminate information. These problems could

be handled when a neutrosophic set is used to express customers' responses in a realistic way, as it handles indeterminate information.

This paper introduced a classification model based on neutrosophic sets to handle the indeterminacy of customer reviews. The outcome indicates that the proposed model has an impact on classification outcome as accuracy increased. This due to the ability of neutrosophic sets to describe degrees of uncertainty that are difficult to be defined by fuzzy and intuitionistic fuzzy sets. For example, if a customer is asked for his opinion on a certain product or service, he may respond that he was satisfied by 80% unsatisfied by 30% and hesitated by 40%. This opinion may be handled adequately by neutrosophic logic, which has true, indeterminate, and false membership functions, depending on how absolute or conditional values are represented.

Based on the above, the researchers observed that the suggested classifying model produced accuracy values ranging from 95.36% to 99.95%, which is greater than the classical model's range of 90.3% to 93.99%. Additionally, the results of sensitive analysis approach in this paper after using the proposed model showed that two factors; value for money and food quality are among the most important factors that affect customer satisfaction while using the food ordering application (Talabat), so this can help the marketing department improve its work by creating promotional campaigns and improving communication with customers. Moreover, enhancing food quality and establishing prudent pricing strategies.

The research restrictions include that the results of this study are based on the data acquired from a single source, at a short period of time. Moreover, this study was carried out with data obtained from (Talabat) application in Egypt; the results of this research might vary from country to another. Finally, future research can be extended using more online applications over long period of time. Additionally, it can utilize neutrosophic multisets or neutrosophic soft sets.

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Robust Seismic Performance Evaluation of Super High-Rise Concrete Structures Using MCDM and Plithogenic-RAM Approach

Kaifeng Xing[1]*, Xiaoping Wang[2], and Peng Chen[3]
[1,2,3]School of Architecture and Civil Engineering, Huangshan University, Huangshan, 245041, Anhui, China
*Corresponding author, E-mail: 13866136299@163.com

Abstract
More reinforced concrete supertall and megatall high-rise structures are being built due to the growing global demand for high-rise buildings, recent advancements in concrete technology, and improvements in construction techniques. When using concrete in these structures, additional care must be taken. Evaluating Super High-Rise Concrete Structures using various criteria is a multi-criteria decision-making (MCDM) challenge. We propose an MCDM method with a set of criteria and alternatives. The Root Assessment Method (RAM) is applied in this study to rank the alternatives. The RAM method is integrated with plithogenic sets to handle uncertain information. Plithogenic operators are used to combine the plithogenic numbers into a single matrix. Seven criteria and nine alternatives are evaluated in this study. The results indicate that the Structural Strength and Stability criterion has the highest weight, while the Non-Structural Component Performance criterion has the lowest weight. Sensitivity analysis was conducted with different criteria weights to observe variations in the rankings of alternatives. The results demonstrate that the rankings of alternatives remain stable under varying weights.

Keywords: Super High-Rise Concrete Structures; Decision Making; RAM Approach; MCDM.

1. Introduction and Background
The rapid growth of the world's population and the accelerating pace of urbanization have led to increasing population density in city centers. With limited lateral expansion available in these densely packed urban areas, cities have resorted to "urban

verticalization" as a solution to accommodate rising demands for residential, commercial, and industrial spaces. This vertical expansion has spurred the construction of taller and more complex high-rise buildings, particularly in crowded and economically thriving urban areas [1], [2]. The phenomenon of urban verticalization has not only reshaped city skylines but also posed unique challenges in structural engineering, material sciences, and construction techniques.

As high-rise buildings reach new heights, the number of supertall (300–600 meters) and megatall (600 meters and above) structures has also seen a dramatic increase. For instance, only three supertall buildings were completed in 2009; by contrast, thirteen supertall buildings were expected to be completed in a single year, signaling a significant upward trend. This growth trajectory continued, and by 2020, projections indicated the completion of 18 supertall structures and three megatall buildings [3], [4]. These developments highlight the critical importance of advancing construction technologies and materials to meet the demands of modern skyscrapers.

One of the foundational elements enabling the construction of these colossal structures is high-performance concrete (HPC). HPC has become indispensable due to its unique properties, such as enhanced strength, stiffness, pumpability, and early-age strength. These attributes make HPC especially suited for supertall and megatall buildings, where structural integrity and economic feasibility are paramount [5], [6]. Additionally, advancements in prefabricated reinforcement cages and slip-and-climb-form work technology have revolutionized the construction process, allowing builders to achieve remarkable speeds of two to three floors per week without compromising safety or quality. These innovations have positioned reinforced concrete (RC) as a competitive alternative to structural steel in terms of both speed and economic efficiency [7], [8].

However, while HPC and RC offer significant advantages, they also demand meticulous attention to design, construction practices, and material selection. The use of concrete in supertall and megatall buildings introduces several challenges. These include ensuring

mix design precision, meeting stringent performance criteria, and addressing construction-related issues such as thermal effects, shrinkage, and creep [9], [10]. Furthermore, the dynamic characteristics of these buildings, such as their response to seismic loads and wind-induced vibrations, necessitate advanced engineering solutions to ensure their safety and resilience.

1.1 The Role of Seismic Performance Evaluation

The evaluation of seismic performance has emerged as a critical aspect of designing super high-rise concrete structures. Given their height and mass, these buildings are particularly susceptible to seismic forces, making it imperative to assess their structural integrity under such conditions. Seismic performance evaluation not only ensures the safety of occupants but also protects significant financial investments in these structures. Modern engineering approaches emphasize a multi-criteria decision-making (MCDM) framework to address the complex interplay of factors influencing seismic performance. This methodology allows for the consideration of diverse criteria such as structural strength, ductility, energy dissipation, and material quality, ensuring a comprehensive assessment of a building's seismic resilience [11], [12].

1.2 MCDM Methodology

Because of the complexity of real-world challenges, decision-making in construction and seismic evaluation often involves conflicting criteria. Multi-criteria decision-making (MCDM) provides a systematic framework to address these issues by breaking down complex problems into smaller, manageable parts that are easier to evaluate. The foundations of MCDM trace back to the 1950s and 1960s, with significant advancements in the 1970s, including the formal introduction of the acronym "MCDM"[13], [14].

MCDM methodologies enable decision-makers to weigh various criteria—often conflicting—to arrive at the best possible solution. Among the many MCDM techniques, the use of plithogenic sets has gained prominence. Plithogenic sets allow the integration

of vague, uncertain, and contradictory information, making them particularly suitable for evaluating complex systems like super high-rise structures [15], [16].

Importance of High-Performance Concrete in Super High-Rise Buildings

High-performance concrete has become the material of choice for constructing super high-rise buildings due to its superior mechanical properties and versatility. Its high compressive strength ensures that structural components can support the enormous vertical loads typical of tall buildings. Additionally, its enhanced durability minimizes maintenance requirements, making it a cost-effective solution over the building's lifecycle. The use of HPC also allows for slender and lightweight designs, which are critical for optimizing the usable floor area in high-rise buildings without compromising structural stability [5], [6].

Moreover, innovations in HPC technology have addressed many of the challenges associated with traditional concrete. For example, the addition of supplementary cementitious materials (SCMs) such as fly ash and silica fumes improves workability and reduces the heat of hydration, mitigating the risk of thermal cracking during the curing process. These advancements are particularly crucial for megatall structures, where the sheer volume of concrete used can exacerbate thermal and shrinkage-related issues [9], [10].

1.3 Challenges and Innovations in High-Rise Construction

The construction of super high-rise buildings involves overcoming numerous engineering and logistical challenges. One of the primary difficulties lies in ensuring redundancy and load path continuity, which are essential for preventing progressive collapse in the event of localized failures. This requires careful consideration of structural redundancy during the design phase, as well as rigorous quality control during construction to ensure the consistent performance of all structural components[15], [16].

In addition, the interaction between a building's foundation and the supporting soil presents significant challenges, particularly in seismic zones. Advanced geotechnical

analyses are required to model the behavior of the foundation-soil system under dynamic loading conditions. These analyses help optimize foundation designs to ensure stability and minimize settlement under seismic forces. Furthermore, the integration of damping systems and tuned mass dampers has proven effective in mitigating the effects of seismic and wind-induced vibrations, enhancing both structural safety and occupant comfort [17], [18].

1.4 Advancing Decision-Making for Seismic Performance

To address the complex demands of seismic performance evaluation, this study employs the Root Assessment Method (RAM) under a plithogenic framework. By combining advanced mathematical tools with expert assessments, the RAM approach provides a comprehensive evaluation of alternatives based on multiple criteria. The use of plithogenic operators allows for the aggregation of uncertain and contradictory information, enabling more accurate and reliable rankings of super high-rise concrete structures. This methodology ensures that critical factors such as structural strength, material quality, and dynamic characteristics are thoroughly considered, providing valuable insights for designers, engineers, and policymakers [19].

1.5 Aim of this study

The aim of this study is organized as follows:

a) We proposed a MCDM methodology for evaluating the Super High-Rise Concrete Structures with a set of criteria and alternatives.
b) The RAM method is used under the plithogenic numbers for the first time to rank the alternatives.
c) We used the plithogenic operators to combine the different plithogenic numbers into one matrix.
d) We performed sensitivity analysis to show the stability of the rank of alternatives with different criteria weights.

1.6 Organization of this study

The rest of this study is organized as follows: Section 2 shows the materials and methods. Section 3 shows the application results. Section 4 shows the discussion of these results. Section 5 shows the conclusions of this study.

2. Materials and Methods

This part presents the steps of the proposed methodology with a set of experts, criteria and alternatives. We use the RAM methodology to rank the alternatives. Figure 1 shows the details of the RAM method under the plithogenic.

2.1 Plithogenic RAM

Plithogeny is the emergence, formation, growth, and evolution of new things from the synthesis of contradictory (dissimilar) or non-contradictory many ancient entities. A set that has multiple members specified by a variety of qualities, each of which has a value, is called a plithogenic set[15], [16], [17]. The RAM approach will be improved in this work by utilizing plithogenic aggregation operators, which are a subset of the plithogenic set, in order to reduce information loss. The Plithogenic RAM approach can be applied in collaborative decision-making scenarios without compromising information, in contrast to the conventional RAM method[18], [19]. The steps of the Plithogenic RAM technique are shown below.

1) Build the decision matrix.

   Suppose an MCDM issue has a set of alternatives $m; A_i(i = 1,2, \dots, m)$ and a set of criteria $n; C_j(j = 1,2, \dots, n)$. These criteria and alternatives are evaluated by the opinions of experts.
2) Replace opinions of experts with the plithogenic numbers. The plithogenic numbers have three parts to reduce the uncertainty and vague information.
3) Combine the decision matrix.

   We combine the plithogenic numbers into one matrix instead of different matrices as:

$$((b_{i1}, b_{i2}, b_{i3}), 1 \le i \le n) \wedge P((t_{i1}, t_{i2}, t_{i3}), 1 \le i \le n)$$
$$= (b_{i1} \wedge_F t_{i1}, 0.5 * (b_{i2} \wedge_F t_{i2}) + 0.5 * (b_{i2} \vee_F t_{i2}), b_{i3} \vee_F t_{i3}), 1 \le i \le n$$

4) Crip values are obtained from the combined matrix. Then the criteria weights by normalized crisp values.

5) Normalize the decision matrix.

   The crisp values in combined decision matrix are normalized by linear sum normalization as:

$$y_{ij} = \frac{r_{ij}}{\sum_{i=1}^{m} r_{ij}}; (i = 1, \dots, m; j = 1, \dots, n)$$

6) Compute the weighted decision matrix as:

$$u_{ij} = y_{ij} w_{ij}$$

7) Compute the sum of weighted normalized matrix for beneficial and non-beneficial criteria such as:

$$q_{+i} = \sum_{j=1}^{n} u_{+ij}$$
$$q_{-i} = \sum_{j=1}^{n} u_{-ij}$$

8) Compute the overall score of every alternative as:

$$S_i = \sqrt[2+q_{-i}]{2 + q_{+i}}$$

9) Rank the alternatives.

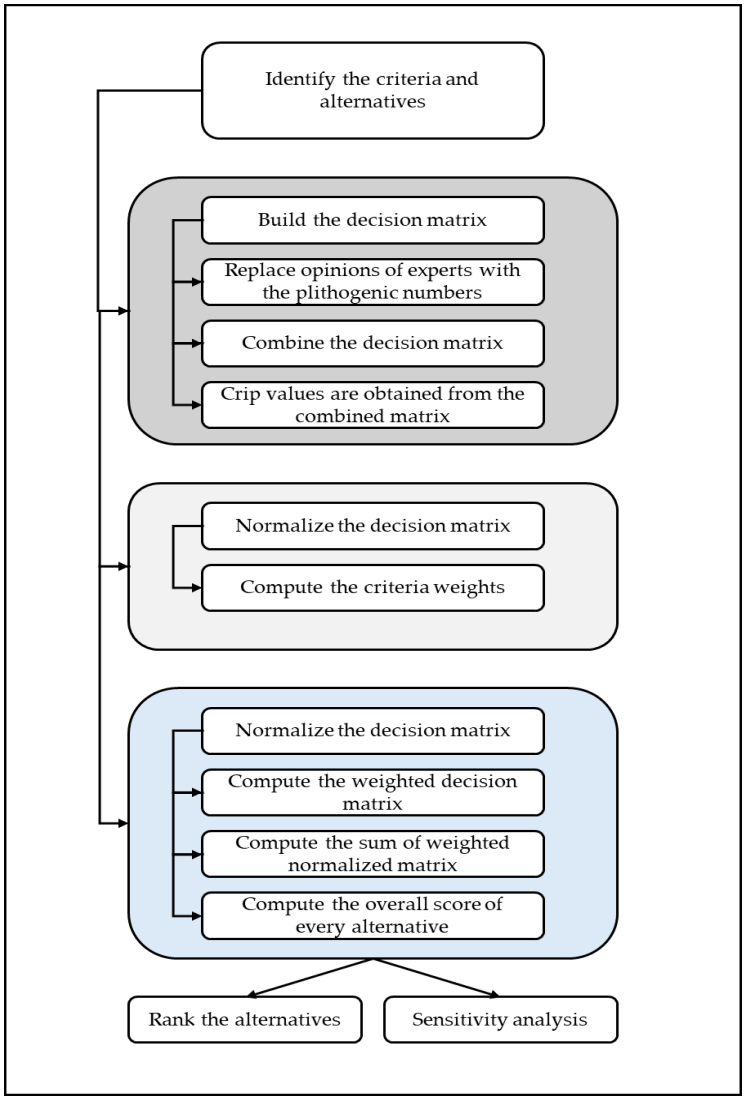
Identify the criteria and alternatives
Build the decision matrix
Replace opinions of experts with the plithogenic numbers
Combine the decision matrix
Crip values are obtained from the combined matrix
Normalize the decision matrix
Compute the criteria weights
Normalize the decision matrix
Compute the weighted decision matrix
Compute the sum of weighted normalized matrix
Compute the overall score of every alternative
Rank the alternatives
Sensitivity analysis

Figure 1. The details of plithogenic RAM approach.

3. Application

Experts have been selected for this consultancy based on their proficiency in the many Super High-Rise Concrete Structures domains and fields they are working in. First, the consulting firm recommended 3 professionals with years of experience in the Super High-Rise Concrete Structures industry. Thus, there are 3 specialists in this study. Expert 1 is a general manager with twenty years of Super High-Rise Concrete Structures experience. The second expert is a consultant. In addition, he has 22 years of Super High-Rise Concrete Structures experience and a doctorate. Expert 3 has 12 years of experience in the Super High-Rise Concrete Structures industry and is an associate professor. The

literature review has established the study's criteria. An academician who was an associate professor later devised the approach. The other members of the consulting firm also talked about the sessions. The literature's criteria were reviewed, and additional criteria were introduced and eliminated. A criteria expert consultant broke down the criteria into their component elements during the criterion-setting process. Furthermore, a thorough analysis of the distance functions was used to construct the criterion. Seven criteria and nine alternatives are evaluated by three experts in this study.

3.1 Explanation of Steps

1. Building the Decision Matrix

The first step involved constructing a decision matrix based on the assessments provided by three experts. These experts evaluated nine alternatives across seven criteria. Each expert's evaluations were recorded, capturing their opinions about the importance and performance of alternatives for each criterion. The criteria included factors such as Structural Strength and Stability (C4) and Material Quality and Properties (C2), which are critical for evaluating super high-rise concrete structures.

The decision matrix represents a structured format where the rows correspond to the alternatives (A1, A2, …, A9), and the columns represent the criteria (C1, C2, …, C7). This matrix serves as the foundation for subsequent calculations and analysis.

2. Replacing Expert Opinions with Plithogenic Numbers

To account for vagueness and uncertainty in the experts' opinions, their evaluations were transformed into plithogenic numbers. A plithogenic number includes three components: truth-membership, indeterminacy-membership, and falsity-membership. This representation enhances the reliability of the evaluation by capturing nuanced and uncertain information. Tables 1–3 illustrate the plithogenic numbers corresponding to each expert's evaluations.

These tables display how each expert's qualitative and uncertain evaluations are quantified and integrated into the decision matrix. For example, for A1 under C1, Expert 1's evaluation might be expressed as (0.65,0.30,0.45), where:

0.65 represents the truth-membership (degree of agreement with the evaluation).

0.30 indicates the indeterminacy-membership (uncertainty in the evaluation).

0.45 reflects the falsity-membership (degree of disagreement).

Table 1. Expert 1 opinions.

	C_1	C_2	C_3	C_4	C_5	C_6	C_7
A_1	(0.65, 0.30, 0.45)	(0.40, 0.70, 0.50)	(0.50, 0.40, 0.60)	(0.65, 0.30, 0.45)	(0.65, 0.30, 0.45)	(0.25, 0.60, 0.80)	(0.65, 0.30, 0.45)
A_2	(0.50, 0.40, 0.60)	(0.65, 0.30, 0.45)	(0.65, 0.30, 0.45)	(0.40, 0.70, 0.50)	(0.50, 0.40, 0.60)	(0.65, 0.30, 0.45)	(0.50, 0.40, 0.60)
A_3	(0.40, 0.70, 0.50)	(0.50, 0.40, 0.60)	(0.50, 0.40, 0.60)	(0.65, 0.30, 0.45)	(0.40, 0.70, 0.50)	(0.50, 0.40, 0.60)	(0.40, 0.70, 0.50)
A_4	(0.25, 0.60, 0.80)	(0.40, 0.70, 0.50)	(0.40, 0.70, 0.50)	(0.50, 0.40, 0.60)	(0.65, 0.30, 0.45)	(0.40, 0.70, 0.50)	(0.65, 0.30, 0.45)
A_5	(0.65, 0.30, 0.45)	(0.25, 0.60, 0.80)	(0.25, 0.60, 0.80)	(0.40, 0.70, 0.50)	(0.50, 0.40, 0.60)	(0.65, 0.30, 0.45)	(0.50, 0.40, 0.60)
A_6	(0.40, 0.70, 0.50)	(0.65, 0.30, 0.45)	(0.65, 0.30, 0.45)	(0.25, 0.60, 0.80)	(0.40, 0.70, 0.50)	(0.50, 0.40, 0.60)	(0.40, 0.70, 0.50)
A_7	(0.25, 0.60, 0.80)	(0.50, 0.40, 0.60)	(0.25, 0.60, 0.80)	(0.65, 0.30, 0.45)	(0.25, 0.60, 0.80)	(0.40, 0.70, 0.50)	(0.25, 0.60, 0.80)
A_8	(0.10, 0.75, 0.85)	(0.40, 0.70, 0.50)	(0.80, 0.10, 0.30)	(0.25, 0.60, 0.80)	(0.65, 0.30, 0.45)	(0.25, 0.60, 0.80)	(0.65, 0.30, 0.45)
A_9	(0.95, 0.05, 0.05)	(0.25, 0.60, 0.80)	(0.95, 0.05, 0.05)	(0.95, 0.05, 0.05)	(0.65, 0.30, 0.45)	(0.65, 0.30, 0.45)	(0.25, 0.60, 0.80)

Table 2. Expert 2 opinions.

	C_1	C_2	C_3	C_4	C_5	C_6	C_7
A_1	(0.10, 0.75, 0.85)	(0.40, 0.70, 0.50)	(0.50, 0.40, 0.60)	(0.65, 0.30, 0.45)	(0.10, 0.75, 0.85)	(0.25, 0.60, 0.80)	(0.40, 0.70, 0.50)
A_2	(0.10, 0.75, 0.85)	(0.10, 0.75, 0.85)	(0.25, 0.60, 0.80)	(0.40, 0.70, 0.50)	(0.50, 0.40, 0.60)	(0.65, 0.30, 0.45)	(0.10, 0.75, 0.85)
A_3	(0.95, 0.05, 0.05)	(0.10, 0.75, 0.85)	(0.10, 0.75, 0.85)	(0.65, 0.30, 0.45)	(0.10, 0.75, 0.85)	(0.10, 0.75, 0.85)	(0.25, 0.60, 0.80)
A_4	(0.10, 0.75, 0.85)	(0.95, 0.05, 0.05)	(0.95, 0.05, 0.05)	(0.10, 0.75, 0.85)	(0.95, 0.05, 0.05)	(0.10, 0.75, 0.85)	(0.10, 0.75, 0.85)
A_5	(0.25, 0.60, 0.80)	(0.10, 0.75, 0.85)	(0.10, 0.75, 0.85)	(0.95, 0.05, 0.05)	(0.10, 0.75, 0.85)	(0.95, 0.05, 0.05)	(0.95, 0.05, 0.05)
A_6	(0.40, 0.70, 0.50)	(0.25, 0.60, 0.80)	(0.25, 0.60, 0.80)	(0.10, 0.75, 0.85)	(0.25, 0.60, 0.80)	(0.10, 0.75, 0.85)	(0.10, 0.75, 0.85)
A_7	(0.25, 0.60, 0.80)	(0.40, 0.70, 0.50)	(0.40, 0.70, 0.50)	(0.25, 0.60, 0.80)	(0.40, 0.70, 0.50)	(0.25, 0.60, 0.80)	(0.25, 0.60, 0.80)
A_8	(0.10, 0.75, 0.85)	(0.50, 0.40, 0.60)	(0.40, 0.70, 0.50)	(0.40, 0.70, 0.50)	(0.65, 0.30, 0.45)	(0.40, 0.70, 0.50)	(0.40, 0.70, 0.50)
A_9	(0.95, 0.05, 0.05)	(0.65, 0.30, 0.45)	(0.50, 0.40, 0.60)	(0.65, 0.30, 0.45)	(0.65, 0.30, 0.45)	(0.65, 0.30, 0.45)	(0.50, 0.40, 0.60)

Table 3. Expert 3 opinions.

	C_1	C_2	C_3	C_4	C_5	C_6	C_7
A_1	(0.40, 0.70, 0.50)	(0.40, 0.70, 0.50)	(0.50, 0.40, 0.60)	(0.65, 0.30, 0.45)	(0.80, 0.10, 0.30)	(0.25, 0.60, 0.80)	(0.40, 0.70, 0.50)
A_2	(0.65, 0.30, 0.45)	(0.40, 0.70, 0.50)	(0.25, 0.60, 0.80)	(0.40, 0.70, 0.50)	(0.50, 0.40, 0.60)	(0.40, 0.70, 0.50)	(0.65, 0.30, 0.45)
A_3	(0.80, 0.10, 0.30)	(0.65, 0.30, 0.45)	(0.40, 0.70, 0.50)	(0.65, 0.30, 0.45)	(0.80, 0.10, 0.30)	(0.65, 0.30, 0.45)	(0.80, 0.10, 0.30)
A_4	(0.65, 0.30, 0.45)	(0.80, 0.10, 0.30)	(0.65, 0.30, 0.45)	(0.40, 0.70, 0.50)	(0.10, 0.75, 0.85)	(0.80, 0.10, 0.30)	(0.65, 0.30, 0.45)
A_5	(0.95, 0.05, 0.05)	(0.65, 0.30, 0.45)	(0.80, 0.10, 0.30)	(0.65, 0.30, 0.45)	(0.40, 0.70, 0.50)	(0.65, 0.30, 0.45)	(0.95, 0.05, 0.05)

A_6	(0.40, 0.70, 0.50)	(0.95, 0.05, 0.05)	(0.65, 0.30, 0.45)	(0.80, 0.10, 0.30)	(0.65, 0.30, 0.45)	(0.95, 0.05, 0.05)	(0.10, 0.75, 0.85)
A_7	(0.25, 0.60, 0.80)	(0.50, 0.40, 0.60)	(0.95, 0.05, 0.05)	(0.65, 0.30, 0.45)	(0.80, 0.10, 0.30)	(0.40, 0.70, 0.50)	(0.25, 0.60, 0.80)
A_8	(0.10, 0.75, 0.85)	(0.40, 0.70, 0.50)	(0.80, 0.10, 0.30)	(0.95, 0.05, 0.05)	(0.65, 0.30, 0.45)	(0.50, 0.40, 0.60)	(0.40, 0.70, 0.50)
A_9	(0.95, 0.05, 0.05)	(0.25, 0.60, 0.80)	(0.95, 0.05, 0.05)	(0.95, 0.05, 0.05)	(0.95, 0.05, 0.05)	(0.65, 0.30, 0.45)	(0.50, 0.40, 0.60)

3. Combining the Plithogenic Numbers into a Single Matrix

After gathering the expert evaluations and representing them as plithogenic numbers (as shown in Tables 1–3), the next step is to aggregate these numbers into a single decision matrix. This step involves combining the opinions of all three experts for each alternative and criterion to form a unified representation.

The aggregation process employs plithogenic operators that effectively handle the inherent vagueness and uncertainty in the data. These operators consider the truth-membership, indeterminacy-membership, and falsity-membership values for each evaluation, synthesizing them into a consolidated matrix. For A1 under C1, if the three experts provided plithogenic evaluations as follows:

Expert 1: (0.65,0.30,0.45); Expert 2: (0.50,0.40,0.60); Expert 3: (0.40,0.70,0.50)

The combined value is calculated using the plithogenic aggregation formula to produce a single plithogenic number for A1 under C1.

This step results in a combined decision matrix that consolidates the opinions of all experts, enabling a cohesive analysis across all alternatives and criteria.

4. Computing the Criteria Weights

After forming the combined decision matrix, the next step is to calculate the criteria's weights. These weights represent the relative importance of each criterion in the decision-making process and are derived using normalized crisp values obtained from the combined matrix.

The process involves the following steps:

1) The plithogenic numbers are converted into crisp values using a score function, which combines the truth-membership, indeterminacy-membership, and falsity-membership values into a single representative score.
2) The crisp values are normalized across all criteria to ensure comparability. This involves dividing each value by the sum of all values in the column for a specific criterion.

The normalized values are then used to calculate the weights for each criterion. These weights reflect how influential each criterion is in determining the overall ranking of alternatives. If the normalized crisp values for C1, C2, and C3 are 0.1305, 0.1357, and 0.1542, the corresponding weights are directly proportional to these values. In Table 4, Structural Strength and Stability (C4) has the highest weight, indicating its critical importance in evaluating super high-rise concrete structures, while Non-Structural Component Performance (C7) has the lowest weight.

Table 4. The weights of criteria.

	Criteria	*Weights*	*Rank*
C_1	Redundancy and Load Path Continuity: Presence of alternative load paths to prevent collapse.	0.130499	2
C_2	Material Quality and Properties: Quality and strength of materials used in construction.	0.135653	3
C_3	Ductility and Energy Dissipation: Deformation and energy absorption capacity during seismic events.	0.154226	6
C_4	Structural Strength and Stability: Ability to withstand seismic forces without failure.	0.156862	7
C_5	Foundation and Soil Interaction: Stability of foundation-soil interaction under seismic loading.	0.152752	5
C_6	Dynamic Characteristics: Compatibility with seismic wave frequencies and damping properties.	0.140183	4
C_7	Non-Structural Component Performance: Resilience of secondary components like cladding and utilities.	0.129827	1

5. Normalizing the Decision Matrix

After aggregating the plithogenic numbers into a single matrix and extracting the crisp values, the next step is to normalize the decision matrix. This step ensures that all criteria values are scaled to a comparable range, typically between 0 and 1, facilitating an unbiased evaluation of alternatives across different criteria. The normalized decision

matrix presented in Table 5 reflects the relative performance of each alternative under each criterion. By transforming all values into a comparable scale, normalization eliminates the influence of differing units or magnitudes among criteria. This ensures that each criterion contributes proportionally to the overall evaluation.

Table 5. The normalization matrix.

	C_1	C_2	C_3	C_4	C_5	C_6	C_7
A_1	0.100132	0.091799	0.105366	0.126883	0.116745	0.059903	0.116584
A_2	0.11012	0.096328	0.085288	0.079387	0.106383	0.125921	0.11069
A_3	0.166104	0.105936	0.074114	0.126883	0.098339	0.102512	0.124728
A_4	0.090157	0.159793	0.132449	0.072869	0.12759	0.107156	0.122802
A_5	0.155843	0.086731	0.090626	0.130223	0.07483	0.159159	0.180843
A_6	0.095424	0.149922	0.110446	0.089103	0.093745	0.130326	0.055504
A_7	0.064348	0.110945	0.115701	0.10859	0.106009	0.081992	0.064681
A_8	0.027244	0.101582	0.133777	0.113757	0.130297	0.091052	0.116584
A_9	0.190629	0.096965	0.152233	0.152306	0.146063	0.141979	0.107584

6. Weighted Normalized Decision Matrix

Once the normalized decision matrix is constructed (Table 5), the criteria weights from Table 4 are applied to compute the weighted normalized decision matrix. This step incorporates the relative importance of each criterion into the evaluation process, ensuring that criteria with higher weights have a greater influence on the rankings.

The weighted normalized value for each alternative is computed using the formula:

$$u_{ij} = y_{ij} w_j$$

Where:

u_{ij}: Weighted normalized value of alternative i under criterion j.

w_j: Weight of criterion j from Table 4.

y_{ij}: Normalized value of alternative iii under criterion j from Table 5.

For C1, with N11=0.684 and w1=0.1305, the weighted normalized value for A1 =0.0893

This calculation is repeated for all criteria and alternatives, producing the weighted normalized matrix shown in Table 6.

Table 6. The weighted normalized decision.

	C_1	C_2	C_3	C_4	C_5	C_6	C_7
A_1	0.014986	0.012615	0.015941	0.018925	0.015722	0.007809	0.017189
A_2	0.01648	0.013238	0.012903	0.011841	0.014327	0.016416	0.01632
A_3	0.024859	0.014558	0.011213	0.018925	0.013244	0.013364	0.01839
A_4	0.013493	0.021959	0.020038	0.010868	0.017183	0.013969	0.018106
A_5	0.023323	0.011919	0.013711	0.019423	0.010078	0.020749	0.026664
A_6	0.014281	0.020603	0.016709	0.01329	0.012625	0.01699	0.008184
A_7	0.00963	0.015246	0.017504	0.016196	0.014277	0.010689	0.009537
A_8	0.004077	0.01396	0.020239	0.016967	0.017548	0.01187	0.017189
A_9	0.028529	0.013325	0.023031	0.022717	0.019671	0.018509	0.015862

Table 7. Rank of alternatives.

	Overall score	*Rank*
A_1	4.408966	4
A_2	4.400375	2
A_3	4.447561	6
A_4	4.463445	7
A_5	4.497974	8
A_6	4.407523	3
A_7	4.371552	1
A_8	4.413962	5
A_9	4.561589	9

7. Computing the Sum of Weighted Normalized Matrix

The sum of the weighted normalized values for each alternative is calculated to assess its overall performance under both beneficial and non-beneficial criteria.

Sum for Beneficial Criteria:

$$q_i^+ = \sum_{j=1}^{n} u_{+ij}$$

Sum for Non-Beneficial Criteria:

$$q_i^- = \sum_{j=1}^{n} u_{-ij}$$

For A1, If the weighted normalized values for beneficial criteria are 0.0893,0.1256,0.0921, then: q_1^+ = 0.0893+0.1256+0.0921= 0.307

For non-beneficial criteria, if the weighted normalized values are 0.0651, 0.0754, then: q_1^- = 0.0651+0.0754= 0.1405

8. Computing the Overall Score

The overall score for each alternative is calculated by dividing the sum of weighted values for beneficial criteria by the sum of weighted values for non-beneficial criteria:

$S_i = \sqrt[2+q_{-i}]{2 + q_{+i}}$ For A1, S_1=2.186

9. Ranking the Alternatives

Once the overall scores (S_i) are calculated for all alternatives, they are ranked in descending order. The alternative with the highest score is ranked first, as it performs closest to the ideal solution. The alternative with the lowest score is ranked last. From Table 7, alternative 9 got the highest overall score and is ranked first and alternative 7, the lowest overall score and is ranked last.

4. Discussion

In this study, three experts evaluated seven criteria and nine alternatives using the RAM method. The experts' opinions were expressed as plithogenic numbers, which were combined into a single matrix using plithogenic operators. The resulting matrix was converted into crisp values using a score function. These values were normalized to calculate the criteria weights, where Structural Strength and Stability had the highest weight, while Non-Structural Component Performance had the lowest weight.

The RAM method was then applied to rank the alternatives. The results showed that Alternative 9 ranked the highest, while Alternative 7 ranked the lowest. To ensure the robustness of the rankings, a sensitivity analysis was conducted by adjusting the criteria weights by 25%. This analysis, depicted in Figure 2, demonstrated that the rankings of alternatives remained stable across different scenarios.

The results of the sensitivity analysis and the rankings are summarized in Table 8, confirming the reliability of the RAM method in evaluating super high-rise concrete structures.

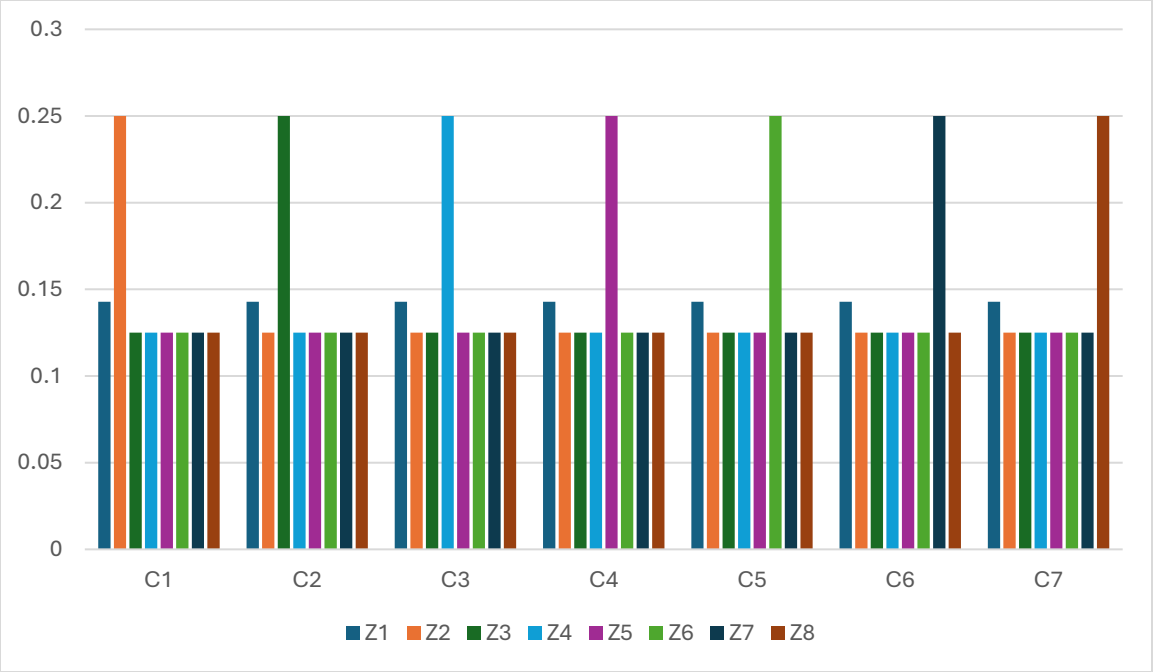

Figure 2. The different criteria weights.

Table 8. The different rank of alternatives.

	Z_1	Z_2	Z_3	Z_4	Z_5	Z_6	Z_7	Z_8
A_1	3	3	3	3	4	4	2	4
A_2	2	4	2	2	2	2	4	3
A_3	6	7	6	5	7	6	6	6
A_4	7	6	8	7	6	8	7	7
A_5	8	8	7	8	8	7	8	8
A_6	4	5	5	4	3	3	5	2
A_7	1	1	1	1	1	1	1	1
A_8	5	2	4	6	5	5	3	5
A_9	9	9	9	9	9	9	9	9

5. Conclusions

This work proposed a MCDM methodology to rank the Super High-Rise Concrete Structures with a set of criteria. Three experts are invited to evaluate the criteria and alternatives. they used the plithogenic numbers in their evaluation. Then we combine these numbers by the plithogenic operators. The criteria weights are computed using the normalized crisp values. Then we applied the steps of the RAM method to rank the alternatives. Seven criteria and nine alternatives are evaluated in this study. The results

show the alternative 9 is the best and alternative 7 is the worst. We conducted the sensitivity analysis to show the rank of alternatives. The results show that the rank of alternatives is stable under different weights.

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Algebraic Properties of Interval -Valued Quadri Partitioned Neutrosophic Fuzzy Matrices and their Application in Multi-Criteria Decision- Making Problem

P.Tharini[1], C. Devi Shyamala Mary[2], P.Tharaniya[3], S. Prathap[4], S. Ramkumar, G. Dhanavel

[1]Department of Mathematics, PPG Institute of Technology, Coimbatore, Tamilnadu, India.
tharinip15@gmail.com
[2]PG & Research Department of Mathematics, St.Joseph's College of Arts & Science (Autonomous), Cuddalore, Tamilnadu, India.
shyam.mary@gmail.com
[3]Department of Mathematics, Rajalakshmi Institute of Technology, Poonamalle, Chennai, Tamilnadu, India.
tharamanisharmi@gmail.com
[4]Department of Mathematics, Panimalar Engineering College, Chennai, Tamil Nadu, India.
prathapmaths@gmail.com
[5]Department of Mathematics, Paavai Engineering College (Autonomous), Namakkal-637 018, Tamilnadu, India.
ramkumars_slm@rediffmail.com
[6]Department of English, IFET College of Engineering, Villupuram, Tamilnadu, India.
dhanavelgd@gmail.com

Abstract – In this study, we introduce two novel matrix concepts in the neutrosophic fuzzy domain: range-symmetric Interval-Valued Quadri Partitioned Neutrosophic Fuzzy Matrices and kernel-symmetric Interval-Valued Quadri Partitioned Neutrosophic Fuzzy Matrices. These matrices are defined analogously to EP-matrices within the complex domain. Initially, we establish fundamental characterizations of range-symmetric matrices and then derive the necessary and sufficient conditions under which an Interval-Valued Quadri Partitioned Neutrosophic Fuzzy Matrices becomes kernel-symmetric . A detailed analysis follows to explore the relationship between range-symmetric and kernel-symmetric Interval-Valued Quadri Partitioned Neutrosophic Fuzzy Matrices. Additionally, we introduce the concepts of Kernel and k-Kernel Symmetric Interval-Valued Quadri Partitioned Neutrosophic Fuzzy Matrices, providing illustrative examples to demonstrate their application. Basic results for kernel-symmetric Interval-Valued Quadri Partitioned Neutrosophic Fuzzy Matrices are derived, highlighting that while k-symmetric implies k- kernel-

symmetric in Interval-Valued Quadri Partitioned Neutrosophic Fuzzy Matrices, the converse does not necessarily hold. We further discuss the connections between kernel-symmetric, k- kernel-symmetric and the Moore-Penrose inverse of Interval-Valued Quadri Partitioned Neutrosophic Fuzzy Matrices, supported by numerical examples. The study culminates in an algorithm tailored for solving multi-criteria decision-making problems using Interval-Valued Quadri Partitioned Neutrosophic Fuzzy Matrices, validated through an illustrative example that demonstrates its practical utility.

Keywords: Interval-Valued Quadri Partitioned Neutrosophic Fuzzy Matrices, range-symmetric Interval-Valued Quadri Partitioned Neutrosophic Fuzzy Matrices, kernel-symmetric Interval-Valued Quadri Partitioned Neutrosophic Fuzzy Matrices, Moore-Penrose inverse, Decision-making.

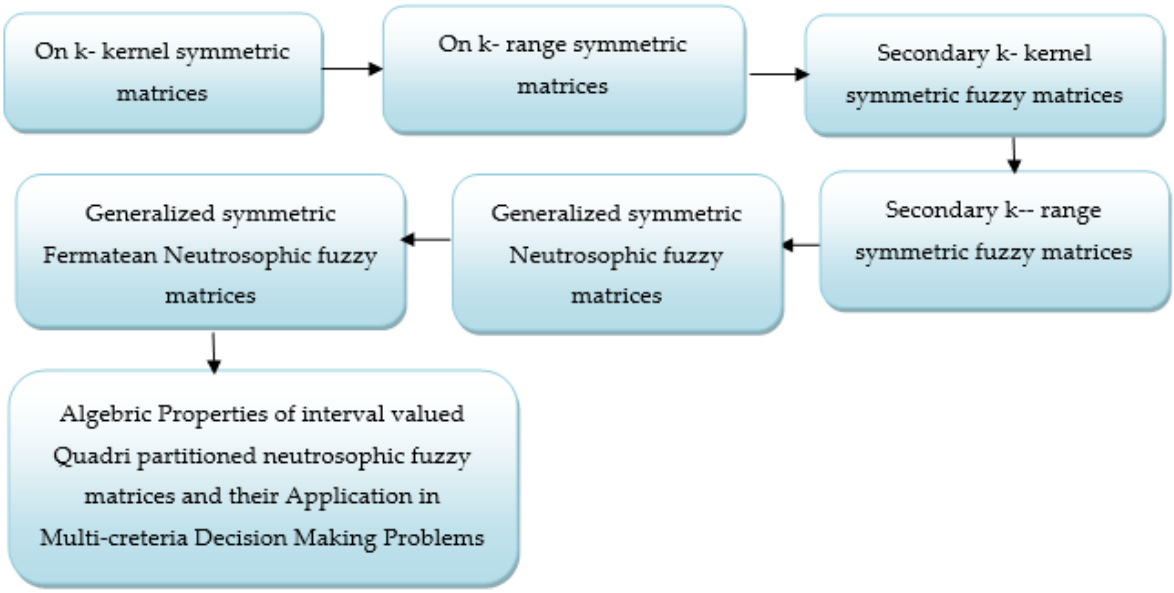
1.Introduction

In recent years, decision-making under uncertainty has gained prominence across various fields due to the growing need to manage vague, imprecise, and indeterminate information effectively. Neutrosophic sets and their extensions provide a flexible framework for modeling such uncertain environments, enabling nuanced representations of truth, indeterminacy, falsity, and hesitation values independently. Among these extensions, Quadri Partitioned Neutrosophic Fuzzy Matrices offer a structured approach to representing multi-dimensional data, making them particularly suitable for applications where handling nuanced and parametric uncertainty systematically is critical. This paper seeks to address the limitations of current fuzzy and neutrosophic matrix approaches in representing and analyzing complex decision-making scenarios. Specifically, it introduces and explores Interval-Valued Quadri Partitioned Neutrosophic Fuzzy Matrices, focusing on their structural properties and applications in predictive and multi-criteria decision-making frameworks.

The theoretical foundations of fuzzy sets, introduced by Zadeh [1] in 1965, revolutionized the mathematical treatment of vagueness in real-world problems by allowing for degrees of membership. Subsequent advancements, such as Atanassov's [2] Intuitionistic Fuzzy Sets and Smarandache's [3] Neutrosophic Sets, further enriched uncertainty modeling by incorporating non-membership and indeterminacy. Neutrosophic sets have spurred the development of neutrosophic matrices, enabling the representation of multi-dimensional uncertainty in systems involving conflicting and ambiguous information. Recent studies have extended fuzzy and neutrosophic matrix theory to specialized forms, such as interval-valued matrices and quadri-partitioned structures, which have demonstrated utility in domains like health risk assessment, supplier evaluation, and financial decision-making.

However, existing research often lacks a thorough exploration of advanced properties, such as secondary symmetry, pseudo-similarity, and k-kernel characteristics, which are crucial for improving the robustness and adaptability of decision models. While Anandhkumar [23],[34] et al. have laid a foundation for understanding range and kernel symmetry in neutrosophic fuzzy matrices, there remains a significant gap in applying these principles to Interval-Valued Quadri Partitioned

Neutrosophic Fuzzy Matrices. Addressing this gap can enhance decision-making accuracy and reliability in high-stakes and uncertain environments.


On k- kernel symmetric matrices
On k- range symmetric matrices
Secondary k- kernel symmetric fuzzy matrices
Secondary k-- range symmetric fuzzy matrices
Generalized symmetric Neutrosophic fuzzy matrices
Generalized symmetric Fermatean Neutrosophic fuzzy matrices
Algebric Properties of interval valued Quadri partitioned neutrosophic fuzzy matrices and their Application in Multi-creteria Decision Making Problems

This study introduces the concept of Interval-Valued Quadri Partitioned Neutrosophic Fuzzy Matrices, extending the principles of range and kernel symmetry to these structures. The research develops necessary and sufficient conditions for range-symmetric, k-range symmetric, kernel symmetric, and k-kernel symmetric matrices, providing a deeper understanding of their algebraic properties. Furthermore, it explores the implications of k-symmetric matrices, demonstrating that k-symmetric Interval-Valued Quadri Partitioned Neutrosophic Fuzzy Matrices are also k-kernel symmetric and k-range symmetric, with specific scenarios where the converse does not hold.

To bridge theory and application, the study constructs an algorithm-based model utilizing these matrices for decision-making problems. The model's applicability is illustrated through a detailed example, showcasing its potential for real-world applications in areas such as financial risk assessment, healthcare, and supply chain management. This work not only fills critical gaps in the literature but also establishes a robust framework for decision-making under uncertainty, paving the way for future research in this evolving field.

The objectives of this study are threefold, focusing on the introduction of new concepts, the development of theoretical foundations, and the creation of application-driven algorithms. The primary aim is to define Interval-Valued Quadri Partitioned Neutrosophic Fuzzy Matrices and explore their underlying mathematical structures, introducing range-symmetric and kernel-symmetric configurations alongside their k-extensions while establishing their algebraic properties. A secondary focus lies in deriving necessary and sufficient conditions for range-symmetric, k-range symmetric, kernel symmetric, and k-kernel symmetric matrices, demonstrating that k-symmetric Interval-Valued Quadri Partitioned Neutrosophic Fuzzy Matrices imply k-kernel symmetry and k-range symmetry, while investigating cases where the converse may not hold. Finally, the study seeks

to develop an algorithm-based model specifically tailored for decision-making problems by leveraging these matrices' unique properties. This model's practical relevance is demonstrated through its application in predictive and multi-criteria decision-making scenarios, thereby bridging the gap between theoretical advancements and real-world utility. Through these objectives, the research aims to establish a comprehensive framework for the effective application of Interval-Valued Quadri Partitioned Neutrosophic Fuzzy Matrices, enhancing both their theoretical depth and practical relevance.

The present study aims to address key limitations in existing fuzzy and neutrosophic matrix theories by developing a comprehensive framework based on Interval-Valued Quadri Partitioned Neutrosophic Fuzzy Matrices. Previous research has successfully applied concepts like Fuzzy Soft Sets, Interval-Valued Fuzzy Soft Sets, and Quadri Partitioned Neutrosophic Soft Sets to solve decision-making problems involving uncertainty and hesitation. However, these models often fail to account for parametric indeterminacy in a neutrosophic environment with dependent components. This work introduces a novel methodology to handle indeterminacy parametrically, offering a more flexible framework for decision-makers. By developing range-symmetric, k-range symmetric, kernel symmetric, and k-kernel symmetric properties for Interval-Valued Quadri Partitioned Neutrosophic Fuzzy Matrices, this study addresses the critical need for refined uncertainty modeling. A comparative analysis of the proposed model with existing soft models highlights its advantages, showcasing its potential to revolutionize decision-making in uncertain environments. This integrated section lays a strong foundation for understanding the study's motivation and objectives, ensuring a seamless transition into the methods and results.

Review of Literature

Decision-making under uncertainty has been a critical area of research across various domains, driven by the increasing complexity of modern systems. The advent of fuzzy set theory by Zadeh in 1965 [1] laid the foundation for handling vagueness and imprecision mathematically. This pioneering work introduced the concept of degrees of membership, revolutionizing the approach to uncertainty. Following this, Intuitionistic Fuzzy Sets by Atanassov [2,22] expanded the framework by incorporating membership, non-membership, and hesitation degrees, enabling more nuanced representations of uncertainty.

The development of neutrosophic sets by Smarandache [3] further extended the landscape of uncertainty modeling by introducing three independent components: truth, indeterminacy, and falsity. This flexibility has made neutrosophic sets particularly useful in applications requiring simultaneous handling of contradictory and incomplete information. Neutrosophic fuzzy matrices, as an extension of this theory, have proven effective in capturing complex relationships and multi-dimensional uncertainties [35–39].

In the realm of fuzzy matrices, foundational studies by Kim and Roush [4] established generalized fuzzy matrix theory, while Meenakshi [5] contributed significantly to their algebraic properties. Special forms of fuzzy matrices, such as κ-Hermitian, κ-real, and EPr matrices [6,7], have been instrumental in addressing structured data analysis in diverse fields. Interval-valued fuzzy

matrices introduced by Shyamal and Pal [12,20] provided an additional layer of robustness by allowing interval representation for matrix elements, making them particularly valuable in areas such as risk analysis and decision support systems [40,43].

More recently, neutrosophic fuzzy matrices have gained attention due to their ability to model multi-layered uncertainties. Research by Anandhkumar et al. [23–29] has delved into properties like pseudo-similarity, secondary symmetry, and partial ordering, providing a refined understanding of these structures. The introduction of quadri-partitioned and interval-valued neutrosophic fuzzy matrices has further expanded their applicability, with studies highlighting their effectiveness in high-stakes domains such as health risk assessment, supplier evaluation, and financial decision-making [36,39,41,42].

Parallel to these developments, soft set theories have emerged as a complementary approach to decision-making. Roy and Maji [46] emphasized the computational utility of fuzzy soft sets, while Yang et al. [47] combined interval-valued fuzzy sets with soft set theory, enriching the theoretical foundation. Neutrosophic fuzzy soft sets, introduced by Said and Smarandache [48,51], and quadripartitioned neutrosophic fuzzy soft sets proposed by Mary [52] have further demonstrated their relevance in addressing indeterminacy and hesitation in decision-making frameworks. J, J., & S, R. [55] have studied Some Operations on Neutrosophic Hypersoft Matrices and Their Applications. Ranulfo Paiva Barbosa and Smarandache [56] have discussed. Pura Vida Neutrosophic Algebra. Anandhkumar et al [57] have studied Determinant Theory of Quadri-Partitioned Neutrosophic Fuzzy Matrices and its Application to Multi-Criteria Decision-Making Problems. Radhika et al [58,59] have focused on Interval Valued Secondary k-Range Symmetric Quadri Partitioned Neutrosophic Fuzzy Matrices with Decision Making and On Schur Complement in k-Kernel Symmetric Block Quadri Partitioned Neutrosophic Fuzzy Matrices, Prathab et al [60] have present Interval Valued Secondary k-Range Symmetric Fuzzy Matrices with Generalized Inverses.

The literature also includes innovative approaches to symmetry in neutrosophic matrices. Studies on k-symmetric and k-kernel matrices by Anandhkumar et al. [27,28] have provided deeper insights into matrix behavior under various algebraic operations. These contributions underscore the potential of advanced neutrosophic matrix properties in enhancing decision-making accuracy and robustness in uncertain environments. Despite these advancements, several research gaps remain. Existing literature often lacks a comprehensive exploration of the interaction between advanced matrix properties and real-world decision frameworks. Moreover, the behavior of interval-valued quadri-partitioned neutrosophic fuzzy matrices under complex conditions, such as multi-attribute or multi-objective criteria, remains underexplored. Addressing these gaps through extended algebraic characterization and empirical validation is essential for advancing the practical utility of these models in high-uncertainty scenarios.

1.1 Notations:

For Interval-Valued Quadri Partitioned Neutrosophic Fuzzy Matrices of $S = <[S^T, S^C, S^I, S^F]_L, [S^T, S^C, S^I, S^F]_U > \in \left(IVQPNFM\right)_n$ where

$[S^T, S^C, S^I, S^F]_L{}^{\mathrm{T}}$: Transpose of Interval-Valued Quadri Partitioned Neutrosophic Fuzzy Matrices $[S^T, S^C, S^I, S^F]_L$.

R($[S^T, S^C, S^I, S^F]_L$) : Row space of Interval-Valued Quadri Partitioned Neutrosophic Fuzzy Matrices $[S^T, S^C, S^I, S^F]_L$.

C($[S^T, S^C, S^I, S^F]_L$) : Column space of Interval-Valued Quadri Partitioned Neutrosophic Fuzzy Matrices $[S^T, S^C, S^I, S^F]_L$.

N($[S^T, S^C, S^I, S^F]_L$) : Null Space of Interval-Valued Quadri Partitioned Neutrosophic Fuzzy Matrices $[S^T, S^C, S^I, S^F]_L$.

$[S^T, S^C, S^I, S^F]_L{}^{+}$: Moore-Penrose inverse of Interval-Valued Quadri Partitioned Neutrosophic Fuzzy Matrices $[S^T, S^C, S^I, S^F]_L$.

2. Preliminaries

Definition: 2.1 [5] (page no 118) (EP- Matrices)} An EP- matrix (or range-symmetric matrix) is a square Fuzzy matrix A whose range is equal to the range of its transpose A^T and it's denoted by R(A) = R(A^T).

Definition 2.2 [53] Let X is an initial universe set and E is a set of parameters. Consider a non-empty set A where A ⊆ E. Let P(X) denote the set of all Quadri Partitioned Neutrosophic soft sets of X. The collection (F, A) is termed the Quadri Partitioned Neutrosophic soft sets over X, where F is a mapping given by F : A⟶ P(X). Here,

$A = \{< x, T_A(x), C_A(x), U_A(x), FA(x) >: x \in U\}$ with T_A , F_A, C_A ,$U_A : X \longrightarrow [0,1]$ and $0 \leq T_A(x) + C_A(x) + U_A(x) + F_A(x) \leq 4$. In this context

- $T_A(x)$ is the TM,
- $C_A(x)$ is CM,
- $U_A(x)$ is IM,
- $F_A(x)$ is the FM,

3. Interval -Valued Quadri Partitioned Neutrosophic Fuzzy Matrices.

The following properties are essential for proving the theorems and examples presented in this work.

(P_1) $K = K^T$, $K^2 = I$ and $\kappa(y) = Ky$ for every

$$S = <[S^T, S^C, S^I, S^F]_L, [S^T, S^C, S^I, S^F]_U > \in \left(IVSQPNFM\right)_n$$

(P_2) $N([S^T, S^C, S^I, S^F]_L) = N([S^T, S^C, S^I, S^F]_L K) = N(K[S^T, S^C, S^I, S^F]_L)$

$N([S^T, S^C, S^I, S^F]_U) = N([S^T, S^C, S^I, S^F]_U K) = N(K[S^T, S^C, S^I, S^F]_U)$

(P_3) $([S^T, S^C, S^I, S^F]_L K)^+ = K[S^T, S^C, S^I, S^F]_L{}^+$ and

$(K[S^T, S^C, S^I, S^F]_L)^+ = [S^T, S^C, S^I, S^F]_L{}^+ K$ exists, if $[S^T, S^C, S^I, S^F]_L{}^+$ exists.

$([S^T, S^C, S^I, S^F]_U K)^+ = K[S^T, S^C, S^I, S^F]_U{}^+$ and $(K[S^T, S^C, S^I, S^F]_U)^+ = [S^T, S^C, S^I, S^F]_U{}^+ K$

exists, if $[S^T, S^C, S^I, S^F]_L{}^+$ exists.

(P_4) $[S^T, S^C, S^I, S^F]_L{}^T$ is a g-inverse of $[S^T, S^C, S^I, S^F]_L$ iff $[S^T, S^C, S^I, S^F]_L{}^+$ exist.

$[S^T, S^C, S^I, S^F]_U{}^T$ is a g-inverse of $[S^T, S^C, S^I, S^F]_U$ iff $[S^T, S^C, S^I, S^F]_U{}^+$ exist.

Definition: 3.1: An Interval-Valued Quadri Partitioned Neutrosophic Fuzzy Matrices S of order m×n is well-defined as $S = [x_{ij}, < T_{ij\mu}, C_{ij\lambda} U_{iju}, F_{ijv} >]_{m\times n}$ where $T_{ij\mu}$, $C_{ij\lambda} U_{iju}$, and F_{ijv} are the subsets of [0,1] which are denoted by $T_{ij\mu} = [T_{ij\mu L}, T_{ij\mu U}]$, $C_{ij\lambda} = [C_{ij\lambda L}, C_{ij\lambda U}]$ and $U_{iju} = [U_{iju L}, U_{iju U}]$ and $F_{ijv} = [F_{ijv L}, F_{ijv U}]$ which maintaining the condition $0 \le T_{ij\mu L} + C_{ij\lambda L} + U_{iju L} + F_{ijv L} \le 4$, $0 \le T_{ij\mu U} + C_{ij\lambda U} + U_{iju U} + F_{ijv U} \le 4$, and $0 \le T_{ij\mu L} \le T_{ij\mu U} \le 1$, $0 \le C_{ij\lambda L} \le C_{ij\lambda U} \le 1$, $0 \le U_{iju L} \le U_{iju U} \le 1$, $0 \le F_{ijv L} \le F_{ijv U} \le 1$.

Example 3.1 Let us consider an Interval-Valued Quadri Partitioned Neutrosophic Fuzzy Matrices.

$$S = \begin{bmatrix} <[1,1],[1,1],[0,0],[0,0]> & <[0.1,0.3],[0.2,0.4],[0.2,0.5],[0.3,0.5]> \\ <[0.1,0.3],[0.2,0.4],[0.2,0.5],[0.3,0.5]> & <[1,1],[1,1],[0,0],[0,0]> \end{bmatrix}$$

Lower Limit Quadri Partitioned Neutrosophic Fuzzy Matrices,

$$[S_T, S_C, S_I, S_F]_L = \begin{bmatrix} <1,1,0,0> & <0.1,0.2,0,2,0.3> \\ <0.1,0.2,0.2,0.3> & <1,1,0,0> \end{bmatrix},$$

Upper Limit Quadri Partitioned Neutrosophic Fuzzy Matrices,

$$[S_T, S_C, S_I, S_F]_U = \begin{bmatrix} <1,1,0,0> & <0.3,0.4,0.5,0.5> \\ <0.3,0.4,0.5,0.5> & <1,1,0,0> \end{bmatrix}$$

Definition: 3.2 Let $S = <[S^T, S^C, S^I, S^F]_L, [S^T, S^C, S^I, S^F]_U > \in (IVSQPNFM)_n$ be a Interval-Valued Quadri Partitioned Neutrosophic Fuzzy Matrices, if R [$[S^T, S^C, S^I, S^F]_L$] = R [$[S^T, S^C, S^I, S^F]_L$ T], then S is called as Lower range symmetric Interval-Valued Quadri Partitioned Neutrosophic Fuzzy Matrices if R[$[S^T, S^C, S^I, S^F]_U$] = R [$[S^T, S^C, S^I, S^F]_U$ T] then S is called as Upper range symmetric Interval-Valued Quadri Partitioned Neutrosophic Fuzzy Matrices.

Example: 3.2 Consider an Interval-Valued Quadri Partitioned Neutrosophic Fuzzy Matrices

S=	$<[0.4,0.5],[0,0.3],[0.7,0.8],$ $[1,1]>$	$<[1,1],[0,0.5],[1,1],$ $[1,1]>$	$<[0.2,0.3],[0.3,0.4],[0.4,0.5],$ $[0.6,0.7]>$
	$<[1,1],[0,0.5],[1,1],$ $[1,1]>$	$<[1,1],[0,1],[1,1],$ $[1,1]>$	$<[1,1],[0,0.9],[1,1],$ $[1,1]>$
	$<[0.2,0.3],[0.3,0.4],[0.4,0.5],$ $[0.6,0.7]>$	$<[1,1],[0,0.9],[1,1],$ $[1,1]>$	$<[1,1],[0,1],[1,1],$ $[1,1]>$

$$[S^T, S^C, S^I, S^F]_L = \begin{bmatrix} <0.4,0,1,0.7,1> & <1,0,1,1> & <0.2,0.3,0.4,0.6> \\ <1,0,1,1> & <1,0,1,1> & <1,0,1,1> \\ <0.2,0.3,0.4,0.6> & <1,0,1,1> & <1,0,1,1> \end{bmatrix}$$

Here, R [$[S^T, S^C, S^I, S^F]_L$] = R [$[S^T, S^C, S^I, S^F]_L$ T]

$$[S^T, S^C, S^I, S^F]_U = \begin{bmatrix} <0.5,0,3,0.8,1> & <1,0.5,1,1> & <0.3,0.4,0.5,0.7> \\ <1,0.5,1,1> & <1,0,1,1> & <1,0.9,1,1> \\ <0.3,0.4,0.5,0.7> & <1,0.9,1,1> & <1,0,1,1> \end{bmatrix}$$

Here, R [$[S^T, S^C, S^I, S^F]_U$] = R [$[S^T, S^C, S^I, S^F]_U$ T]

The following Interval-Valued Quadri Partitioned Neutrosophic Fuzzy Matrices does not satisfies the range symmetric Quadri Partitioned Neutrosophic Fuzzy Matrices condition.

$$[S^T, S^C, S^I, S^F] = \begin{bmatrix} <0.4,0,1,0.9> & <1,0,1,1> & <0.2,0.3,0.4,0.6> \\ <1,0,1,0.9> & <1,0,1,1> & <1,0,1,1> \\ <0.2,0.3,0.4,0.4> & <1,0,1,1> & <1,0,1,1> \end{bmatrix}$$

$$[S^T, S^C, S^I, S^F]^T = \begin{bmatrix} <0.4,0,1,0.9> & <1,0,1,0.9> & <0.2,0.3,0.4,0.4> \\ <1,0,1,1> & <1,0,1,1> & <1,0,1,1> \\ <0.2,0.3,0.4,0.6> & <1,0,1,1> & <1,0,1,1> \end{bmatrix}$$

$[(0.4,0,1,0.9)\ (1,0,1,1)\ (0.2,0.3,0.4,0.6)] \in R([S^T, S^C, S^I, S^F])$,

$[(0.4,0,1,0.9)\ (1,0,1,1)\ (0.2,0.3,0.4,0.6)] \notin R([S^T, S^C, S^I, S^F]^T)$

$[(1,0,1,0.9)\ (1,0,1,1)\ (1,0,1,1)] \in R([S^T, S^C, S^I, S^F])$,

$[(1,0,1,0.9)\ (1,0,1,1)\ (1,0,1,1)] \in R([S^T, S^C, S^I, S^F]^T)$

$[(0.2,0.3,0.4,0.4)\ (1,0,1,1)\ (1,0,1,1)] \in R([S^T, S^C, S^I, S^F])$,

$[(0.2,0.3,0.4,0.4)\ (1,0,1,1)\ (1,0,1,1)] \notin R([S^T, S^C, S^I, S^F]^T)$

$R([S^T, S^C, S^I, S^F]) \notin R([S^T, S^C, S^I, S^F]^T)$

Note: 3.1 For Interval-Valued symmetric Quadri Partitioned Neutrosophic Fuzzy Matrices S with det $[S^T, S^C, S^I, S^F]_L$ > <0,0,1,1> has non- zero rows and non-columns, hereafter N($[S^T, S^C, S^I, S^F]_L$) = <0,0,1,1> = N($[S^T, S^C, S^I, S^F]_L$ T). Furthermore, a symmetric matrix $[S^T, S^C, S^I, S^F]_L$ = $[S^T, S^C, S^I, S^F]_L$ T that is N($[S^T, S^C, S^I, S^F]_L$)= N($[S^T, S^C, S^I, S^F]_L$ T). Similarly, for Interval-Valued symmetric Quadri Partitioned Neutrosophic Fuzzy Matrices S with det $[S^T, S^C, S^I, S^F]_U$ ><0,0,1,1> has non- zero rows and columns, in future N($[S^T, S^C, S^I, S^F]_U$) = <0,0,1,1> = N($[S^T, S^C, S^I, S^F]_U$ T). Furthermore, a symmetric matrix $[S^T, S^C, S^I, S^F]_U$ = $[S^T, S^C, S^I, S^F]_U$ T that is N($[S^T, S^C, S^I, S^F]_U$)= N($[S^T, S^C, S^I, S^F]_U$ T).

Definition: 3.3 Let $S = <[S^T, S^C, S^I, S^F]_L, [S^T, S^C, S^I, S^F]_U> \in (IVSQPNFM)_n$ be a Interval-Valued symmetric Quadri Partitioned Neutrosophic Fuzzy Matrices, if R [$[S^T, S^C, S^I, S^F]_L$] = R [k $[S^T, S^C, S^I, S^F]_L$ T k], then S is called as Lower k- range symmetric Interval-Valued Quadri Partitioned Neutrosophic Fuzzy Matrices if R[$[S^T, S^C, S^I, S^F]_U$]

= R [k$[S^T, S^C, S^I, S^F]_U$ Tk] then S is called as Upper k-range symmetric Interval-Valued symmetric Quadri Partitioned Neutrosophic Fuzzy Matrices.

Definition : 3.4 Let $S = <[S^T, S^C, S^I, S^F]_L, [S^T, S^C, S^I, S^F]_U > \in (IVSQPNFM)_n$ if N($[S^T, S^C, S^I, S^F]_L$) = N($[S^T, S^C, S^I, S^F]_L$ T) and S is called Lower kernel symmetric - Interval-Valued Quadri Partitioned Neutrosophic Fuzzy Matrices where N($[S^T, S^C, S^I, S^F]_L$) = {x/x $[S^T, S^C, S^I, S^F]_L$ = (0,0,1,1) and x ∈ $F_{1\times n}$} if N($[S^T, S^C, S^I, S^F]_U$) = N($[S^T, S^C, S^I, S^F]_U$ T) and S is called Upper kernel symmetric Interval-Valued Quadri Partitioned Neutrosophic Fuzzy Matrices where N($[S^T, S^C, S^I, S^F]_U$) = {x/x$[S^T, S^C, S^I, S^F]_U$ = (0,0,1,1) and x ∈ $F_{1\times n}$} .

Example: 3.3 Consider an Interval-Valued Quadri Partitioned Neutrosophic Fuzzy Matrices

S=	$<[0.4,0.5],[0.4,0.5],[0.6,0.8],[0.9,1]>$	$<[0.6,1],[0.7,0.8],[0.7,1],[0.3,1]>$	$<[0.5,0.6],[0.6,0.6],[0.7,0.8],[0.4,0.7]>$
	$<[0.6,1],[0.8,0.8],[0.7,1],[0.5,1]>$	$<[0.7,1],[0.9,1],[0.2,1],[0.3,1]>$	$<[0.3,1],[0.7,0.9],[0.2,1],[0.4,1]>$
	$<[0.7,0.7],[0.6,0.7],[0.7,0.8],[0.3,0.7]>$	$<[0.5,1],[0.6,0.9],[0.6,1],[0.8,1]>$	$<[0.5,1],[0.5,1],[0.6,1],[0.4,1]>$

$$[S^T, S^C, S^I, S^F]_L = \begin{bmatrix} <0.4,0.4,0.6,0.9> & <1,0.8,1,1> & <0.6,0.6,0.8,0.7> \\ <0.6,0.8,0.7,0.5> & <0.7,0.9,0.2,0.3> & <0.3,0.7,0.2,0.5> \\ <0.7,0.6,0.7,0.3> & <0.5,0.6,0.6,0.8> & <0.5,0.5,0.6,0.4> \end{bmatrix}$$

$$N([S^T, S^C, S^I, S^F]_L) = N([S^T, S^C, S^I, S^F]_L^T) = (0,0,1,1).$$

$$[S^T, S^C, S^I, S^F]_U = \begin{bmatrix} <0.5,0.5,0.8,0.1> & <0.6,0.7,0.7,0.3> & <0.5,0.6,0.7,0.4> \\ <1,0.8,1,1> & <1,1,1,1> & <1,0.9,1,1> \\ <0.7,0.7,0.8,0.7> & <1,0.9,1,1> & <1,1,1,1> \end{bmatrix}$$

$$N([S^T, S^C, S^I, S^F]_L) = N([S^T, S^C, S^I, S^F]_L^T) = (0,0,1,1).$$

Definition 3.5. Permutation Quadri Partitioned Neutrosophic Fuzzy Matrices: If each row and each column contains correctly one <1,1,0,0> and all other entries are <0,0,1,1> in a square Quadri Partitioned Neutrosophic Fuzzy Matrices, it is known as Quadri Partitioned Neutrosophic Fuzzy Matrices.

Example: 3.4 Consider a Quadri Partitioned Neutrosophic Fuzzy Matrices,

$$K = \begin{bmatrix} (1,1,0,0) & (0,0,1,1) & (0,0,1,1) \\ (0,0,1,1) & (1,1,0,0) & (0,0,1,1) \\ (0,0,1,1) & (0,0,1,1) & (1,1,0,0) \end{bmatrix}$$

4. Theorems and Results

Theorem:4.1 For an Interval-Valued Quadri Partitioned Neutrosophic Fuzzy Matrices $S =< [S^T, S^C, S^I, S^F]_L, [S^T, S^C, S^I, S^F]_U >$, $T =< [T^T, T^C, T^I, T^F]_L, [T^T, T^C, T^I, T^F]_U >$ and K be a Quadri Partitioned Neutrosophic Fuzzy permutation Matrices if $N([S^T, S^C, S^I, S^F]_L) = N([T^T, T^C, T^I, T^F]_L)$ $\Leftrightarrow$ $N(K[S^T, S^C, S^I, S^F]_L K^T) = N(K[T^T, T^C, T^I, T^F]_L K^T)$ and $N([S^T, S^C, S^I, S^F]_U) = N([T^T, T^C, T^I, T^F]_U)$ $\Leftrightarrow$ $N(K[S^T, S^C, S^I, S^F]_U K^T) = N(K[T^T, T^C, T^I, T^F]_U K^T)$.

Proof: Let $w \in N(K[S^T, S^C, S^I, S^F]_L K^T)$

$\Rightarrow w(K[S^T, S^C, S^I, S^F]_L K^T) = (0,0,1,1)$

$\Rightarrow yK^T = (0,0,1,1)$ where $y = wK([S^T, S^C, S^I, S^F]_L)$

$\Rightarrow y \in N(K^T)$

$\det K = \det K^T > (0,0,1,1)$

Then, $N(K^T) = (0,0,1,1)$

Hereafter, $y = (0,0,1,1) \Rightarrow wK([S^T, S^C, S^I, S^F]_L) = (0,0,1,1)$

$\Rightarrow wK \in N([S^T, S^C, S^I, S^F]_L) = N([T^T, T^C, T^I, T^F]_L)$

$\Rightarrow wK([T^T, T^C, T^I, T^F]_L)K^T = (0,0,1,1) \Rightarrow w \in N(K([T^T, T^C, T^I, T^F]_L)K^T)$

$N(K[S^T, S^C, S^I, S^F]_L K^T) \subseteq N(K[T^T, T^C, T^I, T^F]_L K^T)$

Similarly, $N(K[T^T, T^C, T^I, T^F]_L K^T) \subseteq N(K[S^T, S^C, S^I, S^F]_L K^T)$

Therefore, $N([S^T, S^C, S^I, S^F]_L) = N([T^T, T^C, T^I, T^F]_L)$

$\Leftrightarrow N(K[S^T, S^C, S^I, S^F]_L K^T) = N(K[T^T, T^C, T^I, T^F]_L K^T)$

Conversely, if $N(K[S^T, S^C, S^I, S^F]_L K^T) = N(K[T^T, T^C, T^I, T^F]_L K^T)$,

N $[S^T, S^C, S^I, S^F]_L$ = N (K^T (K$[S^T, S^C, S^I, S^F]_L$ K^T) K) = N (K^T (K$([T^T, T^C, T^I, T^F]_L)$ K^T) K)

$N([S^T, S^C, S^I, S^F]_L) = N([T^T, T^C, T^I, T^F]_L)$

Similarly, $N([S^T, S^C, S^I, S^F]_U) = N([T^T, T^C, T^I, T^F]_U)$

$\Leftrightarrow N(K[S^T, S^C, S^I, S^F]_U K^T) = N(K[T^T, T^C, T^I, T^F]_U K^T)$.

Example: 4.1 Consider an Interval-Valued Quadri Partitioned Neutrosophic Fuzzy Matrices

$$[S^T, S^C, S^I, S^F]_L = \begin{bmatrix} <0.3,0.3,0.4,0.9> & <0.5,0.3,0.6,0.8> & <0.2,0.3,0.4,0.4> \\ <0.5,0.4,0.7,0.6> & <0.6,0.4,0.8,0.1> & <0.8,0.2,0.5,0.1> \\ <0.2,0.4,0.4,0.4> & <0.4,0.4,0.5,0.9> & <0.4,0.5,0.6,0.5> \end{bmatrix}$$

$$K = \begin{bmatrix} (1,1,0,0) & (0,0,1,1) & (0,0,1,1) \\ (0,0,1,1) & (1,1,0,0) & (0,0,1,1) \\ (0,0,1,1) & (0,0,1,1) & (1,1,0,0) \end{bmatrix}$$

$$[T^T, T^C, T^I, T^F]_L = \begin{bmatrix} <0.5,0.3,0.7,0.9> & <0.4,0.3,0.6,0.4> & <0.3,0.4,0.4,0.7> \\ <0.5,0.7,0.5,0.7> & <0.4,0.8,0.2,0.8> & <0.2,0.5,0.2,0.3> \\ <0.2,0.4,0.4,0.5> & <0.4,0.4,0.5,0.9> & <0.4,0.5,0.6,0.8> \end{bmatrix}$$

$$K[S^T, S^C, S^I, S^F]_L K^T = \begin{bmatrix} (1,1,0,0) & (0,0,1,1) & (0,0,1,1) \\ (0,0,1,1) & (1,1,0,0) & (0,0,1,1) \\ (0,0,1,1) & (0,0,1,1) & (1,1,0,0) \end{bmatrix}$$

$$\begin{bmatrix} <0.3,0.3,0.4,0.9> & <0.5,0.3,0.6,0.8> & <0.2,0.3,0.4,0.4> \\ <0.5,0.4,0.7,0.6> & <0.6,0.4,0.8,0.1> & <0.8,0.2,0.5,0.1> \\ <0.2,0.4,0.4,0.4> & <0.4,0.4,0.5,0.9> & <0.4,0.5,0.6,0.5> \end{bmatrix}$$

$$\begin{bmatrix} (1,1,0,0) & (0,0,1,1) & (0,0,1,1) \\ (0,0,1,1) & (1,1,0,0) & (0,0,1,1) \\ (0,0,1,1) & (0,0,1,1) & (1,1,0,0) \end{bmatrix}$$

$$= \begin{bmatrix} <0.3,0.3,0.4,0.9> & <0.5,0.3,0.6,0.8> & <0.2,0.3,0.4,0.4> \\ <0.5,0.4,0.7,0.6> & <0.6,0.4,0.8,0.1> & <0.8,0.2,0.5,0.1> \\ <0.2,0.4,0.4,0.4> & <0.4,0.4,0.5,0.9> & <0.4,0.5,0.6,0.5> \end{bmatrix}$$

Let $w \in N(K[S^T, S^C, S^I, S^F]_L K^T)$

$w = [(0,0,1,1) \quad (0,0,1,1) \quad (0,0,1,1)]$

$\Rightarrow w(K[S^T, S^C, S^I, S^F]_L K^T) = [(0,0,1,1) \quad (0,0,1,1) \quad (0,0,1,1)]$

$$\begin{bmatrix} <0.3,0.3,0.4,0.9> & <0.5,0.3,0.6,0.8> & <0.2,0.3,0.4,0.4> \\ <0.5,0.4,0.7,0.6> & <0.6,0.4,0.8,0.1> & <0.8,0.2,0.5,0.1> \\ <0.2,0.4,0.4,0.4> & <0.4,0.4,0.5,0.9> & <0.4,0.5,0.6,0.5> \end{bmatrix} = (0,0,1,1)$$

$\Rightarrow yK^T = (0,0,1,1)$ where $y = wK([S^T, S^C, S^I, S^F]_L)$. Hence, $y = (0,0,1,1)$

$$K[S^T, S^C, S^I, S^F]_L = \begin{bmatrix} <0.3,0.3,0.4,0.9> & <0.5,0.3,0.6,0.8> & <0.2,0.3,0.4,0.4> \\ <0.5,0.4,0.7,0.6> & <0.6,0.4,0.8,0.1> & <0.8,0.2,0.5,0.1> \\ <0.2,0.4,0.4,0.4> & <0.4,0.4,0.5,0.9> & <0.4,0.5,0.6,0.5> \end{bmatrix}$$

$\Rightarrow wK([S^T, S^C, S^I, S^F]_L) = (0,0,1,1)$

$\Rightarrow wK \in N([S^T, S^C, S^I, S^F]_L) = N([T^T, T^C, T^I, T^F]_L)$

$\Rightarrow wK([T^T, T^C, T^I, T^F]_L)K^T = (0,0,1,1)$

$N([S^T, S^C, S^I, S^F]_L) = N([T^T, T^C, T^I, T^F]_L)$

$\Leftrightarrow N(K[S^T, S^C, S^I, S^F]_L \; K^T) = N(K[T^T, T^C, T^I, T^F]_L K^T)$.

Similarly, $N([S^T, S^C, S^I, S^F]_U) = N([T^T, T^C, T^I, T^F]_U)$

$\Leftrightarrow N(K[S^T, S^C, S^I, S^F]_U \; K^T) = N(K[T^T, T^C, T^I, T^F]_U K^T)$.

Theorem:4.2 For an Interval-Valued Quadri Partitioned Neutrosophic Fuzzy Matrices $S = <[S^T, S^C, S^I, S^F]_L, [S^T, S^C, S^I, S^F]_U > \in (IVSQPNFM)_n$ and K be Quadri Partitioned Neutrosophic Fuzzy Permutation Matrices if $N([S^T, S^C, S^I, S^F]_L) = N([S^T, S^C, S^I, S^F]_L{}^T)$

$\Leftrightarrow$N(K$[S^T, S^C, S^I, S^F]_L$ K^T) = N(K $[S^T, S^C, S^I, S^F]_L{}^T$ K^T) and N(K$[S^T, S^C, S^I, S^F]_U$ K^T) = N(K

$[S^T, S^C, S^I, S^F]_U{}^T$ K^T) $\Leftrightarrow$

$$N([S^T, S^C, S^I, S^F]_U) = N([S^T, S^C, S^I, S^F]_U{}^T)$$

Proof: The proof follows a structure like that of Theorem 9.1

Example: 4.2 Consider an Quadri Partitioned Neutrosophic Fuzzy permutation Matrices

$$K = \begin{bmatrix} (1,1,0,0) & (0,0,1,1) & (0,0,1,1) \\ (0,0,1,1) & (1,1,0,0) & (0,0,1,1) \\ (0,0,1,1) & (0,0,1,1) & (1,1,0,0) \end{bmatrix}$$

$$[S^T, S^C, S^I, S^F]_L = \begin{bmatrix} <0.7,0.3,0.5,0.9> & <0.5,0.3,0.6,0.7> & <0.2,0.3,0.4,0.8> \\ <0.4,0.7,0.6,0.6> & <0.4,0.8,0.1,0.7> & <0.2,0.5,0.1,0.3> \\ <0.2,0.4,0.4,0.5> & <0.4,0.4,0.5,0.8> & <0.4,0.5,0.6,0.4> \end{bmatrix}$$

$$[S^T, S^C, S^I, S^F]_U = \begin{bmatrix} <0.8,0.4,0.6,1> & <0.6,0.4,0.7,0.8> & <0.3,0.5,0.5,0.9> \\ <0.5,0.8,0.7,0.7> & <0.5,0.9,0.2,0.8> & <0.3,0.6,0.2,0.4> \\ <0.3,0.5,0.5,0.6> & <0.5,0.5,0.6,0.9> & <0.5,0.6,0.7,0.5> \end{bmatrix}$$

Theorem: 4.3 For $S = <[S^T, S^C, S^I, S^F]_L, [S^T, S^C, S^I, S^F]_U>$ is kernel symmetric Interval-Valued Quadri Partitioned Neutrosophic Fuzzy Matrices, then N($[S^T, S^C, S^I, S^F]_L$ $[S^T, S^C, S^I, S^F]_L{}^T$) = N($[S^T, S^C, S^I, S^F]_L$) = N($[S^T, S^C, S^I, S^F]_L{}^T$ $[S^T, S^C, S^I, S^F]_L$) and N($[S^T, S^C, S^I, S^F]_U$ $[S^T, S^C, S^I, S^F]_U{}^T$) = N($[S^T, S^C, S^I, S^F]_U$) = N($[S^T, S^C, S^I, S^F]_U{}^T$ $[S^T, S^C, S^I, S^F]_U$).

Example: 4.3 Consider an Interval-Valued Quadri Partitioned Neutrosophic Fuzzy Matrices

$$[S^T, S^C, S^I, S^F]_L = \begin{bmatrix} <0.1,0.3,0.5,0.8> & <0.4,0.3,0.6,0.7> & <0.2,0.3,0.4,0.3> \\ <0.4,0.7,0.6,0.4> & <0.4,0.7,0.1,0.5> & <0.2,0.5,0.1,0.2> \\ <0.2,0.3,0.5,0.6> & <0.4,0.3,0.5,0.9> & <0.4,0.2,0.1,0.6> \end{bmatrix}$$

$$[S^T, S^C, S^I, S^F]_U = \begin{bmatrix} <0.2,0.5,0.5,0.9> & <0.5,0.4,0.7,0.9> & <0.4,0.5,0.7,0.4> \\ <0.5,0.8,0.7,0.5> & <0.5,0.7,0.2,0.6> & <0.2,0.7,0.2,0.4> \\ <0.2,0.5,0.7,0.8> & <0.5,0.4,0.5,0.9> & <0.5,0.2,0.1,0.6> \end{bmatrix}$$

$$K=\begin{bmatrix}(1,1,0,0) & (0,0,1,1) & (0,0,1,1)\\ (0,0,1,1) & (1,1,0,0) & (0,0,1,1)\\ (0,0,1,1) & (0,0,1,1) & (1,1,0,0)\end{bmatrix}$$

Theorem:4.4 For a Interval-Valued symmetric Quadri Partitioned Neutrosophic Fuzzy Matrices $S=<[S^T,S^C,S^I,S^F]_L,[S^T,S^C,S^I,S^F]_U>, T=<[T^T,T^C,T^I,T^F]_L,[T^T,T^C,T^I,T^F]_U>$ and K be a Quadri Partitioned Neutrosophic Fuzzy permutation Matrices, R ($[S^T,S^C,S^I,S^F]_L$) = R ($[T^T,T^C,T^I,T^F]_L$) $\Leftrightarrow$ R (K $[S^T,S^C,S^I,S^F]_L$ K^T) = R (K $[T^T,T^C,T^I,T^F]_L$ K^T) and R ($[S^T,S^C,S^I,S^F]_U$) = R ($[T^T,T^C,T^I,T^F]_U$) $\Leftrightarrow$ R (K $[S^T,S^C,S^I,S^F]_U$ K^T) = R (K $[T^T,T^C,T^I,T^F]_U$ K^T).

Proof: Let R ($[S^T,S^C,S^I,S^F]_L$) = R ($[T^T,T^C,T^I,T^F]_L$)

Then, R ($[S^T,S^C,S^I,S^F]_L$ K^T) = R ($[S^T,S^C,S^I,S^F]_L$) K^T

= R ($[S^T,S^C,S^I,S^F]_L$) K^T

= R ($[S^T,S^C,S^I,S^F]_L$ K^T)

Let $z\in\left\{R(K[S^T,S^C,S^I,S^F]_L K^T)\right\}$

$z=w(K[S^T,S^C,S^I,S^F]_L K^T)$ for some $w\in V^n$

$z=r[S^T,S^C,S^I,S^F]_L K^T,\ r=wK$

$z\in R\left([S^T,S^C,S^I,S^F]_L K^T\right)=R\left([T^T,T^C,T^I,T^F]_L\left(K^T\right)\right)$

$z=u[T^T,T^C,T^I,T^F]_L K^T$ for some $u\in V^n$

$z=\left(uK^T\right)K[T^T,T^C,T^I,T^F]_L K^T$

$z=vK[T^T,T^C,T^I,T^F]_L K^T$ for some $v\in V^n$

$z\in R(K[T^T,T^C,T^I,T^F]_L K^T)$

Therefore, $R(K[S^T,S^C,S^I,S^F]_L K^T)\subseteq R(K[T^T,T^C,T^I,T^F]_L K^T)$

Similarly, $R(K[T^T,T^C,T^I,T^F]_L K^T) \subseteq R(K[S^T,S^C,S^I,S^F]_L K^T)$

Therefore, R(K$[S^T,S^C,S^I,S^F]_L$ K^T) = R(K$[T^T,T^C,T^I,T^F]_L$ K^T)

Similarly, ($[S^T,S^C,S^I,S^F]_U$) = R ($[T^T,T^C,T^I,T^F]_U$) ⇔ R (K $[S^T,S^C,S^I,S^F]_U$ K^T) = R (K $[T^T,T^C,T^I,T^F]_U$ K^T).

Conversely, Let R(K$[S^T,S^C,S^I,S^F]_L$ K^T) = R(K$[T^T,T^C,T^I,T^F]_L$ K^T)

$$R([S^T,S^C,S^I,S^F]_L) = R[K^T(K[S^T,S^C,S^I,S^F]_L K^T)K]$$

$$= R[K^T(K[T^T,T^C,T^I,T^F]_L K^T)K]$$

$$= R([T^T,T^C,T^I,T^F]_L)$$

$$R([S^T,S^C,S^I,S^F]_L) = R([T^T,T^C,T^I,T^F]_L)$$

Similarly,

$$R([S^T,S^C,S^I,S^F]_U) = R([T^T,T^C,T^I,T^F]_U)$$

Example: 4.4 Consider an Interval-Valued symmetric Quadri Partitioned Neutrosophic Fuzzy Matrices

$$[S^T,S^C,S^I,S^F]_L = \begin{bmatrix} <0.2,0.4,0.7,0.9> & <0.9,0.9,0.9,0.9> & <0.2,0.3,0.4,0.8> \\ <0.4,0.3,0.6,0.6> & <0.4,0.7,0.1,0.6> & <0.3,0.4,0.5,0.9> \\ <0.1,0.2,0.3,0.4> & <0.7,0.8,0.2,0.1> & <0.8,0.5,0.7,0.9> \end{bmatrix}$$

$$[T^T,T^C,T^I,T^F]_L = \begin{bmatrix} <0.1,0.2,0.3,0.4> & <0.7,0.8,0.2,0.1> & <0.8,0.5,0.7,0.9> \\ <0.4,0.3,0.6,0.5> & <0.4,0.7,0.1,0.6> & <0.3,0.4,0.5,0.9> \\ <0.2,0.4,0.7,0.9> & <0.9,0.9,0.9,0.9> & <0.2,0.3,0.4,0.8> \end{bmatrix}$$

$$K = \begin{bmatrix} (1,1,0,0) & (0,0,1,1) & (0,0,1,1) \\ (0,0,1,1) & (1,1,0,0) & (0,0,1,1) \\ (0,0,1,1) & (0,0,1,1) & (1,1,0,0) \end{bmatrix}$$

Theorem:4.5 For $S = <[S^T,S^C,S^I,S^F]_L,[S^T,S^C,S^I,S^F]_U>$ be the Interval-Valued symmetric Quadri Partitioned Neutrosophic Fuzzy Matrices and K be a Quadri Partitioned Neutrosophic Fuzzy permutation Matrices, R($[S^T,S^C,S^I,S^F]_L$) = R($[S^T,S^C,S^I,S^F]_L$) ⇔ R(K$[S^T,S^C,S^I,S^F]_L$ K^T) =

R(K $[S^T, S^C, S^I, S^F]_L$ ^{T} K^T) and R($[S^T, S^C, S^I, S^F]_U$) = R($[S^T, S^C, S^I, S^F]_U$) $\Leftrightarrow$ R(K $[S^T, S^C, S^I, S^F]_U$ K^T) = R(K $[S^T, S^C, S^I, S^F]_U$ ^{T} K^T).

Proof: The proof follows a structure like that of Theorem 4.4

Example: 4.5 Consider an Interval-Valued symmetric Quadri Partitioned Neutrosophic Fuzzy Matrices

$$[S^T, S^C, S^I, S^F]_L = \begin{bmatrix} <0.2,0.1,0.3,0.9> & <0.3,0.3,0.2,0.8> & <0.4,0.5,0.6,0.7> \\ <0.3,0.3,0.2,0.8> & <0.4,0.7,0.1,0.3> & <0.4,0.5,0.2,0.7> \\ <0.4,0.5,0.6,0.7> & <0.4,0.5,0.2,0.7> & <0.3,1,1,0.6> \end{bmatrix}$$

Theorem:4.6 For a Interval-Valued symmetric Quadri Partitioned Neutrosophic Fuzzy Matrices $S = <[S^T, S^C, S^I, S^F]_L, [S^T, S^C, S^I, S^F]_U>$, $T = <[T^T, T^C, T^I, T^F]_L, [T^T, T^C, T^I, T^F]_U>$ and K be a Quadri Partitioned Neutrosophic Fuzzy permutation Matrices C($[S^T, S^C, S^I, S^F]_L$) = C($[T^T, T^C, T^I, T^F]_L$) $\Leftrightarrow$ C(K$[S^T, S^C, S^I, S^F]_L$ K^T) = C(K $[T^T, T^C, T^I, T^F]_L$ K^T) and C($[S^T, S^C, S^I, S^F]_U$) = C($[T^T, T^C, T^I, T^F]_U$) $\Leftrightarrow$ C(K $[S^T, S^C, S^I, S^F]_U$ K^T) = C(K $[T^T, T^C, T^I, T^F]_U$ K^T).

Proof: The proof follows a structure like that of Theorem 4.4

5. k-Kernel Symmetric Interval-Valued Quadri Partitioned Neutrosophic Fuzzy Matrices

Definition: 5.1 Let $S = <[S^T, S^C, S^I, S^F]_L, [S^T, S^C, S^I, S^F]_U>$ is called Lower k-kernel symmetric Interval-Valued symmetric Quadri Partitioned Neutrosophic Fuzzy Matrices if N($[S^T, S^C, S^I, S^F]_L$) = N(K$[S^T, S^C, S^I, S^F]_L$ ^{T}K) and Upper k-Kernel symmetric Interval-Valued Quadri Partitioned Neutrosophic Fuzzy Matrices if N($[S^T, S^C, S^I, S^F]_U$) = N(K $[S^T, S^C, S^I, S^F]_U$ ^{T}K).

Note:5.1 Let $S = <[S^T, S^C, S^I, S^F]_L, [S^T, S^C, S^I, S^F]_U>$ is k-Symmetric Interval-Valued symmetric Quadri Partitioned Neutrosophic Fuzzy Matrices implies it is k-kernel - Interval-Valued Quadri Partitioned Neutrosophic Fuzzy Matrices, i.e $[S^T, S^C, S^I, S^F]_L$ = K($[S^T, S^C, S^I, S^F]_L$ ^{T})K,

spontaneously implies N($[S^T, S^C, S^I, S^F]_L$) = N(K$[S^T, S^C, S^I, S^F]_L$ ^{T} K) but converse need not be true.

Similarly, $[S^T, S^C, S^I, S^F]_U$ = K($[S^T, S^C, S^I, S^F]_U$ ^{T})K, spontaneously implies N($[S^T, S^C, S^I, S^F]_U$) = N(K$[S^T, S^C, S^I, S^F]_U$ ^{T} K) however, the converse does not necessarily hold.

Example 5.1 We show that kernel symmetric implies k-kernel symmetric; however, the converse does not necessarily apply

Example: 5.1 Consider an Interval-Valued Quadri Partitioned Neutrosophic Fuzzy Matrices

$$[S^T, S^C, S^I, S^F]_L = \begin{bmatrix} (0,0,0.5) & (0,0,0.4) & (0.3,0.4,0.5) \\ (0.5,0.4,0.6) & (0.1,0.4,0.6) & (0,0,0.4) \\ (0.4,0.5,0.3) & (0.3,0.4,0.5) & (0,0,0.3) \end{bmatrix}$$

$$K = \begin{bmatrix} (0,0,1) & (0,0,1) & (1,1,0) \\ (0,0,1) & (1,1,0) & (0,0,1) \\ (1,1,0) & (0,0,1) & (0,0,1) \end{bmatrix}$$

Therefore, $[S^T, S^C, S^I, S^F]_L \neq$ K$[S^T, S^C, S^I, S^F]_L$ ^{T} K

But, N ($[S^T, S^C, S^I, S^F]_L$) = N(K$[S^T, S^C, S^I, S^F]_L$ ^{T} K) = $(0,0,1,1)$

Similarly, $[S^T, S^C, S^I, S^F]_U \neq$ K$[S^T, S^C, S^I, S^F]_U$ ^{T} K

But, N ($[S^T, S^C, S^I, S^F]_U$) = N(K$[S^T, S^C, S^I, S^F]_U$ ^{T} K) = $(0,0,1,1)$

Theorem: 5.1 The following conditions are equivalent for $S = <[S^T, S^C, S^I, S^F]_L, [S^T, S^C, S^I, S^F]_U > \in (IVSQPNFM)_n$

(i) N($[S^T, S^C, S^I, S^F]_L$) = N(K$[S^T, S^C, S^I, S^F]_L$ ^{T}K)

N($[S^T, S^C, S^I, S^F]_U$) = N(K$[S^T, S^C, S^I, S^F]_U$ ^{T}K)

(ii) N(K$[S^T, S^C, S^I, S^F]_L$) = N((K$[S^T, S^C, S^I, S^F]_L$)^T)

N(K$[S^T, S^C, S^I, S^F]_U$) = N((K$[S^T, S^C, S^I, S^F]_U$)^T)

(iii) N($[S^T, S^C, S^I, S^F]_L$ K) = N(($[S^T, S^C, S^I, S^F]_L$ K)^T)

$N([S^T, S^C, S^I, S^F]_U K) = N(([S^T, S^C, S^I, S^F]_U K)^T)$

(iv) $N([S^T, S^C, S^I, S^F]_L{}^T) = N(K[S^T, S^C, S^I, S^F]_L)$

$N([S^T, S^C, S^I, S^F]_U{}^T) = N(K[S^T, S^C, S^I, S^F]_U)$

(v) $N([S^T, S^C, S^I, S^F]_L) = N(([S^T, S^C, S^I, S^F]_L K)^T)$

$N([S^T, S^C, S^I, S^F]_U) = N(([S^T, S^C, S^I, S^F]_U K)^T)$

(vi) $[S^T, S^C, S^I, S^F]_L^+$ is k- kernel symmetric Quadri Partitioned Neutrosophic Fuzzy Matrices and $[S^T, S^C, S^I, S^F]_U^+$ is k- kernel symmetric Quadri Partitioned Neutrosophic Fuzzy Matrices

(vii) $N([S^T, S^C, S^I, S^F]_L) = N([S^T, S^C, S^I, S^F]_L{}^+ K)$

$N([S^T, S^C, S^I, S^F]_U) = N([S^T, S^C, S^I, S^F]_U K)$

(viii) $K\ [S^T, S^C, S^I, S^F]_L^+ [S^T, S^C, S^I, S^F]_L = [S^T, S^C, S^I, S^F]_L [S^T, S^C, S^I, S^F]_L{}^+ K$

$K\ [S^T, S^C, S^I, S^F]_U^+ [S^T, S^C, S^I, S^F]_U = [S^T, S^C, S^I, S^F]_U [S^T, S^C, S^I, S^F]_U{}^+ K$

(ix) $[S^T, S^C, S^I, S^F]_L^+ [S^T, S^C, S^I, S^F]_L K = K[S^T, S^C, S^I, S^F]_L [S^T, S^C, S^I, S^F]_L^+$

$[S^T, S^C, S^I, S^F]_U^+ [S^T, S^C, S^I, S^F]_U K = K[S^T, S^C, S^I, S^F]_U [S^T, S^C, S^I, S^F]_U^+$

Proof: (i) $\Leftrightarrow$ (ii)

$\Leftrightarrow N([S^T, S^C, S^I, S^F]_L) = N(K[S^T, S^C, S^I, S^F]_L{}^T K)$

$\Leftrightarrow N(K[S^T, S^C, S^I, S^F]_L) = N([S^T, S^C, S^I, S^F]_L{}^T K)$ (By P_2) ($K^2 = I$)

$\Leftrightarrow N(K[S^T, S^C, S^I, S^F]_L) = N((K[S^T, S^C, S^I, S^F]_L)^T)$ (Because , $(KP)^T = P^T K^T = P^T K$)

$\Leftrightarrow K[S^T, S^C, S^I, S^F]_L$ is kernel symmetric Quadri Partitioned Neutrosophic Fuzzy Matrices

Similarly, $N([S^T, S^C, S^I, S^F]_U) = N(K[S^T, S^C, S^I, S^F]_U{}^T K)$.

Condition (ii) is true

Condition (i) $\Leftrightarrow$ (iii)

$\Leftrightarrow$ N($[S^T, S^C, S^I, S^F]_L$) = N(K$[S^T, S^C, S^I, S^F]_L$ ^{T}K)

$\Leftrightarrow$ N($[S^T, S^C, S^I, S^F]_L$ K) = N(K$[S^T, S^C, S^I, S^F]_L$ ^{T}) (By P_2) ($K^2 = I$)

$\Leftrightarrow$ N($[S^T, S^C, S^I, S^F]_L$ K) = N(($[S^T, S^C, S^I, S^F]_L$ K) ^{T}) (Because , $(PK)^T = K^T P^T = KP^T$)

$[S^T, S^C, S^I, S^F]_L$ K is kernel symmetric Quadri Partitioned Neutrosophic Fuzzy Matrices

Similarly, N($[S^T, S^C, S^I, S^F]_U$ K) = N(($[S^T, S^C, S^I, S^F]_U$ K)^T).

Therefore, (iii) holds

(ii) $\Leftrightarrow$ (iv)

$\Leftrightarrow$ N(K$[S^T, S^C, S^I, S^F]_L$) = N(K$[S^T, S^C, S^I, S^F]_L$)^T =N($[S^T, S^C, S^I, S^F]_L$ ^{T}K)

$\Leftrightarrow$ N(K$[S^T, S^C, S^I, S^F]_L$) = N($[S^T, S^C, S^I, S^F]_L$ ^{T})

Similarly, N($[S^T, S^C, S^I, S^F]_U$ ^{T}) = N(K$[S^T, S^C, S^I, S^F]_U$), (By P_2)

Therefore, (iv) holds

(iii) $\Leftrightarrow$ (v)

$\Leftrightarrow$ N($[S^T, S^C, S^I, S^F]_L$ K) = N(($[S^T, S^C, S^I, S^F]_L$ K)^T)

$\Leftrightarrow$ N($[S^T, S^C, S^I, S^F]_L$) = N(($[S^T, S^C, S^I, S^F]_L$ K)^T) (By P_2)

Similarly, N($[S^T, S^C, S^I, S^F]_U$) = N(($[S^T, S^C, S^I, S^F]_U$ K)^T)

(ii) $\Leftrightarrow$ (vi)

$\Leftrightarrow$ N(K$[S^T, S^C, S^I, S^F]_L$) = N(K$[S^T, S^C, S^I, S^F]_L$)^T

$\Leftrightarrow$ N(K$[S^T, S^C, S^I, S^F]_L$) = N($[S^T, S^C, S^I, S^F]_L$ ^{T}K) (By P_2)

$\Leftrightarrow$ N(K$[S^T, S^C, S^I, S^F]_L$) = N($[S^T, S^C, S^I, S^F]_L$ $^+$K)

Since N(K$[S^T, S^C, S^I, S^F]_L$ $^+$K) = N($^+$K) $[S^T, S^C, S^I, S^F]_L$

$\Leftrightarrow$ N(K$[S^T, S^C, S^I, S^F]_L$) = N($[S^T, S^C, S^I, S^F]_L{}^+$)

$\Leftrightarrow$ $[S^T, S^C, S^I, S^F]_L{}^+$ is k-kernel symmetric Quadri Partitioned Neutrosophic Fuzzy Matrices

Similarly, $[S^T, S^C, S^I, S^F]_U{}^+$ is k- kernel symmetric Quadri Partitioned Neutrosophic Fuzzy Matrices. Condition (i) $\Leftrightarrow$ (vii)

$\Leftrightarrow$ N($[S^T, S^C, S^I, S^F]_L$) = N(K$[S^T, S^C, S^I, S^F]_L{}^T$K)

$\Leftrightarrow$ N() = N(K$[S^T, S^C, S^I, S^F]_L{}^T$K) = N($[S^T, S^C, S^I, S^F]_L{}^T$K)

$\Leftrightarrow$ N($[S^T, S^C, S^I, S^F]_L$) = N(K$[S^T, S^C, S^I, S^F]_L$)^T

$\Leftrightarrow$ N($[S^T, S^C, S^I, S^F]_L$) = N($[S^T, S^C, S^I, S^F]_L{}^+$K) (By P_2)

Similarly, N($[S^T, S^C, S^I, S^F]_U$) = N($[S^T, S^C, S^I, S^F]_U$ K),

(i) $\Leftrightarrow$ (viii)

$[S^T, S^C, S^I, S^F]_L$ K is kernel symmetric Quadri Partitioned Neutrosophic Fuzzy Matrices

$\Leftrightarrow$ ($[S^T, S^C, S^I, S^F]_L$ K)($[S^T, S^C, S^I, S^F]_L$ K)$^+$ = ($[S^T, S^C, S^I, S^F]_L$ K)$^+$($[S^T, S^C, S^I, S^F]_L$ K)

$\Leftrightarrow$ ($[S^T, S^C, S^I, S^F]_L$ K)(K$[S^T, S^C, S^I, S^F]_L{}^+$) = (K$[S^T, S^C, S^I, S^F]_L{}^+$)($[S^T, S^C, S^I, S^F]_L$ K)

$\Leftrightarrow$ $[S^T, S^C, S^I, S^F]_L$ $[S^T, S^C, S^I, S^F]_L{}^+$ = K$[S^T, S^C, S^I, S^F]_L{}^+$$[S^T, S^C, S^I, S^F]_L$ K

$\Leftrightarrow$ $[S^T, S^C, S^I, S^F]_L$ $[S^T, S^C, S^I, S^F]_L{}^+$K = K$[S^T, S^C, S^I, S^F]_L{}^+$$[S^T, S^C, S^I, S^F]_L$

(viii) $\Leftrightarrow$ (ix)

$\Leftrightarrow$ K$[S^T, S^C, S^I, S^F]_L{}^+$$[S^T, S^C, S^I, S^F]_L$ = $[S^T, S^C, S^I, S^F]_L$ $[S^T, S^C, S^I, S^F]_L{}^+$ K

$\Leftrightarrow$ $[S^T, S^C, S^I, S^F]_L{}^+$ $[S^T, S^C, S^I, S^F]_L$ K = K$[S^T, S^C, S^I, S^F]_L$ $[S^T, S^C, S^I, S^F]_L{}^+$.

Similarly, $[S^T, S^C, S^I, S^F]_U{}^+$ $[S^T, S^C, S^I, S^F]_U$ K = K$[S^T, S^C, S^I, S^F]_U$ $[S^T, S^C, S^I, S^F]_U{}^+$

Therefore, (ix) is holds.

6. An Algorithm Based on IVQPNSS in a Decision-Making Problem

Definition 6.1. Let Y = {y_1,y_2,.....,y_n} be an initial universe and E = {e_1,e_2,.....,e_m} be a set of parameters. Then, for an **IVQPNSS** (η,E) over Y the degree of **TM** and the degree of **CM** of an element y_i to η (e_j) denoted by $T_{\eta(e_j)}(y_i)=\left[\underline{T}_{\eta(e_j)}(y_i),\overline{T}_{\eta(e_j)}(y_i)\right]$ and $C_{\eta(e_j)}(y_i)=\left[\underline{C}_{\eta(e_j)}(y_i),\overline{C}_{\eta(e_j)}(y_i)\right]$ respectively. Then, their corresponding score functions are denoted and defined by the following:

$$S_{T_{\eta(e_j)}}(y_i)=\sum_{K=1}^{N}\left[\left(\underline{T}_{\eta(e_j)}(y_i)+\overline{T}_{\eta(e_j)}(y_i)\right)-\left(\underline{T}_{\eta(e_j)}(y_k)+\overline{T}_{\eta(e_j)}(y_k)\right)\right]$$

$$S_{C_{\eta(e_j)}}(y_i)=\sum_{K=1}^{N}\left[\left(\underline{C}_{\eta(e_j)}(y_i)+\overline{C}_{\eta(e_j)}(y_i)\right)-\left(\underline{C}_{\eta(e_j)}(y_k)+\overline{C}_{\eta(e_j)}(y_k)\right)\right]$$

Definition 6.2. Let Y = {y_1,y_2,.....,y_n} be an initial universe and E = {e_1,e_2,.....,e_m} be a set of parameters. Then, for an **IVQPNSS** (η,E) over X the degree of **UM** and the degree of − **FM** of an element xi to η (e_j) denoted by $U_{\eta(e_j)}(y_i)=\left[\underline{U}_{\eta(e_j)}(y_i),\overline{U}_{\eta(e_j)}(y_i)\right]$ and $F_{\eta(e_j)}(y_i)=\left[\underline{F}_{\eta(e_j)}(y_i),\overline{F}_{\eta(e_j)}(y_i)\right]$ respectively. Then, their corresponding score functions are denoted and defined by the following:

$$S_{U_{\eta(e_j)}}(y_i)=-\sum_{K=1}^{N}\left[\left(\underline{U}_{\eta(e_j)}(y_i)+\overline{U}_{\eta(e_j)}(y_i)\right)-\left(\underline{U}_{\eta(e_j)}(y_k)+\overline{U}_{\eta(e_j)}(y_k)\right)\right]$$

$$S_{F_{\eta(e_j)}}(y_i)=\sum_{K=1}^{N}\left[\left(\underline{F}_{\eta(e_j)}(y_i)+\overline{F}_{\eta(e_j)}(y_i)\right)-\left(\underline{F}_{\eta(e_j)}(y_k)+\overline{F}_{\eta(e_j)}(y_k)\right)\right]$$

Definition 6.3. Let X = {y_1,y_2,.....,y_n} be an initial universe and E = {e_1,e_2,.....,e_m} be a set of parameters. For an **IVQPNSS** (η,E) over Y, the scores of the **TM**, CM, **UM**, and **FM** of y_i for each e_j be denoted by $S_{T_{\eta(e_j)}}(y_i)$, $S_{C_{\eta(e_j)}}(y_i)$, $S_{U_{\eta(e_j)}}(y_i)$ and $S_{F_{\eta(e_j)}}(y_i)$ respectively. Then, the total score of x_i for each e_j is denoted by $T_{\eta(e_j)}(y_i)=S_{T_{\eta(e_j)}}(y_i)+S_{C_{\eta(e_j)}}(y_i)+S_{U_{\eta(e_j)}}(y_i)+S_{F_{\eta(e_j)}}(y_i)$

Based on the above definitions, we give the steps of the proposed algorithm as follows:

Algorithm: Step 1: Input Matrix Representation

In this initial step, the decision-maker provides the data in the form of an interval-valued Quadri Partitioned Neutrosophic soft set. The universal set Y = {y_1,y_2,.....,y_n} represents the elements under consideration, and the parameter set E = {e_1,e_2,.....,e_m} denotes the criteria or attributes. The data is structured into a tabular matrix, where each element represents the interval-valued Quadri Partitioned Neutrosophic soft value corresponding to a specific element y_i and criterion e_j. This step establishes the foundation for further computation by organizing the data systematically.

Step 2: Compute Individual Components: Using the interval-valued Quadri Partitioned Neutrosophic matrix from Step 1, Definitions 11.1 and 11.2 are applied to calculate the components $S_{T_{\eta(e_j)}}(y_i)$, $S_{C_{\eta(e_j)}}(y_i)$, $S_{U_{\eta(e_j)}}(y_i)$ and $S_{F_{\eta(e_j)}}(y_i)$. These represent truth, indeterminacy, falsity, and hesitation degrees, respectively, for each element x_i under the criterion e_j. This computation ensures that the nuanced aspects of each element's evaluation are captured.

Step 3: Score Calculation for Individual Criteria

With the components computed in Step 2, Definition 11.3 is utilized to determine the score $\square_{T_{\eta(e_j)}}(x_i)$ for each element x_i under each criterion e_j. The score reflects a composite evaluation of the truth, indeterminacy, falsity, and hesitation degrees. This step condenses the multi-dimensional information into a single score for each element-criterion pair, facilitating easier comparison.

Step 4: Overall Score Computation

The overall score u_i for each element x_i is calculated by aggregating the scores $v_i = \square_{T_{\eta(e_1)}}(y_i) + \square_{T_{\eta(e_2)}}(y_i) + \square_{T_{\eta(e_3)}}(y_i) + \ldots + \square_{T_{\eta(e_m)}}(y_i)$ across all criteria e_j. This provides a holistic evaluation of each element concerning all the given parameters. The aggregation formula ensures that the influence of all criteria is considered, offering a balanced view of each element's performance.

Step 5: Optimal Choice Identification

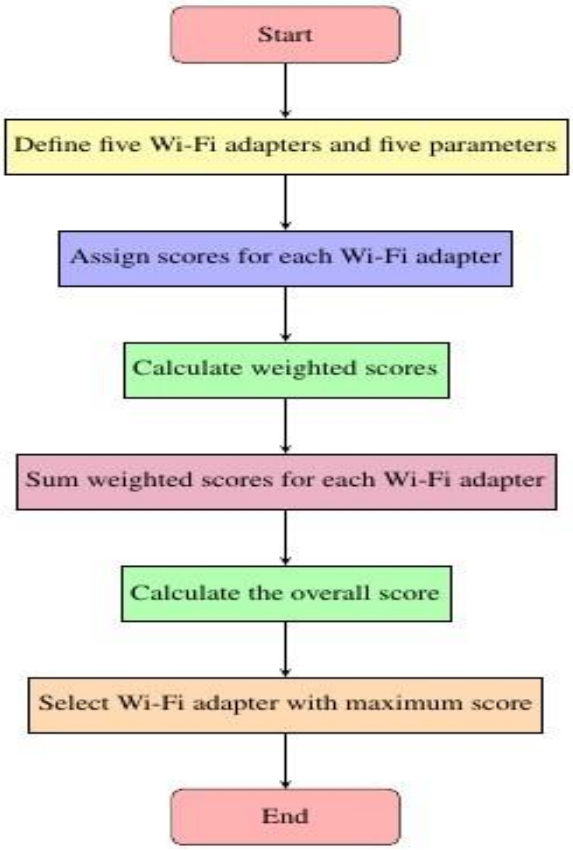
After computing the overall scores for all elements, the element x_k with the maximum score $v_k = \max_{xi \in X}\{v_i\}$. is identified as the optimal choice. This step concludes the decision-making process by pinpointing the best candidate based on the computed scores.

Step 6: Handling Ties and Reassessment

In cases where a tie occurs, indicating that two or more elements share the highest score, either all tied elements are considered as optimal choices, or the decision-making process is revisited. Reassessment involves consulting experts and potentially redefining the parameters or input values, followed by repeating the previous steps. This ensures that the final decision is both robust and aligned with expert judgment.

7. To demonstrate the practical application of the algorithm, we present the following example.

We consider the following example. Illustrative Example to implement the proposed algorithm successfully in a real life context, we consider the following problem: Suppose Mr. Z wants to purchase a Wi-Fi adapters to carry out his official work. But he has limited knowledge to select a good quality WiFi adapters. The selection process is complicated due to various conflicting factors involved during decision making. To solve the purpose, Mr. Z consulted with some experts cum decision makers having IT backgrounds. The decision makers evaluated the five Wi-Fi adapters according to the fixed criteria. To select the best alternative, the evaluation procedure is executed as follows:


Start
Define five Wi-Fi adapters and five parameters
Assign scores for each Wi-Fi adapter
Calculate weighted scores
Sum weighted scores for each Wi-Fi adapter
Calculate the overall score
Select Wi-Fi adapter with maximum score
End

Step1. Consider a set of five Wi-Fi adapters be $Y=\{y_1,y_2,y_3,y_4,y_5\}$and a set of parameters be $E=\{e_i:i=1,2,3,4,5\}$,where

e_1= Speed (Bandwidth),

e_2 = Compatibility,

e_3 = Range and Coverage,

e_4 = Interface Type, and

e_5 = Portability and Design.

Based on the opinions of the decision makers, the decision matrix of the set of five alternatives and five evaluation criteria under the interval valued Neutrosophic soft set environment is shown in Table1.

Here's an example of assigning weights to the five parameters based on their importance in the decision making process for selecting a Wi-Fi adapter.

Step1: Begin by considering a set of five Wi-Fi adapters represented as $Y=\{y_1,y_2,y_3,y_4,y_5\}$, along with a set of assessment criteria denoted by $E = \{e_i :i=1,2,3,4,5\}$

Speed Bandwidth:40 This is often themost important parameter as it affects data transfer rates and performance.

Compatibility:20

Ensures the adapter works with your system and network standards, crucial for usability.

Range and Coverage: 15 Determines the signal strength and connection stability across distances.

Interface Type: 15 USB or PCIe; affects ease of installation, portability, and data transfer rates. Portability and Design: 10 Impacts how convenient the adapter is for travel or integration into a setup.

Step 2: Assign Scores for Each Course Rate each course for each parameter on a scale from 0 to 1 (where 0 is the lowest and 1 is the highest). Table: 1. Tabular representation of interval valued Neutrosophic Fuzzy Matrices to describe the f ive Wi-Fi adapters.

Step 3. The score of the truth-membership degrees f shown in Table: 2. The score of the indeterminacy-membership degrees is shown in Table: 3, Score of the false-membership degrees Table: 4 and Total score function Table: 5.

Table: 1 Scores for Each Wi-Fi adapters

X/E	e_1	e_2	e_3	e_4	e_5
y_1	< [0.4,0.5], [0.2,0.4], [0.3,0.8], [0.6,0.7]>	< [0.4,0.5], [0.2,0.4], [0.3,0.8], [0.6,0.7]>	< [0.4,0.5], [0.2,0.4], [0.3,0.8], [0.6,0.7]>	< [0.4,0.5], [0.2,0.4], [0.3,0.8], [0.6,0.7]>	< [0.4,0.5], [0.2,0.4], [0.3,0.8], [0.6,0.7]>
y_2	< [0.3,0.4], [0.5,0.6], [0.6,0.9], [0.8,0.9]>	< [0.1,0.2], [0.2,0.4], [0.5,0.6], [0.8,0.9]>	< [0.4,0.5], [0.3,0.4], [0.6,0.8], [0.7,0.7]>	< [0.3,0.5], [0.2,0.6], [0.3,0.7], [0.5,0.7]>	< [0.7,0.9], [0.3,0.4], [0.5,0.9], [0.7,0.9]>
y_3	< [0.3,0.6], [0.5,0.6], [0.7,0.8], [0.8,0.9]>	< [0.6,0.7], [0.2,0.4], [0.4,0.8], [0.7,0.8]>	< [0.1,0.5], [0.3,0.4], [0.5,0.8], [0.7,0.7]>	< [0.6,0.6], [0.3,0.4], [0.7,0.8], [0.8,0.9]>	< [0.1,0.2], [0.4,0.8], [0.5,0.8], [0.1,0.6]>
y_4	< [0.1,0.7], [0.5,0.9], [0.6,0.9], [0.7,0.9]>	< [0.2,0.5], [0.6,0.7], [0.3,0.8], [0.7,0.8]>	< [0.3,0.5], [0.2,0.4], [0.5,0.8], [0.8,0.9]>	< [0.2,0.6], [0.2,0.4], [0.6,0.8], [0.1,0.7]>	< [0.2,0.5], [0.2,0.7], [0.6,0.8], [0.6,0.9]>
y_5	< [0.4,0.7], [0.1,0.4], [0.3,0.9], [0.6,0.8]>	< [0.1,0.5], [0.2,0.3], [0.4,0.8], [0.6,0.9]>	< [0.4,0.4], [0.3,0.4], [0.5,0.8], [0.6,0.7]>	< [0.4,0.5], [0.3,0.4], [0.3,0.5], [0.6,0.6]>	< [0.6,0.7], [0.2,0.4], [0.4,0.8], [0.5,0.7]>

Step 2. The score of the truth-membership degrees $S_{T_{\eta(e_j)}}(y_i)$ for (η ,E) is exposed in Table 2.

X/E	e_1	e_2	e_3	e_4	e_5
y_1	0.1	0.7	0.5	-0.1	-.0.3
y_2	-0.9	-2.3	0.5	0.6	3.2
y_3	0.1	2.7	-1	1.4	-3.3
y_4	-0.4	1.1	0	-0.6	-.0.8

y_5	1.1	-0.8	-1.6	-0.1	1.7

The score of the contradiction-membership degrees $S_{C_{\eta(e_j)}}(y_i)$ for (η,E) is exposed in Table 3.

Table 4. Tabular illustration of the score of contradiction-membership degree

X/E	e_1	e_2	e_3	e_4	e_5
y_1	-1.4	-0.8	-1	-1	-1.8
y_2	1.1	-0.8	-0.5	-0.6	-1.3
y_3	1.1	-0.8	-0.5	-1.1	1.2
y_4	2.6	2.7	-1	-2.2	-0.3
y_5	1.1	-1.3	-0.5	-1.1	-1.8

The score of the unknown-membership degrees $S_{U_{\eta(e_j)}}(y_i)$ for (η,E) is exposed in Table 4.

Table 5. Tabular illustration of the score of unknown-membership degree

X/E	e1	e2	e3	e4	e5
y_1	1.1	1.7	1.5	0.9	0.7
y_2	3.1	1.7	3	0.4	2.2
y_3	3.1	2.2	2.5	2.9	1.7
y_4	3.1	1.7	2.5	2.4	2.2
y_5	1.6	2.2	2.5	-0.6	1.2

The score of the false-membership degrees $S_{F_{\eta(e_j)}}(y_i)$ for (η,E) is exposed in Table 5.

Table 6. Tabular illustration of the score of false-membership degree

X/E	e_1	e_2	e_3	e_4	e_5
x_1	2.1	2.7	2.5	1.9	1.7
x_2	4.1	4.7	3	1.4	3.2
x_3	3.1	3.7	3	3.9	-1.3
x_4	3.1	1.7	3.5	-0.6	2.7
x_5	2.6	3.7	2.5	1.4	1.2

Step 3. By using Table 3 to Table 6, the score $\eta_{(ej)}(x_i)$ for (η,E) is exhibited in Table 6.

Table 7. Tabular illustration of the score $\eta_{(ej)}(x_i)$.

X/E	e_1	e_2	e_3	e_4	e_5
x_1	1.9	4.3	3.5	1.7	0.4
x_2	7.4	3.3	6	1.8	7.3
x_3	7.4	7.8	4	6	-1.7
x_4	8.4	7.2	5	-1	3.8
x_5	6.4	3.8	2.9	-0.4	2.3

Step 4. Now, we calculate the overall score given as:

$v_1 = 11.8$, $v_2 = 25.8$, $v_3 = 23.5$, $v_4 = 23.4$, $v_5 = 15$.

Step 5. Thus, $v_k = \max_{y_i \in Y} \{ v_1, v_2, v_3, v_4, v_5\} = v_2$. Therefore, v_2 is the optimal choice object for the decision maker. If v_2 is not available in the market, then he/she will choose v_3 or v_4 .

Step 6. As there is no tie in the optimal choice so there is no need to reassess data in the given problem.

13. Conclusion and Future work

In conclusion, this study makes a significant contribution to the field of neutrosophic fuzzy matrices by introducing two innovative concepts: range symmetric Interval valued Quadri Partitioned Neutrosophic Fuzzy matrices and kernel symmetric Interval valued Quadri Partitioned Neutrosophic Fuzzy matrices. Through rigorous characterization and analysis, we establish essential conditions for the kernel-symmetry of Interval-Valued Symmetric Quadri Partitioned Neutrosophic Fuzzy Matrices and elucidate the intricate relationships between range-symmetry and kernel-symmetry. The development of the concepts of Kernel and k-Kernel Symmetric Interval valued Quadri Partitioned Neutrosophic Fuzzy matrices enhances the theoretical understanding of these matrix types, while the derivation of basic results underscores the distinctions between k-symmetry and k-Kernel symmetry. Additionally, the study sheds light on the interplay between kernel symmetry, k-kernel symmetry, and the Moore-Penrose inverse, providing valuable insights into their applications in practical scenarios. By proposing an algorithm for multi-criteria decision-making using Interval valued Quadri Partitioned Neutrosophic Fuzzy matrices, validated with a real-world example, we demonstrate the practical implications of our findings. Overall, this research lays a robust foundation for further exploration in the neutrosophic fuzzy domain and highlights the potential of these matrix concepts in addressing complex decision-making challenges. Future work can explore several avenues to enhance the findings of this study. First, researchers could extend the concepts of Range-Symmetric and Kernel-Symmetric Interval-Valued Quadri Partitioned Neutrosophic Fuzzy Matrices to encompass higher-order matrices or other fuzzy structures, thereby deepening our understanding of their properties. Additionally, practical applications of the developed algorithms in diverse real-world multi-criteria decision-making scenarios, such as finance and healthcare, could validate their effectiveness. There is also potential to create numerical methods and computational tools to facilitate efficient calculations and manipulations of these matrices, making them more accessible for practical use. Furthermore, theoretical extensions investigating relationships between various classes of neutrosophic fuzzy matrices could provide insights into their interconnectedness. Lastly, combining range symmetric Interval valued Quadri Partitioned Neutrosophic Fuzzy matrices and kernel symmetric Interval valued Quadri Partitioned Neutrosophic Fuzzy matrices with other mathematical models, like interval-valued fuzzy sets, may yield hybrid approaches that better capture uncertainty and imprecision in decision-making processes.

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University of New Mexico

Distance Measure for Fermatean Neutrosophic Sets: A Comparative Framework for Qualitative Evaluation

S. Bhuvaneshwari[1], C. Antony Crispin Sweety[2], Broumi Said [3,4],

[1] Avinashilingam Institute for Home Science and Higher Education for Women. Coimbatore. Tamil Nadu. India; bhuvi14495@gamil.com

[2] Avinashilingam Institute for Home Science and Higher Education for Women. Coimbatore. Tamil Nadu. India; riosweety@gmail.com

[3] Laboratory of Information Processing, Faculty of Science Ben M'Sik, University of Hassan II. Casablanca. Morocco; broumisaid78@gmail.com

[4] STIE team,Regional Center for the Professions of Education and Training (C.R.M.E.F). Casablanca-Settat. Morocco; broumisaid78@gmail.com

Abstract: This work introduces a novel distance measure specifically designed for Fermatean Neutrosophic sets (FNS), extending upon existing measures such as the intuitionistic fuzzy distance and single-valued Neutrosophy distance measures. The study proposes a Hypothetical Framework for Fermatean Neutrosophic Sets to evaluate minimum distances and compares six different distance measures. These measures are applied within the contexts of both Fermatean Neutrosophic Sets and Neutrosophic Sets. The primary objective is to identify the shortest distance, enabling precise and appropriate decision-making. The study highlights the importance of false membership in FNS, which alters the ranking of FNS and establishes the Fermatean Neutrosophic set as a more accurate tool for evaluation in specific cases.

Keywords: Fermatean Neutrosophic set; Euclidean Distance (ED), Normalized Euclidean Distance (N-ED), Hamming Distance (HD), Normalized Hamming Distance (N-HD), Sine Metric Single-Valued Neutrosophic Distance Measure (SMSVNDM), Tangent Matric Fermatean Neutrosophic Distance Measure (TMFNDM).

1. Introduction

Distance measures are vital for assessing similarity or dissimilarity between elements, enabling accurate decision-making, ranking, and analysis in uncertain or complex scenarios. They are widely used in fields like medical diagnosis, pattern recognition, and risk analysis to ensure precise evaluations and optimal solutions [1-4].

Zadeh [5] introduced fuzzy set theory, which models vagueness through membership degrees, laying the foundation for advanced uncertainty modelling. Atanassov [6] introduced intuitionistic fuzzy sets, which account for both membership and non-membership degrees, enabling better handling of uncertainty. Distance measures play a key role in assessing similarity or dissimilarity within fuzzy, intuitionistic, and neutrosophic frameworks, which have been widely applied in areas like site selection (Rouyendegh et al. [7]) and education (Citil [8]).

Smarandache [9] introduced neutrosophy, extending intuitionistic fuzzy sets by incorporating indeterminacy, offering greater flexibility in managing incomplete and inconsistent information (Smarandache) [10]. Gong et al. [11] further enhanced decision analysis with spherical distance measures for intuitionistic fuzzy sets, improving similarity assessments. Building upon existing methodologies, this work introduces a novel distance measure tailored specifically to Fermatean Neutrosophic sets [12].

Fermatean Neutrosophic sets extend the framework of Neutrosophic sets [13], introducing additional parameters to capture uncertainty, indeterminacy, and falsity inherent in medical data. By integrating insights from intuitionistic fuzzy distance measures [14] and single-valued Neutrosophy (SVN) distance measures [15], this new distance measure offers a comprehensive approach to evaluating medical conditions within the Fermatean Neutrosophic framework. Recent works on Fermatean neutrosophic sets include foundational matrix operations [21], a framework for green biomedical waste management [22], and a correlation-based measure for electric vehicle selection [23], showcasing their real-world applicability.

Here, we defined distance measures for the Fermatean neutrosophic set to handle higher membership values in a set. Also, we can use this distance measure to handle all the lower case of neutrosophic sets and their extensions like fuzzy, intuitionistic fuzzy, Pythagorean fuzzy, spherical fuzzy, and Fermatean fuzzy.

Throughout this exploration, we have scrutinize six distinct distance measures, including Euclidean Distance, Normalized Euclidean Distance, Hamming Distance, Normalized Hamming Distance, SMSVNDM Distance, and TMFNDM Distance. By comparing and contrasting these measures within the context of Fermatean Neutrosophic sets and traditional Neutrosophic sets, we seek to elucidate the advantages and limitations of each approaches.

2. Preliminaries

Definition 2.1. [19]

The HD between two SVN sets I and J is defined as

$$d_H(I,J)=\frac{1}{3}\sum_{j=1}^{n}\left(\left|\mu_I(x_j)-\mu_J(x_j)\right|+\left|\sigma_I(x_j)-\sigma_J(x_j)\right|+\left|\gamma_I(x_j)-\gamma_J(x_j)\right|\right) \quad (1)$$

Definition 2.2. [19]

The ED between two SVN sets I and J is defined as

$$d_E(I,J)=\left\{\frac{1}{3}\sum_{j=1}^{n}\left(\left(\mu_I(x_j)-\mu_J(x_j)\right)^2+\left(\sigma_I(x_j)-\sigma_J(x_j)\right)^2+\left(\gamma_I(x_j)-\gamma_J(x_j)\right)^2\right)\right\}^{\frac{1}{2}} \quad (2)$$

Definition 2.3. [19]

The N-HD between two SVN sets I and J is defined as

$$d_{n-H}(I,J)=\frac{1}{3n}\sum_{j=1}^{n}\left(\left|\mu_I(x_j)-\mu_J(x_j)\right|+\left|\sigma_I(x_j)-\sigma_J(x_j)\right|+\left|\gamma_I(x_j)-\gamma_J(x_j)\right|\right) \quad (3)$$

Definition 2.4. [19]

The N-ED between two SVN sets I and J is defined as

$$d_{n-E}(I,J)=\left\{\frac{1}{3n}\sum_{j=1}^{n}\left(\left(\mu_I(x_j)-\mu_J(x_j)\right)^2+\left(\sigma_I(x_j)-\sigma_J(x_j)\right)^2+\left(\gamma_I(x_j)-\gamma_J(x_j)\right)^2\right)\right\}^{\frac{1}{2}} \quad (4)$$

Definition 2.5. [15]

Let $A=\{x_1,x_2,...,x_n\}$ be a universal set. Let $I=\{x_i,\mu_I(x_i),\sigma_I(x_i),\gamma_I(x_i):x_i\in A\}$ and $J=\{x_i,\mu_J(x_i),\sigma_J(x_i),\gamma_J(x_i):x_i\in A\}$ be two SVNSs on A. Then define a mapping $d:SVNS(A)\times SVNS(A)\to[0,1]$ as: $d_{SMSVN}(I,J)$

$$=\frac{5}{3n}\sum_{i=1}^{n}\frac{\sin\left\{\frac{\pi}{6}\left|\mu_I(x_i)-\mu_J(x_i)\right|\right\}+\sin\left\{\frac{\pi}{6}\left|\sigma_I(x_j)-\sigma_J(x_j)\right|\right\}+\sin\left\{\frac{\pi}{6}\left|\gamma_I(x_j)-\gamma_J(x_j)\right|\right\}}{1+\sin\left\{\frac{\pi}{6}\left|\mu_I(x_i)-\mu_J(x_i)\right|\right\}+\sin\left\{\frac{\pi}{6}\left|\sigma_I(x_j)-\sigma_J(x_j)\right|\right\}+\sin\left\{\frac{\pi}{6}\left|\gamma_I(x_j)-\gamma_J(x_j)\right|\right\}} \quad (5)$$

Definition 2.6 [6]

Let X be a non-empty set (universe). A FNS $\tilde{S}$ on X is an object of the form:

$$\tilde{S}=\{<s,(T_{\tilde{S}}(s),I_{\tilde{S}}(s),F_{\tilde{S}}(s))>|\ s\in X\}$$

where, $T_{\tilde{S}}(s),I_{\tilde{S}}(s),F_{\tilde{S}}(s):X\to[0,1]$, $\quad 0\le T_{\tilde{S}}^{3}(s)+F_{\tilde{S}}^{3}(s)\le 1, 0\le I_{\tilde{S}}^{3}(s)\le 1$ then

$$0\le T_{\tilde{S}}^{3}(s)+I_{\tilde{S}}^{3}(s)+F_{\tilde{S}}^{3}(s)\le 2 \quad \forall s\in X$$

$T_{\tilde{S}}(s)$ is the degree of membership, $I_{\tilde{S}}(s)$ is the degree of inderminancy and $F_{\tilde{S}}(s)$ is the degree of non-membership. Here $T_{\tilde{S}}(s)$ and $F_{\tilde{S}}(s)$ are dependent components and $I_{\tilde{S}}(s)$ is an independent component.

3. Distance Measure on Fermatean Neutrosophic Sets

The distance measure on Fermatean Neutrosophic Sets defined in this section is a generalization of the single-valued neutrosophic distance measure and intuitionistic fuzzy distance measure. It includes several derived properties.

Definition 3.1.

Let $A=\{x_1,x_2,...,x_n\}$ be a universal set. Let $I=\{x_i,T_I(x_i),I_I(x_i),F_I(x_i):x_i\in A\}$

and $J=\left\{x_i,T_J\left(x_i\right),I_J\left(x_i\right),F_J\left(x_i\right):x_i\in A\right\}$ be two Fermatean Neutrosophic Sets on A. Then define a mapping $d:FNS\left(A\right)\times FNS\left(A\right)\to\left[0,1\right]$ as:

$$d_{TMFN}\left(I,J\right)=\frac{2}{\sqrt{3}n}\sum_{i=1}^{n}\begin{bmatrix}\tan\left\{\frac{\pi}{6}\left|T_I\left(x_i\right)-T_J\left(x_i\right)\right|\right\}\\+\tan\left\{\frac{\pi}{6}\left|I_I\left(x_i\right)-I_J\left(x_i\right)\right|\right\}\\+\tan\left\{\frac{\pi}{6}\left|F_I\left(x_i\right)-F_J\left(x_i\right)\right|\right\}\end{bmatrix} \tag{6}$$

Theorem 3.2.

The FNSs hold the following properties. I,J,K.

i. $d_{TMFN}\left(I,J\right)\geq 0$ for all $I,J\in FNS\left(A\right)$.

ii. $d_{TMFN}\left(I,J\right)=0$ if and only if $I=J$ for all $I,J\in FNS\left(A\right)$.

iii. $d_{TMFN}\left(I,J\right)=d_{TMFN}\left(J,I\right)$ for all $I,J\in FNS\left(X\right)$.

iv. If $I\subseteq J\subseteq K$ for all $I,J,K\in FNS\left(A\right)$, then $d_{TMFN}\left(I,K\right)\geq d_{TMFN}\left(I,J\right)$ and $d_{TMFN}\left(I,K\right)\geq d_{TMFN}\left(J,K\right)$.

Proof:

Part (i):

If $I,J\in FNS\left(A\right)$, then $0\leq T_I\left(x_i\right)\leq 1,\ 0\leq I_I\left(x_i\right)\leq 1, 0\leq F_I\left(x_i\right)\leq 1, \forall x_i\in A$

$\Rightarrow 0\leq\left|T_I\left(x_i\right)-T_J\left(x_i\right)\right|\leq 1, 0\leq\left|I_I\left(x_i\right)-I_J\left(x_i\right)\right|\leq 1$, and $0\leq\left|F_I\left(x_i\right)-F_J\left(x_i\right)\right|\leq 1$.

$$\Rightarrow 0\leq\tan\left\{\frac{\pi}{6}\left|T_I\left(x_i\right)-T_J\left(x_i\right)\right|\right\}\leq\frac{1}{\sqrt{3}},$$

$$0\leq\tan\left\{\frac{\pi}{6}\left|I_I\left(x_i\right)-I_J\left(x_i\right)\right|\right\}\leq\frac{1}{\sqrt{3}},$$

$$0\leq\tan\left\{\frac{\pi}{6}\left|F_I\left(x_i\right)-F_J\left(x_i\right)\right|\right\}\leq\frac{1}{\sqrt{3}}$$

Then

$$0 \le \tan\left\{\frac{\pi}{6}\left|T_I(x_i)-T_J(x_i)\right|\right\}+\tan\left\{\frac{\pi}{6}\left|I_I(x_i)-I_J(x_i)\right|\right\}+\tan\left\{\frac{\pi}{6}\left|F_I(x_i)-F_J(x_i)\right|\right\} \le \sqrt{3} \quad ,$$

$\forall x_i \in X$

Multiple by 2

$$0 \le \frac{2}{\sqrt{3}}\begin{bmatrix}\tan\left\{\frac{\pi}{6}\left|T_I(x_i)-T_J(x_i)\right|\right\} \\ +\tan\left\{\frac{\pi}{6}\left|I_I(x_i)-I_J(x_i)\right|\right\} \\ +\tan\left\{\frac{\pi}{6}\left|F_I(x_i)-F_J(x_i)\right|\right\}\end{bmatrix} \le 2 \text{ , } \forall x_i \in X \text{ .}$$

This implies

$$0 \le \frac{2}{\sqrt{3}n}\sum_{i=1}^{n}\begin{bmatrix}\tan\left\{\frac{\pi}{6}\left|T_I(x_i)-T_J(x_i)\right|\right\} \\ +\tan\left\{\frac{\pi}{6}\left|I_I(x_i)-I_J(x_i)\right|\right\} \\ +\tan\left\{\frac{\pi}{6}\left|F_I(x_i)-F_J(x_i)\right|\right\}\end{bmatrix} \le 2$$

Thus, $0 \le d_{TMNS}(I,J) \le 2$.

Part (ii):

$d_{TMNS}(I,J)=0$

$$\Leftrightarrow 0 \le \frac{2}{\sqrt{3}n}\sum_{i=1}^{n}\begin{bmatrix}\tan\left\{\frac{\pi}{6}\left|T_I(x_i)-T_J(x_i)\right|\right\} \\ +\tan\left\{\frac{\pi}{6}\left|I_I(x_i)-I_J(x_i)\right|\right\} \\ +\tan\left\{\frac{\pi}{6}\left|F_I(x_i)-F_J(x_i)\right|\right\}\end{bmatrix} \le 2$$

$$\Leftrightarrow 0 \le \frac{2}{\sqrt{3}}\begin{bmatrix}\tan\left\{\frac{\pi}{6}\left|T_I(x_i)-T_J(x_i)\right|\right\} \\ +\tan\left\{\frac{\pi}{6}\left|I_I(x_i)-I_J(x_i)\right|\right\} \\ +\tan\left\{\frac{\pi}{6}\left|F_I(x_i)-F_J(x_i)\right|\right\}\end{bmatrix} = 0 \text{ , } \forall x_i \in X$$

$$\Leftrightarrow \begin{bmatrix} \tan\left\{\frac{\pi}{6}\left|T_I(x_i)-T_J(x_i)\right|\right\} \\ +\tan\left\{\frac{\pi}{6}\left|I_I(x_i)-I_J(x_i)\right|\right\} \\ +\tan\left\{\frac{\pi}{6}\left|F_I(x_i)-F_J(x_i)\right|\right\} \end{bmatrix} = 0, \forall x_i \in X$$

$$\Leftrightarrow \tan\left\{\frac{\pi}{6}\left|T_I(x_i)-T_J(x_i)\right|\right\} = 0,$$

$$\tan\left\{\frac{\pi}{6}\left|I_I(x_i)-I_J(x_i)\right|\right\} = 0, , \forall x_i \in X$$

$$\tan\left\{\frac{\pi}{6}\left|F_I(x_i)-F_J(x_i)\right|\right\} = 0$$

$$\Leftrightarrow \left\{\frac{\pi}{6}\left|T_I(x_i)-T_J(x_i)\right|\right\} = 0,$$

$$\left\{\frac{\pi}{6}\left|I_I(x_i)-I_J(x_i)\right|\right\} = 0, \quad , \ \forall x_i \in X$$

$$\left\{\frac{\pi}{6}\left|F_I(x_i)-F_J(x_i)\right|\right\} = 0$$

$$\Leftrightarrow \left|T_I(x_i)-T_J(x_i)\right| = 0, \left|I_I(x_i)-I_J(x_i)\right| = 0, \left|F_I(x_i)-F_J(x_i)\right| = 0, \forall x_i \in A$$

$$\Leftrightarrow T_I(x_i) = T_J(x_i), I_I(x_i) = I_J(x_i), F_I(x_i) = F_J(x_i), \forall x_i \in A$$

$$\Leftrightarrow A = B.$$

Part (iii):

$$d_{TMNS}(I,J) = \frac{2}{\sqrt{3}n}\sum_{i=1}^{n}\begin{bmatrix} \tan\left\{\frac{\pi}{6}\left|T_I(x_i)-T_J(x_i)\right|\right\} \\ +\tan\left\{\frac{\pi}{6}\left|I_I(x_i)-I_J(x_i)\right|\right\} \\ +\tan\left\{\frac{\pi}{6}\left|F_I(x_i)-F_J(x_i)\right|\right\} \end{bmatrix}$$

$$= \frac{2}{\sqrt{3}n}\sum_{i=1}^{n}\tan\left\{\frac{\pi}{6}\left|T_J(x_i)-T_I(x_i)\right|\right\} + \tan\left\{\frac{\pi}{6}\left|I_J(x_i)-I_I(x_i)\right|\right\} + \tan\left\{\frac{\pi}{6}\left|F_J(x_i)-F_I(x_i)\right|\right\}$$

$$d_{TMNS}(J,I).$$

Part (iv):

If $I \subseteq J \subseteq K$ then $T_I(x_i) \le T_J(x_i) \le T_K(x_i)$, $I_I(x_i) \le I_J(x_i) \le I_K(x_i)$ and $F_I(x_i) \ge F_J(x_i) \ge F_K(x_i)$, $\forall x_i \in X$. This implies to the following inequalities

$$\left|T_I(x_i) - T_K(x_i)\right| \ge \left|T_I(x_i) - T_J(x_i)\right|, \ \left|T_I(x_i) - T_K(x_i)\right| \ge \left|T_J(x_i) - T_K(x_i)\right|$$

$$\left|I_I(x_i) - I_K(x_i)\right| \ge \left|I_I(x_i) - I_J(x_i)\right|, \ \left|I_I(x_i) - I_K(x_i)\right| \ge \left|I_J(x_i) - I_K(x_i)\right|$$

and $\left|F_I(x_i) - F_K(x_i)\right| \ge \left|F_I(x_i) - F_K(x_i)\right|$, $\left|F_I(x_i) - F_K(x_i)\right| \ge \left|F_J(x_i) - F_K(x_i)\right|$. From these inequalities we have

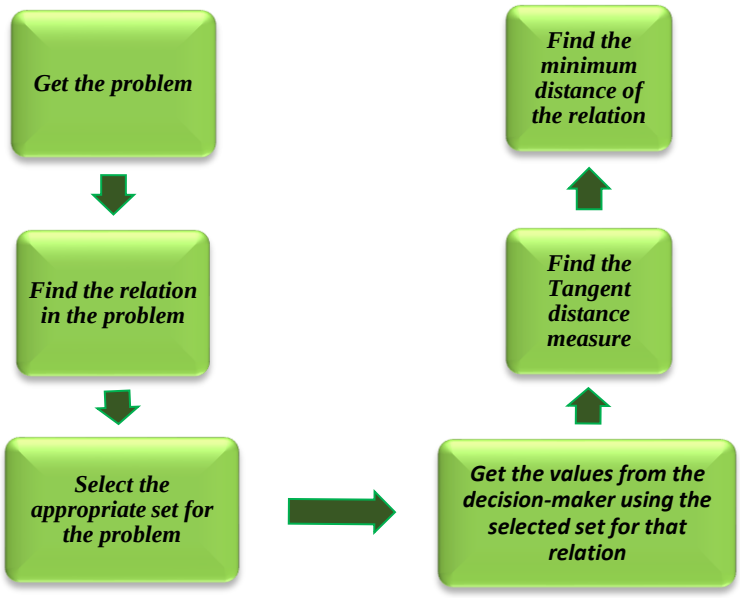
$$\tan\left\{\frac{\pi}{6}\left|T_I(x_i) - T_K(x_i)\right|\right\} + \tan\left\{\frac{\pi}{6}\left|I_I(x_i) - I_K(x_i)\right|\right\} + \tan\left\{\frac{\pi}{6}\left|F_I(x_i) - F_K(x_i)\right|\right\}$$
$$\ge \tan\left\{\frac{\pi}{6}\left|T_I(x_i) - T_J(x_i)\right|\right\} + \tan\left\{\frac{\pi}{6}\left|I_I(x_i) - I_J(x_i)\right|\right\} + \tan\left\{\frac{\pi}{6}\left|F_I(x_i) - F_J(x_i)\right|\right\}$$

$$\frac{2}{\sqrt{3}n}\sum_{i=1}^{n}\tan\left\{\frac{\pi}{6}\left|T_I(x_i) - T_K(x_i)\right|\right\} + \tan\left\{\frac{\pi}{6}\left|I_I(x_i) - I_K(x_i)\right|\right\} + \tan\left\{\frac{\pi}{6}\left|F_I(x_i) - F_K(x_i)\right|\right\}$$
$$\ge \frac{2}{\sqrt{3}n}\sum_{i=1}^{n}\tan\left\{\frac{\pi}{6}\left|T_I(x_i) - T_J(x_i)\right|\right\} + \tan\left\{\frac{\pi}{6}\left|I_I(x_i) - I_J(x_i)\right|\right\} + \tan\left\{\frac{\pi}{6}\left|F_I(x_i) - F_J(x_i)\right|\right\}$$

Hence $d_{TMNS}(I,K) \ge d_{TMNS}(I,J)$ is proven. Likewise $d_{TMNS}(I,K) \ge d_{TMNS}(J,K)$ can prove, and hence the theorem.

4. A Comparative Analysis of Distance Measures

In this section, we first construct Distance Measures using Fermatean Neutrosophic sets and Neutrosophic sets. Further, we have given an illustrative example for defined Distance Measures. So, we consider an example of disease, and their associated symptoms and treatment. We examine six different Distance Measures in this context, which include Fermatean Neutrosophic Set and Neutrosophic sets: Euclidean Distance, Normalised Euclidean Distance, Hamming Distance, Normalised Hamming Distance, SMSVNDM Distance, and TMFNDM Distance. We aim to find the minimum distance to ensure that patients receive the right treatment. The flowchart for the Distance Measures is presented below:


Get the problem
Find the relation in the problem
Select the appropriate set for the problem
Get the values from the decision-maker using the selected set for that relation
Find the Tangent distance measure
Find the minimum distance of the relation

Step 1: Problem:

Let $S=\{S_1,S_2,S_3,S_4,S_5,S_6,S_7,S_8,S_9,S_{10}\}$ be the set of patients, $C=\{C_1,C_2,C_3,C_4,C_5\}$ be the set of Treatment Strategies and $R=\{R_1,R_2,R_3,R_4,R_5,R_6\}$ be the set of Symptoms.

Table 4.1. a. shows the information about Treatment Strategies and the Symptoms of patient using Fermatean Neutrosophic Set

	R_1	R_2	R_3	R_4	R_5	R_6
C_1	(0.4,0.8,0.7)	(0.5,0.5,0.9)	(0.3,0.7,0.4)	(0.65,0.65,0.7)	(0.35,0.95,0.9)	(0.7,0.7,0.65)
C_2	(0.3 0.8 0.9)	(0.55,0.55,0.95)	(0.7,0.8,0.3)	(0.6,0.75,0.75)	(0.9,0.8,0.45)	(0.65,0.65,0.7)
C_3	(0.4,0.7,0.7)	(0.4,0.8,0.7)	(0.9,0.7,0.3)	(0.55,0.8,0.8)	(0.85,0.75,0.5)	(0.6,0.75,0.75)
C_4	(0.4,0.6,0.9)	(0.3,0.8,0.9)	(0.7,0.9,0.2)	(0.5,0.85,0.85)	(0.8,0.7,0.55)	(0.55,0.8,0.8)
C_5	(0.3,0.7,0.4)	(0.4,0.7,0.7)	(0.5,0.5,0.9)	(0.45,0.9,0.9)	(0.75,0.75,0.6)	(0.5,0.85,0.85)

Table 4.1. b. shows the information about Treatment Strategies and the Symptoms of patient using Neutrosophic Set

	R_1	R_2	R_3	R_4	R_5	R_6
C_1	(0.4,0.8,0.6)	(0.5,0.5,0.5)	(0.3,0.7,0.7)	(0.65,0.65,0.35)	(0.35,0.95,0.65)	(0.7,0.7,0.3)
C_2	(0.3,0.8,0.7)	(0.55,0.55,0.45)	(0.7,0.8,0.3)	(0.6,0.75,0.4)	(0.9,0.8,0.1)	(0.65,0.65,0.35)
C_3	(0.4,0.7,0.6)	(0.4,0.8,0.6)	(0.9,0.7,0.1)	(0.55,0.8,0.45)	(0.85,0.75,0.15)	(0.6,0.75,0.4)
C_4	(0.4,0.6,0.6)	(0.3,0.8,0.7)	(0.7,0.9,0.3)	(0.5,0.85,0.5)	(0.8,0.7,0.2)	(0.55,0.8,0.45)
C_5	(0.3,0.7,0.7)	(0.4,0.7,0.6)	(0.5,0.5,0.5)	(0.45,0.9,0.55)	(0.75,0.75,0.25)	(0.5,0.85,0.5)

Table 4.2.a. Shows the information about the patient's Symptoms using Fermatean Neutrosophic Set

	R_1	R_2	R_3	R_4	R_5	R_6
S_1	(0.8,0.7,0.55)	(0.7,0.9,0.65)	(0.6,0.75,0.75)	(0.7,0.8,0.6)	(0.95,0.85,0.4)	(0.4,0.95,0.95)
S_2	(0.75,0.75,0.6)	(0.65,0.9,0.7)	(0.55,0.8,0.8)	(0.85,0.9,0.7)	(0.9,0.8,0.45)	(0.35,0.95,0.9)
S_3	(0.7,0.7,0.65)	(0.6,0.9,0.75)	(0.5,0.85,0.85)	(0.9,0.9,0.3)	(0.85,0.75,0.5)	(0.9,0.8,0.45)
S_4	(0.65,0.65,0.7)	(0.55,0.8,0.8)	(0.45,0.9,0.9)	(0.95,0.95,0.4)	(0.8,0.7,0.55)	(0.85,0.75,0.5)
S_5	(0.6,0.75,0.75)	(0.5,0.85,0.85)	(0.4,0.95,0.95)	(0.9,0.9,0.45)	(0.75,0.65,0.6)	(0.8,0.7,0.55)
S_6	(0.55,0.8,0.8)	(0.45,0.9,0.9)	(0.95,0.85,0.4)	(0.85,0.85,0.5)	(0.7,0.9,0.65)	(0.75,0.75,0.6)
S_7	(0.5,0.85,0.85)	(0.4,0.95,0.95)	(0.9,0.8,0.45)	(0.8,0.8,0.55)	(0.65,0.9,0.7)	(0.7,0.7,0.65)
S_8	(0.45,0.9,0.9)	(0.35,0.95,0.9)	(0.85,0.75,0.5)	(0.75,0.75,0.6)	(0.6,0.9,0.75)	(0.65,0.65,0.7)
S_9	(0.4,0.95,0.95)	(0.9,0.8,0.45)	(0.8,0.7,0.55)	(0.7,0.7,0.65)	(0.55,0.8,0.8)	(0.6,0.75,0.75)
S_{10}	(0.95,0.85,0.4)	(0.85,0.75,0.5)	(0.75,0.65,0.6)	(0.65,0.65,0.7)	(0.5,0.85,0.85)	(0.55,0.8,0.8)

Table 4.2.b. Shows the information about the patient's Symptoms using Neutrosophic Set

	R_1	R_2	R_3	R_4	R_5	R_6
S_1	(0.8,0.7,0.2)	(0.7,0.9,0.3)	(0.6,0.75,0.4)	(0.7,0.8,0.3)	(0.95,0.85,0.05)	(0.4,0.95,0.6)
S_2	(0.75,0.75,0.25)	(0.65,0.9,0.35)	(0.55,0.8,0.45)	(0.85,0.9,0.15)	(0.9,0.8,0.1)	(0.35,0.95,0.65)
S_3	(0.7,0.7,0.3)	(0.6,0.9,0.4)	(0.5,0.85,0.5)	(0.9,0.9,0.2)	(0.85,0.75,0.15)	(0.9,0.8,0.1)
S_4	(0.65,0.65,0.35)	(0.55,0.8,0.45)	(0.45,0.9,0.55)	(0.95,0.95,0.05)	(0.8,0.7,0.2)	(0.85,0.75,0.15)
S_5	(0.6,0.75,0.4)	(0.5,0.85,0.5)	(0.4,0.95,0.6)	(0.9,0.9,0.1)	(0.75,0.65,0.25)	(0.8,0.7,0.2)
S_6	(0.55,0.8,0.45)	(0.45,0.9,0.55)	(0.95,0.85,0.05)	(0.85,0.85,0.15)	(0.7,0.9,0.3)	(0.75,0.75,0.25)
S_7	(0.5,0.85,0.5)	(0.4,0.95,0.6)	(0.9,0.8,0.1)	(0.8,0.8,0.2)	(0.65,0.9,0.35)	(0.7,0.7,0.3)
S_8	(0.45,0.9,0.55)	(0.35,0.95,0.65)	(0.85,0.75,0.15)	(0.75,0.75,0.25)	(0.6,0.9,0.4)	(0.65,0.65,0.35)
S_9	(0.4,0.95,0.6)	(0.9.0.8,0.1)	(0.8,0.7,0.2)	(0.7,0.7,0.3)	(0.55,0.8,0.45)	(0.6,0.75,0.4)
S_{10}	(0.95,0.85,0.05)	(0.85,0.75,0.15)	(0.75,0.65,0.25)	(0.65,0.65,0.35)	(0.5,0.85,0.5)	(0.55,0.8,0.45)

Step 2: Distance measure:

The ED measure $d_E(I,J)$ determines the shortest distance between each patient's symptoms (Tables 4.2.a and 4.2.b) and treatment strategies (Tables 4.1.a and 4.1.b), with the results presented in Tables 4.3.a and 4.3.b.Here, 'a' represents Fermatean Neutrosophic Sets and 'b' represents Neutrosophic Sets throughout the paper.

Table 4.3.a. $d_E(I,J)$

	C_1	C_2	C_3	C_4	C_5
S_1	.747774	.543906	.513971	1.814754	.510718
S_2	.736546	.61101	.53929	.6	.500833

S_3	.715309	.653835	.65192	.720532	.691616
S_4	.654472	.607591	.625833	.665207	.62849
S_5	.578072	.597216	.627827	.645497	.606218
S_6	.581664	.423281	.376386	.446281	.677003
S_7	.53929	.414327	.375278	.426224	.643558
S_8	.5058	.422295	.403113	.432049	.623832
S_9	.560506	.527573	.499166	.606905	.639661
S_{10}	.604152	.707696	.6	.728583	.619139

Table 4.3.b. $d_E\left(I,J\right)$

	C_1	C_2	C_3	C_4	C_5
S_1	.764308	.558271	.543906	1.624038	.593717
S_2	.744983	.580948	.57735	.564948	.602771
S_3	.685565	.588076	.635085	.63705	.70946
S_4	.608961	.538516	.603462	.570088	.640963
S_5	.498331	.51559	.587367	.526783	.595819
S_6	.695821	.465475	.361709	.467262	.630476
S_7	.63901	.44441	.32914	.423281	.570088
S_8	.587367	.438748	.32914	.403113	.521217
S_9	.595819	.469929	.522813	.618466	.602771

S_{10}	.691014	.701189	.680074	.737111	.754431

Tables 4.4.a and 4.4.b present the shortest distances between each patient (Tables 4.2.a and 4.2.b) and each treatment strategy (Tables 4.1.a and 4.1.b) calculated using the **N-ED Measure** $d_{n-E}(I,J)$.

Table 4.4.a. $d_{n-E}(I,J)$

	C_1	C_2	C_3	C_4	C_5
S_1	.431728	.314024	.296742	1.047749	.294863
S_2	.425245	.352767	.311359	.34641	.289156
S_3	.412984	.377492	.376386	.415999	.399305
S_4	.377859	.350793	.361325	.384057	.362859
S_5	.33375	.344803	.362476	.372678	.35
S_6	.335824	.244381	.217307	.25766	.390868
S_7	.311359	.239212	.216667	.24608	.371558
S_8	.292024	.243812	.232737	.249444	.36017
S_9	.323608	.304594	.288194	.350397	.369309
S_{10}	.348807	.408588	.34641	.420648	.35746

Table 4.4.b. $d_{n-E}(I,J)$

	C_1	C_2	C_3	C_4	C_5
S_1	.441273	.322318	.314024	.937639	.342783
S_2	.430116	.33541	.333333	.326173	.34801

S_3	.395811	.339526	.366667	.367801	.409607
S_4	.351584	.310913	.348409	.32914	.37006
S_5	.287711	.297676	.339116	.304138	.343996
S_6	.401732	.268742	.208833	.269774	.364005
S_7	.368932	.25658	.190029	.244381	.32914
S_8	.339116	.253311	.190029	.232737	.300925
S_9	.343996	.271314	.301846	.357071	.34801
S_{10}	.398957	.404832	.392641	.425572	.435571

HD Measure $d_H(I,J)$ is applied in Tables 4.5.a and 4.5.b to calculate the shortest distance between each patient (Tables 4.2.a and 4.2.b) and each treatment strategy (Tables 4.1.a and 4.1.b).

Table 4.5.a. $d_H(I,J)$

	C_1	C_2	C_3	C_4	C_5
S_1	1.55	1.2	1.016667	4.333333	1.05
S_2	1.55	1.166667	1.05	4.466667	0.983333
S_3	1.533333	1.316667	1.166667	1.35	1.233333
S_4	1.366667	1.25	1.133333	1.15	1.166667
S_5	1.216667	1.2	1.183333	1.166667	1.15
S_6	1	0.883333	0.8	0.916667	1.4
S_7	0.916667	0.8	0.783333	0.933333	1.383333

S_8	0.916667	0.766667	0.85	0.933333	1.383333
S_9	1.05	1	0.883333	1.133333	1.35
S_{10}	1.133333	1.35	1.166667	1.416667	1.2

Table 4.5.b $d_H(I,J)$

	C_1	C_2	C_3	C_4	C_5
S_1	1.583333	1.066667	1.116667	3.65	1.216667
S_2	1.633333	1.116667	1.233333	3.7	1.2
S_3	1.5	1.183333	1.233333	1.25	1.3
S_4	1.316667	1.133333	1.183333	1.066667	1.216667
S_5	1.1	1.116667	1.166667	1.083333	1.133333
S_6	1.25	1	0.75	1	1.283333
S_7	1.1	0.916667	0.633333	0.95	1.2
S_8	1.033333	0.85	0.666667	0.883333	1.166667
S_9	1.066667	0.916667	0.833333	1.083333	1.266667
S_{10}	1.283333	1.266667	1.283333	1.366667	1.483333

Tables 4.6.a and 4.6.b use the **N-HD Measure** $d_{n-H}(I,J)$ to calculate the shortest distance between each patient (Tables 4.2.a and 4.2.b) and each treatment strategy (Tables 4.1.a and 4.1.b).

Table 4.6.a $d_{n-H}(I,J)$

	C_1	C_2	C_3	C_4	C_5

S_1	.516667	.4	.338889	1.444444	.35
S_2	.516667	.388889	.35	1.488889	.327778
S_3	.511111	.438889	.388889	.45	.411111
S_4	.455556	.416667	.377778	.383333	.388889
S_5	.405556	.4	.394444	.388889	.383333
S_6	.333333	.294444	.266667	.305556	.466667
S_7	.305556	.266667	.261111	.311111	.461111
S_8	.305556	.255556	.283333	.311111	.461111
S_9	.35	.333333	.294444	.377778	.45
S_{10}	.377778	.45	.388889	.472222	.4

Table 4.6.b $d_{n-H}(I,J)$

	C_1	C_2	C_3	C_4	C_5
S_1	.527778	.355556	.372222	1.216667	.405556
S_2	.544444	.372222	.411111	1.233333	.4
S_3	.5	.394444	.411111	.416667	.433333
S_4	.438889	.377778	.394444	.355556	.405556
S_5	.366667	.372222	.388889	.361111	.377778
S_6	.416667	.333333	.25	.333333	.427778
S_7	.366667	.305556	.211111	.316667	.4

S_8	.344444	.283333	.222222	.294444	.388889
S_9	.355556	.305556	.277778	.361111	.422222
S_{10}	.427778	.422222	.427778	.455556	.494444

SMSVNDM $d_{SMSVN}(I,J)$ is applied in Tables 4.7.a and 4.7.b to calculate the shortest distance between each patient (Tables 4.2.a and 4.2.b) and each treatment strategy (Tables 4.1.a and 4.1.b).

Table 4.7.a $d_{SMSVN}(I,J)$

	C_1	C_2	C_3	C_4	C_5
S_1	.464307	.379313	.343275	.874075	.355207
S_2	.467474	.367576	.344268	.885766	.337049
S_3	.464007	.411326	.355037	.415623	.385918
S_4	.421581	.397855	.348495	.356958	.369734
S_5	.379576	.385099	.364244	.367209	.368729
S_6	.330478	.309958	.284596	.317735	.426397
S_7	.304578	.281881	.279889	.323697	.429843
S_8	.310954	.262432	.297704	.322444	.435095
S_9	.3364	.324272	.29669	.364477	.423775
S_{10}	.353776	.410054	.371496	.4165	.386503

Table 4.7.b $d_{SMSVN}(I,J)$

	C_1	C_2	C_3	C_4	C_5

S_1	.469595	.337518	.367908	.803274	.390976
S_2	.490022	.350467	.394929	.808717	.386716
S_3	.458565	.376773	.380913	.393788	.402805
S_4	.411555	.367563	.370281	.342259	.383257
S_5	.354198	.370483	.366593	.356914	.363875
S_6	.378302	.342953	.267461	.342778	.396847
S_7	.33707	.314587	.228695	.329424	.381951
S_8	.327754	.284782	.239657	.308627	.381117
S_9	.330455	.305328	.271538	.345515	.400498
S_{10}	.383377	.382179	.388731	.396423	.437752

The **TMFNDM** $d_{TMFN}(I,J)$ is used in Tables 4.8.a and 4.8.b to calculate the shortest distance between each patient (Tables 4.2.a and 4.2.b) and each treatment strategy (Tables 4.1.a and 4.1.b).

Table 4.8.a $d_{TMFN}(I,J)$

	C_1	C_2	C_3	C_4	C_5
S_1	.95103	.73298	.61939	2.77556	.63946
S_2	.9496	.71299	.64042	2.87073	.59897
S_3	.93827	.80546	.71558	.8302	.75801
S_4	.83509	.76368	.69464	.70746	.71434
S_5	.74384	.73386	.72561	.71758	.70299

S_6	.6133	.53647	.48535	.55705	.8562
S_7	.56139	.48621	.47516	.56657	.84436
S_8	.56002	.46632	.51589	.56665	.84355
S_9	.64143	.61013	.53876	.6938	.82489
S_{10}	.69333	.82934	.71326	.86955	.7348

Table 4.8.b $d_{TMFN}\left(I, J\right)$

	C_1	C_2	C_3	C_4	C_5
S_1	.48593	.32588	.34009	1.16465	.37139
S_2	.49996	.34084	.37563	1.18268	.36643
S_3	.4583	.36104	.37696	.38222	.3992
S_4	.40142	.34494	.36139	.32555	.37245
S_5	.33515	.33955	.35635	.32983	.3463
S_6	.3849	.30378	.2275	.30384	.39198
S_7	.33817	.27852	.19204	.28827	.36557
S_8	.3168	.25841	.20213	.26798	.35468
S_9	.3265	.27896	.25502	.33261	.38657
S_{10}	.39393	.39009	.3938	.42044	.45638

Comparison:

Six different distance measures have been employed to calculate the distance between each patient and each treatment strategy. The comparison of these distance measures is shown in the following table (Table 4.9.a and Table 4.9.b). In this table, the minimum distance for each measure is

highlighted with a yellow shade, while the Tangent Metric Fermatean Neutrosophic Distance Measure is highlighted with a red shade.

Table 4.9.a Comparison of distance measures

	C_1	C_2	C_3	C_4	C_5
S_1	0.747774476	0.543906	0.513971	1.814754	0.510718
	0.294863434	0.314024	0.296742	1.047749	0.294863
	1.016666667	1.2	1.016667	4.333333	1.05
	0.338888889	0.4	0.338889	1.444444	0.35
	0.343275464	0.379313	0.343275	0.874075	0.355207
	0.95103165	0.732979	0.619392	2.775565	0.639462
S_2	0.736545993	0.61101	0.53929	0.6	0.500833
	0.28915586	0.352767	0.311359	0.34641	0.289156
	1.166666667	1.166667	1.05	4.466667	0.983333
	0.516666667	0.388889	0.35	1.488889	0.327778
	0.467474217	0.367576	0.344268	0.885766	0.337049
	0.94959805	0.712986	0.640418	2.870727	0.598968
S_3	0.715308791	0.653835	0.65192	0.720532	0.691616
	0.412983723	0.377492	0.376386	0.415999	0.399305
	1.533333333	1.316667	1.166667	1.35	1.233333
	0.511111111	0.438889	0.388889	0.45	0.411111
	0.4640068	0.411326	0.355037	0.415623	0.385918
	0.938269576	0.805462	0.71558	0.830203	0.758013
S_4	0.654471797	0.607591	0.625833	0.665207	0.62849
	0.377859468	0.350793	0.361325	0.384057	0.362859
	1.366666667	1.25	1.133333	1.15	1.166667
	0.455555556	0.416667	0.377778	0.383333	0.388889
	0.421580569	0.397855	0.348495	0.356958	0.369734
	0.835094884	0.763681	0.69464	0.707462	0.714335
S_5	0.578071507	0.597216	0.627827	0.645497	0.606218
	0.33374974	0.344803	0.362476	0.372678	0.35
	1.216666667	1.2	1.183333	1.166667	1.15
	0.405555556	0.4	0.394444	0.388889	0.383333
	0.379576186	0.385099	0.364244	0.367209	0.368729

	0.743837083	0.733858	0.725612	0.717584	0.702993
S_6	0.581664279	0.423281	0.376386	0.446281	0.677003
	0.335824028	0.244381	0.217307	0.25766	0.390868
	1	0.883333	0.8	0.916667	1.4
	0.333333333	0.294444	0.266667	0.305556	0.466667
	0.330477572	0.309958	0.284596	0.317735	0.426397
	0.613302537	0.536471	0.485349	0.55705	0.856196
S_7	0.539289656	0.414327	0.375278	0.426224	0.643558
	0.311359028	0.239212	0.216667	0.24608	0.371558
	0.916666667	0.8	0.783333	0.933333	1.383333
	0.305555556	0.266667	0.261111	0.311111	0.461111
	0.304578182	0.281881	0.279889	0.323697	0.429843
	0.561392765	0.486208	0.475156	0.566565	0.844364
S_8	0.505799697	0.422295	0.403113	0.432049	0.623832
	0.292023591	0.243812	0.232737	0.249444	0.36017
	0.916666667	0.766667	0.85	0.933333	1.383333
	0.305555556	0.255556	0.283333	0.311111	0.461111
	0.310954344	0.262432	0.297704	0.322444	0.435095
	0.560016442	0.466317	0.515891	0.566649	0.843551
S_9	0.560505724	0.527573	0.499166	0.606905	0.639661
	0.323608131	0.304594	0.288194	0.350397	0.369309
	1.05	1	0.883333	1.133333	1.35
	0.35	0.333333	0.294444	0.377778	0.45
	0.336400426	0.324272	0.29669	0.364477	0.423775
	0.64143395	0.61013	0.538757	0.693804	0.824894
S_{10}	0.604152299	0.707696	0.6	0.728583	0.619139
	0.348807492	0.408588	0.34641	0.420648	0.35746
	1.133333333	1.35	1.166667	1.416667	1.2
	0.377777778	0.45	0.388889	0.472222	0.4
	0.353775568	0.410054	0.371496	0.4165	0.386503
	0.693329647	0.82934	0.713259	0.869545	0.734797

Table 4.9.b Comparison of distance measures

	C_1	C_2	C_3	C_4	C_5

S_1	0.764307966	0.558271	0.543906	1.624038	0.593717
	0.441273	0.322318	0.314024	0.937639	0.342783
	1.583333	1.066667	1.116667	3.65	1.216667
	0.527778	0.355556	0.372222	1.216667	0.405556
	0.469595	0.337518	0.367908	0.803274	0.390976
	0.48593	0.32588	0.34009	1.16465	0.37139
S_2	0.744983	0.580948	0.57735	0.564948	0.602771
	0.430116	0.33541	0.333333	0.326173	0.34801
	1.633333	1.116667	1.233333	3.7	1.2
	0.544444	0.372222	0.411111	1.233333	0.4
	0.490022	0.350467	0.394929	0.808717	0.386716
	0.49996	0.34084	0.37563	1.18268	0.36643
S_3	0.685565	0.588076	0.635085	0.63705	0.70946
	0.395811	0.339526	0.366667	0.367801	0.409607
	1.5	1.183333	1.233333	1.25	1.3
	0.5	0.394444	0.411111	0.416667	0.433333
	0.458565	0.376773	0.380913	0.393788	0.402805
	0.4583	0.36104	0.37696	0.38222	0.3992
S_4	0.608961	0.538516	0.603462	0.570088	0.640963
	0.351584	0.310913	0.348409	0.32914	0.37006
	1.316667	1.133333	1.183333	1.066667	1.216667
	0.438889	0.377778	0.394444	0.355556	0.405556
	0.411555	0.367563	0.370281	0.342259	0.383257
	0.40142	0.34494	0.36139	0.32555	0.37245
S_5	0.498331	0.51559	0.587367	0.526783	0.595819
	0.287711	0.297676	0.339116	0.304138	0.343996
	1.1	1.116667	1.166667	1.083333	1.133333
	0.366667	0.372222	0.388889	0.361111	0.377778
	0.354198	0.370483	0.366593	0.356914	0.363875
	0.33515	0.33955	0.35635	0.32983	0.3463
S_6	0.695821	0.465475	0.361709	0.467262	0.630476
	0.401732	0.268742	0.208833	0.269774	0.364005
	1.25	1	0.75	1	1.283333
	0.416667	0.333333	0.25	0.333333	0.427778
	0.378302	0.342953	0.267461	0.342778	0.396847

	0.3849	0.30378	0.2275	0.30384	0.39198
S_7	0.63901	0.44441	0.32914	0.423281	0.570088
	0.368932	0.25658	0.190029	0.244381	0.32914
	1.1	0.916667	0.633333	0.95	1.2
	0.366667	0.305556	0.211111	0.316667	0.4
	0.33707	0.314587	0.228695	0.329424	0.381951
	0.33817	0.27852	0.19204	0.28827	0.36557
S_8	0.587367	0.438748	0.32914	0.403113	0.521217
	0.339116	0.253311	0.190029	0.232737	0.300925
	1.033333	0.85	0.666667	0.883333	1.166667
	0.344444	0.283333	0.222222	0.294444	0.388889
	0.327754	0.284782	0.239657	0.308627	0.381117
	0.3168	0.25841	0.20213	0.26798	0.35468
S_9	0.595819	0.469929	0.522813	0.618466	0.602771
	0.343996	0.271314	0.301846	0.357071	0.34801
	1.066667	0.916667	0.833333	1.083333	1.266667
	0.355556	0.305556	0.277778	0.361111	0.422222
	0.330455	0.305328	0.271538	0.345515	0.400498
	0.3265	0.27896	0.25502	0.33261	0.38657
S_{10}	0.691014	0.701189	0.680074	0.737111	0.754431
	0.398957	0.404832	0.392641	0.425572	0.435571
	1.283333	1.266667	1.283333	1.366667	1.483333
	0.427778	0.422222	0.427778	0.455556	0.494444
	0.383377	0.382179	0.388731	0.396423	0.437752
	0.39393	0.39009	0.3938	0.42044	0.45638

Step 3: Comparative result

The consolidated results are presented in Table 4.9.a and Table 4.9.b, and further summarized in Table 4.10.a and Table 4.10.b. The shortest distance values for each patient and treatment strategy are shown in Table 4.10 below. Therefore, the $d_{TMFN}(I,J)$ offers the most consistent and straightforward solution for measuring distance using Frematean neutrosophic set values and neutrosophic set values.

Table 4.10.a Comparison (Fermatean neutrosophic set)

	$d_E(I,J)$	$d_{n-E}(I,J)$	$d_H(I,J)$	$d_{n-H}(I,J)$	$d_{SMSVN}(I,J)$	$d_{TMFN}(I,J)$
S_1	C5	C5	C3	C1&C3	C1&C3	C3
S_2	C5	C1&C5	C5	C5	C5	C5
S_3	C3	C3	C3	C3	C3	C3
S_4	C2	C2	C3	C3	C3	C3
S_5	C1	C1	C5	C5	C3	C5
S_6	C3	C3	C3	C3	C3	C3
S_7	C3	C3	C3	C3	C3	C3
S_8	C3	C3	C2	C2	C2	C2
S_9	C3	C3	C3	C3	C3	C3
S_{10}	C3	C3	C5	C1	C1	C1

Table 4.10.b Comparison (Neutrosophic set)

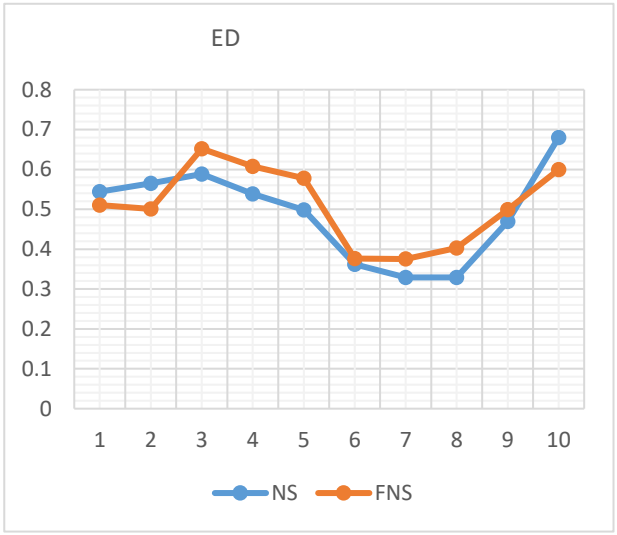
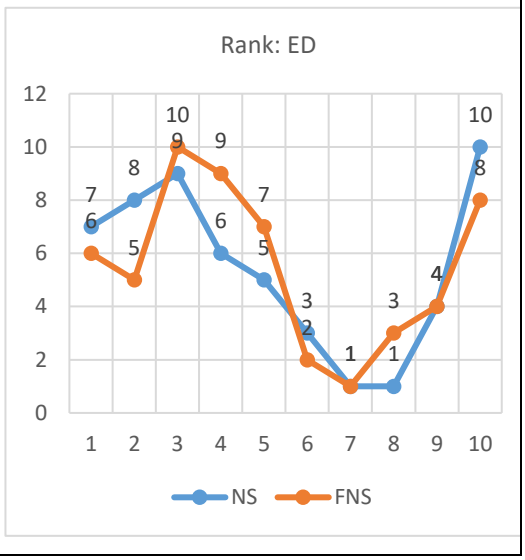
	$d_E(I,J)$	$d_{n-E}(I,J)$	$d_H(I,J)$	$d_{n-H}(I,J)$	$d_{SMSVN}(I,J)$	$d_{TMFN}(I,J)$
S_1	C3	C3	C2	C2	C2	C2
S_2	C4	C4	C2	C2	C2	C2
S_3	C2	C2	C2	C2	C2	C2
S_4	C2	C2	C4	C4	C4	C4
S_5	C2	C1	C4	C4	C1	C4
S_6	C3	C3	C3	C3	C3	C3
S_7	C3	C3	C3	C3	C3	C3

S_8	C3	C3	C3	C3	C3	C3
S_9	C2	C2	C3	C3	C3	C3
S_{10}	C3	C3	C2	C2	C2	C2

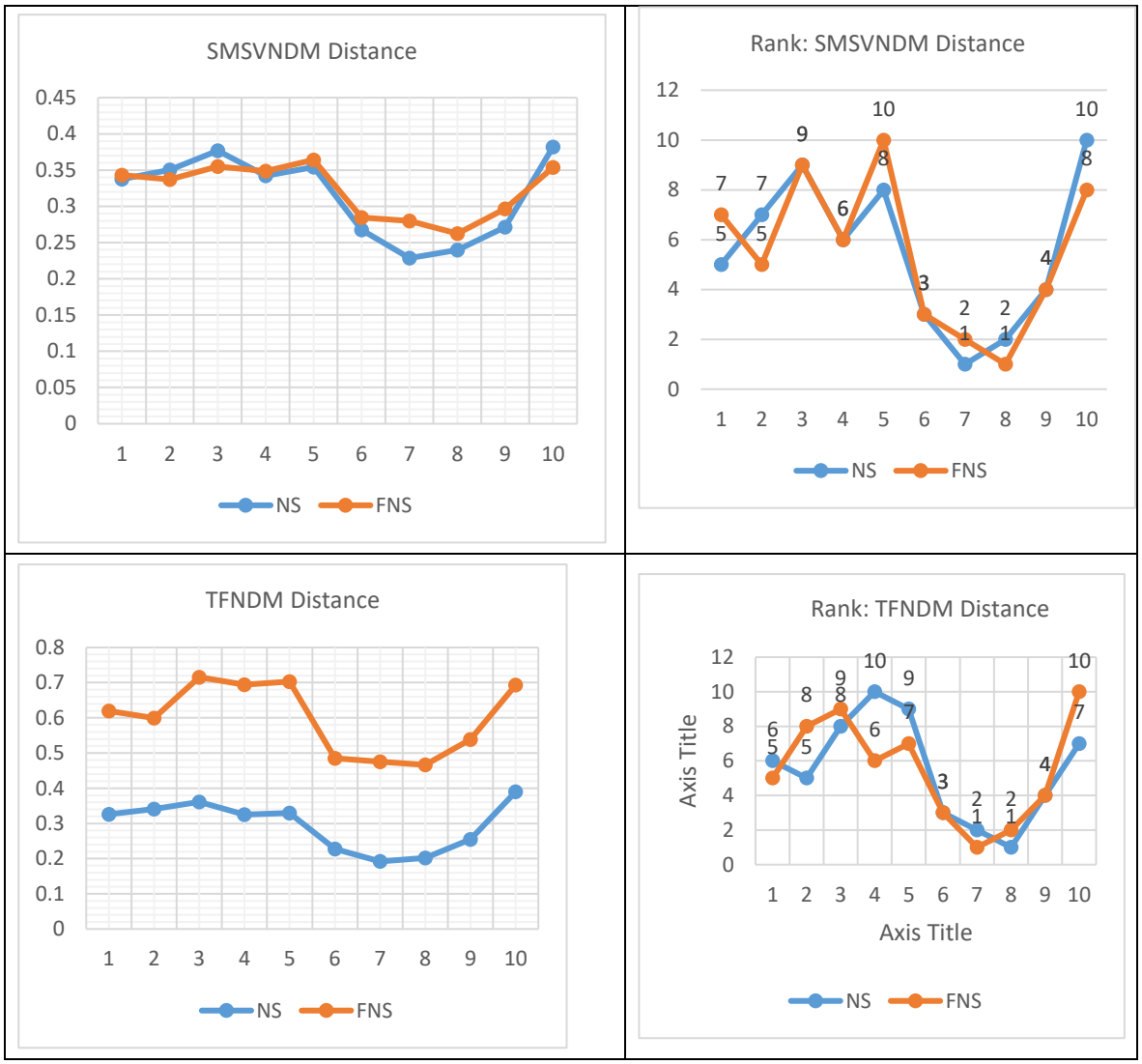
The above table shows that the most frequently occurring shortest distance is found in the TMFNDM. Hence TMFNDM generalized case of other six distance measure.

Step 4: Graphical result

In this step, we compare the results using the FNS and NS values. Also, we present graphical representations of six distinct distance measurements comparing each patient to each Treatment Strategy given in the model example. Additionally, we provide graphs below illustrating rankings using FNS and NS. Top of Form

Distance Measure	**Rank of the Distance Measure**
	

N-ED
0.45
0.4
0.35
0.3
0.25
0.2
0.15
0.1
0.05
0
1 2 3 4 5 6 7 8 9 10
NS FNS
Rank: N-ED
12
10
8
6
4
2
0
1 2 3 4 5 6 7 8 9 10
NS FNS
HD
1.4
1.2
1
0.8
0.6
0.4
0.2
0
1 2 3 4 5 6 7 8 9 10
NS FNS
Rank: HD
1.4
1.2
1
0.8
0.6
0.4
0.2
0
1 2 3 4 5 6 7 8 9 10
NS FNS
N-HD
0.45
0.4
0.35
0.3
0.25
0.2
0.15
0.1
0.05
0
1 2 3 4 5 6 7 8 9 10
NS FNS
Rank: N-HD
12
10
8
6
4
2
0
1 2 3 4 5 6 7 8 9 10
NS FNS



In the graphical representation, we observed a difference in ranking between the Neutrosophic and Fermatean Neutrosophic Sets. The Fermatean Neutrosophic Set permits higher membership degree values, resulting in comparatively higher false values, which influence the outcome.

Conclusion

This study presents a novel distance measure tailored for FNSs, with the TMFNDM serving as a generalized case. A comparative analysis showed that the graphical rankings differ between neutrosophic and Fermatean neutrosophic sets, primarily due to the latter's allowance for higher membership degree values. Fermatean Neutrosophic Set values lead to comparatively higher false values, significantly influencing the results. The findings emphasize that selecting an appropriate set is crucial for achieving optimal solutions in scenarios requiring high membership values. The Tangent Metric Fermatean Neutrosophic Distance Measure provides superior outcomes in such cases. Fields like medicine, research, and disaster management often involve high membership values, making this distance measure particularly relevant and effective.

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A novel plithogenic MCDM framework for credit evaluation of bidding and tendering in construction projects with contradictions

Maohua Yang[1]*, Jiajia Yu[2], Yuhe Wang[3]
[1,2,3]Chongqing Jianzhu College, 400072, Chongqing, China
*Corresponding author, E-mail: 944168748@qq.com

Abstract: This paper introduces a novel approach to decision-making by integrating Plithogenic modeling methodology with multi-criteria decision-making (MCDM) techniques. Traditional Plithogenic decision-making frameworks primarily rely on degrees of appurtenance as their foundation. However, this study presents a new paradigm that leverages degrees of contradiction to rank alternatives, offering a unique and innovative perspective compared to conventional methods. The research employs the PROMETHEE method to evaluate and rank alternatives, with the assessment carried out by four experienced experts using Plithogenic numbers. A total of ten criteria and five alternatives were analyzed in this study. The findings highlight that the Financial Stability criterion holds the highest importance, while the Risk Management Strategy criterion is weighted the lowest. This demonstrates the capability of the proposed method to address complex decision-making scenarios in the construction bidding and tendering context.

Keywords: Multi-Criteria Decision-Making, Uncertainty, Plithogenic Sets, Bidding and Tendering, Construction Projects.

1. Introduction

Tendering is a fundamental process in the construction industry, serving as a formal mechanism to transform an estimate into a bid for carrying out a specific project at a proposed price. Defined as "the process of preparing and submitting for acceptance a conforming offer to carry out work for a price," tendering plays a pivotal role in selecting contractors for building projects, assessing the state of the market, and ensuring compliance with regulations promoting free and open competition [1-2]. Traditionally, the purpose of tendering has been to allow contractors to secure opportunities to execute construction projects. However, tendering processes extend beyond contractors and clients, often involving subcontractors, suppliers, and consultants in an interconnected network of negotiations. While price competition frequently forms the basis of these processes, it is not the sole criterion for selection [3-4].

Over recent years, the rapid evolution of e-commerce technologies has introduced transformative changes to tendering procedures. Digital platforms now facilitate the preparation, publication, and management of tender documents while streamlining the process of bid submissions and award notifications. These advancements are crucial for organizations operating in dynamic environments, such as the construction industry, where success often hinges on identifying opportunities quickly and forming effective consortiums [5-6]. Furthermore, e-commerce tools

have allowed for greater transparency and coordination, enabling tendering authorities to better navigate the complexities of modern procurement [7-8].
The integration of advanced mathematical frameworks, such as plithogenic sets, into tendering processes has represented a significant leap forward in decision-making capabilities. Introduced by Smarandache in 2018, plithogenic sets extend traditional models such as crisp, fuzzy, intuitionistic fuzzy, and neutrosophic sets by incorporating degrees of contradiction alongside appurtenance. This extension allows for more robust evaluations of complex, multi-criteria scenarios, making plithogenic sets particularly useful in construction tendering where criteria such as cost, technical quality, and compliance often conflict [9-10]. For instance, a contractor with low cost but limited technical expertise might be evaluated differently compared to one offering higher cost with superior quality. Plithogenic frameworks help decision-makers resolve such conflicts effectively [11-12].

Applications of plithogenic sets have expanded across various industries, showcasing their adaptability in handling complex decision-making scenarios. In the construction sector, plithogenic models have been utilized to evaluate contractor bids by addressing contradictions between financial stability and innovation. For instance, Zavadskas et al. (2020) demonstrated the effectiveness of these models in optimizing procurement strategies, ensuring fair and accurate selection of contractors [13-14]. Similarly, Grida et al. (2020) applied plithogenic methods to assess IoT-based supply chain performance, incorporating conflicting attributes such as cost efficiency and reliability to enhance overall decision-making accuracy [15].

Beyond construction, plithogenic sets have shown utility in the healthcare sector. Abdel-Basset et al. (2019) introduced a plithogenic-based evaluation framework for hospital care systems, addressing diverse attributes such as patient satisfaction, operational efficiency, and service quality [16]. This model enabled healthcare administrators to make balanced decisions, even in the presence of conflicting priorities. Moreover, Ulutaş and Topal (2022) demonstrated the application of plithogenic sets in renewable energy selection by evaluating sources based on sustainability, cost, and technological feasibility, thereby highlighting the versatility of these methods in environmental decision-making [17-18].

Furthermore, combining plithogenic sets with established methodologies like PROMETHEE has significantly enhanced their applicability. PROMETHEE, known for its systematic ranking capabilities in multi-criteria decision-making, is further refined by integrating plithogenic operators that account for contradictions among criteria. Martin et al. (2021) utilized this combination in smart material selection, evaluating attributes like durability, cost, and environmental impact while addressing inherent trade-offs among them [19-20]. Similarly, Choukolaei et al. (2023) employed plithogenic PROMETHEE methods to develop GIS-based disaster response strategies, providing a structured approach to prioritizing actions during crises such as earthquakes [21-22].

Despite their effectiveness, the adoption of plithogenic sets in decision-making processes is still underexplored in certain fields. Current research has primarily focused on theoretical

advancements and specific case studies, leaving room for broader applications and innovative developments. By integrating advanced mathematical concepts with practical applications, plithogenic sets offer a promising solution to the increasing complexity of decision-making in modern industries [23-24].

This study builds on the existing body of knowledge by proposing a novel MCDM framework for evaluating construction tendering and bidding practices. By incorporating plithogenic sets, the framework addresses the complexities of conflicting criteria while utilizing advanced tools like PROMETHEE for ranking alternatives. The findings aim to contribute to the ongoing development of sophisticated decision-making systems, paving the way for more effective and reliable practices in tendering processes [25].

1.1 Main Motivation of This Study

The motivation for this study stems from the growing complexity of tendering and bidding processes in the construction industry, where decisions often involve evaluating multiple conflicting criteria such as cost, financial stability, technical expertise, and innovation. Traditional decision-making approaches are often insufficient to handle the nuanced and interconnected nature of these factors, leading to suboptimal outcomes. Plithogenic sets, with their ability to incorporate degrees of appurtenance, contradiction, and uncertainty, present a promising alternative for improving decision-making accuracy.

This study aims to advance the application of plithogenic sets within multi-criteria decision-making (MCDM) frameworks, particularly in scenarios involving construction projects where decisions must balance multiple priorities. The integration of plithogenic sets with established methodologies like PROMETHEE offers a more robust tool for evaluating alternatives, especially in environments characterized by uncertainty and conflicting data. Ultimately, this research is motivated by the need to create a framework that enhances the efficiency and reliability of tendering decisions, benefiting key stakeholders in the construction sector.

1.2 The main contributions of this study are as follows:

1. Proposing a novel MCDM approach to address decision-making challenges in evaluating bidding and tendering processes in construction projects.
2. Utilizing plithogenic sets to handle uncertainty and conflicting information effectively within the decision-making framework.
3. Employing the PROMETHEE (Preference Ranking Organization Method for Enrichment Evaluation) method to rank alternatives based on expert evaluations of ten criteria and five alternatives.

2. Definitions and Key Concepts

2.1 Informal Definition of the Plithogenic Set

The plithogenic set represents an advanced framework for modeling systems where multiple attributes and values interact. It is grounded in the concept of plithogeny, which refers to the creation and evolution of new entities from the interaction of contradictory or complementary components. Key characteristics:

I. *Multi-Attribute Representation*: Each element in the set is defined by one or more attributes, and each attribute can take multiple values.
II. *Degree of Appurtenance*: Each attribute value is associated with a degree of appurtenance that quantifies how strongly an element belongs to the set.
III. *Contradiction Degree*: A measure of dissimilarity between attribute values, which refines operations such as intersection and union.

2.2 Formal Definition of the Plithogenic Set

a. Attribute Value Spectrum

Let $A=\{\alpha_1, \alpha_2, \ldots, \alpha_m\}$ represent a non-empty set of attributes. Each attribute $\alpha \in A$ has a spectrum S, which is the set of all possible values for that attribute. The spectrum S may be:

- *Discrete*: $S=\{s_1, s_2, \ldots, s_l\}$
- *Countable Infinite*: $S=\{s_1, s_2, \ldots\}$
- *Uncountable*: S= [a, b] (e.g., a continuous interval of real numbers).

b. Attribute Value Range

For a specific application, a subset $V \subseteq S$ is identified as the attribute's value range. For example, if the attribute is "color," V could be {red, green, blue}

c. Dominant Attribute Value

The dominant value $v_D \in V$ is the most critical value for a given application. For example, if evaluating ripeness, "yellow" might be the dominant color for bananas.

d. Degree of Appurtenance

Each attribute value $v \in V$ is assigned a degree of appurtenance d(x,v) for an element x in the plithogenic set P: $d: P \times V \rightarrow [0,1]^z$, where z=1 for fuzzy sets, z=2 for intuitionistic fuzzy sets, and z=3 for neutrosophic sets.

e. Contradiction Degree

A contradiction degree c (v1, v2) is introduced to measure the dissimilarity between two attribute values $v1, v2 \in V$. This function satisfies:

- c (v1, v1) =0 (self-contradiction is zero),
- c (v1, v2) =c (v1, v2) (symmetry).

For example, if the attribute is "size," the contradiction degree between "small" and "large" may be c (small, large) =1, indicating maximum dissimilarity.

2.3 Mathematical Definition of Plithogenic Set

A plithogenic set P is represented as: $P=(U, \alpha, V, d, c)$, where:

U is the universe of discourse,
A is the attribute,
V is the range of attribute values,
d is the degree of appurtenance function,
c is the contradiction degree function.

2.3.1 Plithogenic Intersection

The intersection of two elements $x,y \in P$ is influenced by the contradiction degree between their attribute values. Mathematically:

$d(x \cap y,v)=(1-c(v1,v2)) \cdot t\_norm(d(x,v1),d(y,v2))+c(v1,v2) \cdot t\_conorm(d(x,v1),d(y,v2))$,where *t_norm and t_conorm are* standard fuzzy logic operators.

2.3.2 Plithogenic Union

The union is defined as:

$d(x \cup y,v)=(1-c(v1,v2)) \cdot t\_conorm(d(x,v1),d(y,v2))+c(v1,v2) \cdot t\_norm(d(x,v1),d(y,v2))$.

2.3.3 Plithogenic Complement

The complement of element x is influenced by the contradiction of each attribute value with the dominant value: $d(\neg x,v)=1-d(x,v)$.

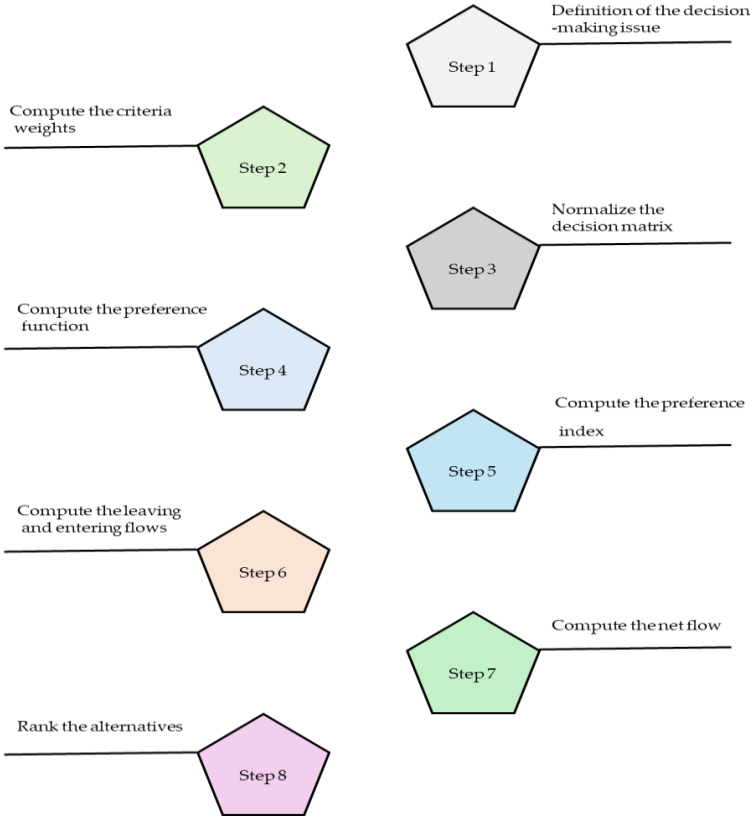

Figure 1. Overview framework of decision-making.

3. The Main Steps of the Framework

This section shows the steps of the decision-making methodology based on plithogenic sets to compute the criteria weights and rank the alternatives. Figure 1 presents the main steps of the proposed Framework.

1. Definition of the decision-making issue

The decision matrix with initial values is built with plithogenic terms such as:

$$r = \begin{bmatrix} x_{11} & \cdots & x_{1n} \\ \vdots & \ddots & \vdots \\ x_{m1} & \cdots & x_{mn} \end{bmatrix} \quad (1)$$

2. Compute the criteria weights.
3. Normalize the decision matrix.
The PROMETHEE method is used to normalize the decision matrix between the criteria and alternatives such as:

$$u_{ij} = \frac{x_{ij} - \min x_{ij}}{\max x_{ij} - \min x_{ij}} \quad (2)$$

$$u_{ij} = \frac{\max x_{ij} - x_{ij}}{\max x_{ij} - \min x_{ij}} \quad (3)$$

4. Compute the preference function as:

$$d_j(A_i, A_{i'}) = f_i(A_i) - f_i(A_{i'}) \quad (4)$$

Where the values of the preference function are calculated as:

$$\begin{cases} if\ d_j \leq 0 & 0 \\ if\ d_j > 0 & f_i(A_i) - f_i(A_{i'}) \end{cases} \quad (5)$$

5. Compute the preference index as:

$$q(A_i, A_{i'}) = \sum_{j=1}^{n} d_j(A_i, A_{i'}) w_j \quad (6)$$

6. Compute the leaving and entering flows as:

$$G^+(A_i) = \frac{1}{m-1} \sum q(A_i, A_{i'}) \quad (7)$$

$$G^-(A_i) = \frac{1}{m-1} \sum q(A_{i'}, A_i) \quad (8)$$

7. Compute the net flow as:

$$G(A_i) = G^+(A_i) - G^-(A_i) \quad (9)$$

8. Rank the alternatives.

4. Proposed Plithogenic Decision-Making Framework

This section details the proposed decision-making methodology using plithogenic sets to compute criteria weights and rank alternatives effectively. The framework incorporates the PROMETHEE method, a widely recognized MCDM technique, to handle uncertainty and contradictions in complex decision-making scenarios. The phases involved in the framework are described below.

Phase 4.1. Definition of the Decision-Making Issue:

The first step in the methodology is to clearly define the problem, criteria, and alternatives. In this study, ten criteria are (C1 - Resource Availability, C2 - Innovation and Technological Adoption, C3 - Project Execution Record, C4 - Financial Stability, C5 - Technical Capability and Expertise, C6 - Safety and Quality Assurance, C7 - Adherence to Legal and Regulatory Compliance, C8 - Risk Management Strategy, C9 - Reputation and References, C10 - Bidding Price Competitiveness)—are evaluated across five alternatives. The criteria are categorized as positive or negative based on their nature, and the initial decision matrix is constructed using plithogenic numbers. These numbers represent degrees of appurtenance, non-membership, and contradiction for each alternative against each criterion.

Phase 4.2. Computation of Criteria Weights

In this step, plithogenic operators are used to aggregate the expert evaluations and combine them into a unified matrix. The normalized crisp values of the plithogenic numbers are then calculated

to derive the criteria weights. These weights, shown in Table 5, reflect the relative importance of each criterion in the decision-making process.

Phase 4.3. Normalization of the Decision Matrix.
The decision matrix is standardized to ensure comparability between criteria. Positive and negative criteria are normalized using appropriate equations. For instance, for positive criteria, normalization follows the formula as

$$u_{ij} = \frac{x_{ij} - \min(x_{ij})}{\max(x_{ij}) - \min(x_{ij})}$$

outlined in Equation (2). This step ensures all values are scaled consistently, preparing the data for further analysis.

Phase 4.4. Computation of the Preference Function.
The preference between each pair of alternatives is determined based on their performance under a specific criterion. This is calculated using Equation (4), where the difference between the performance scores of two alternatives is evaluated. Preference values are then calculated to quantify the advantage of one alternative over another, as shown in Table 6.

Phase 4.5. Computation of the Preference Index.
The preference values are integrated with the criteria weights to determine an overall preference score for each pair of alternatives. This is achieved using Equation (6), which sums the weighted preference values across all criteria. The resulting indices capture the relative performance of alternatives, providing a foundation for further ranking.

Phase 4.6. Computation of the Leaving and Entering Flows.
The leaving flow $G^{+}(Ai)$, calculated using Equation (7), represents how much an alternative outperforms others, while the entering flow $G^{-}(Ai)$, computed using Equation (8), indicates how much it is outperformed. These flows offer a clear visualization of the competitive standing of each alternative.

Phase 4.7. Net Flow Calculation.
The net performance of each alternative is determined by subtracting the entering flow from the leaving flow, as shown in Equation (9). Alternatives with higher net flow values are considered better performers, as their strengths outweigh their weaknesses compared to others.

Phase 4.8. Ranking of Alternatives
Finally, the alternatives are ranked based on their net flow values, with the results displayed in Table 7. This step provides decision-makers with a clear and actionable hierarchy of alternatives, enabling them to identify the most suitable option for the given decision-making scenario.

4.1 Application of the Framework
To illustrate the proposed framework, a real-world case study was conducted involving the evaluation of five alternatives across ten criteria. Expert judgments were collected and converted

into plithogenic numbers, as shown in Tables 1–4. Criteria weights were derived from normalized values, as presented in Table 5. The PROMETHEE method was then applied to calculate preference values (Table 6), leaving and entering flows, and the final net flows (Table 7). The analysis revealed that Alternative 2 ranked the highest, indicating its suitability, while Alternative 4 ranked the lowest due to weaker overall performance.

This framework demonstrates the effectiveness of plithogenic sets in addressing multi-criteria decision-making challenges, providing a structured approach that integrates expert insights, handles contradictions, and delivers clear rankings for complex situations.

Table 1: Initial Criteria and Alternatives Matrix (Version 1)

	A_1	A_2	A_3	A_4	A_5
C_1	(0.30, 0.40, 0.80)	(0.50, 0.40, 0.60)	(0.70, 0.30, 0.10)	(0.30, 0.40, 0.80)	(0.10, 0.70, 0.80)
C_2	(0.70, 0.30, 0.10)	(0.30, 0.40, 0.80)	(0.10, 0.70, 0.80)	(0.70, 0.30, 0.10)	(0.30, 0.40, 0.80)
C_3	(0.50, 0.40, 0.60)	(0.70, 0.30, 0.10)	(0.30, 0.40, 0.80)	(0.10, 0.70, 0.80)	(0.70, 0.30, 0.10)
C_4	(0.70, 0.30, 0.10)	(0.70, 0.30, 0.10)	(0.70, 0.30, 0.10)	(0.30, 0.40, 0.80)	(0.10, 0.70, 0.80)
C_5	(0.90, 0.10, 0.10)	(0.70, 0.30, 0.10)	(0.30, 0.40, 0.80)	(0.10, 0.70, 0.80)	(0.50, 0.40, 0.60)
C_6	(0.30, 0.40, 0.80)	(0.70, 0.30, 0.10)	(0.30, 0.40, 0.80)	(0.10, 0.70, 0.80)	(0.70, 0.30, 0.10)
C_7	(0.70, 0.30, 0.10)	(0.30, 0.40, 0.80)	(0.70, 0.30, 0.10)	(0.30, 0.40, 0.80)	(0.10, 0.70, 0.80)
C_8	(0.10, 0.70, 0.80)	(0.70, 0.30, 0.10)	(0.30, 0.40, 0.80)	(0.10, 0.70, 0.80)	(0.70, 0.30, 0.10)
C_9	(0.70, 0.30, 0.10)	(0.30, 0.40, 0.80)	(0.10, 0.70, 0.80)	(0.30, 0.40, 0.80)	(0.50, 0.40, 0.60)
C_{10}	(0.30, 0.40, 0.80)	(0.70, 0.30, 0.10)	(0.30, 0.40, 0.80)	(0.10, 0.70, 0.80)	(0.70, 0.30, 0.10)

Table 2: Initial Criteria and Alternatives Matrix (Version 2)

	A_1	A_2	A_3	A_4	A_5
C_1	(0.10, 0.70, 0.80)	(0.10, 0.70, 0.80)	(0.70, 0.30, 0.10)	(0.90, 0.10, 0.10)	(0.10, 0.70, 0.80)
C_2	(0.30, 0.40, 0.80)	(0.10, 0.70, 0.80)	(0.70, 0.30, 0.10)	(0.90, 0.10, 0.10)	(0.30, 0.40, 0.80)
C_3	(0.10, 0.70, 0.80)	(0.70, 0.30, 0.10)	(0.90, 0.10, 0.10)	(0.30, 0.40, 0.80)	(0.90, 0.10, 0.10)
C_4	(0.70, 0.30, 0.10)	(0.10, 0.70, 0.80)	(0.70, 0.30, 0.10)	(0.90, 0.10, 0.10)	(0.70, 0.30, 0.10)
C_5	(0.90, 0.10, 0.10)	(0.10, 0.70, 0.80)	(0.10, 0.70, 0.80)	(0.70, 0.30, 0.10)	(0.90, 0.10, 0.10)
C_6	(0.30, 0.40, 0.80)	(0.10, 0.70, 0.80)	(0.70, 0.30, 0.10)	(0.90, 0.10, 0.10)	(0.30, 0.40, 0.80)
C_7	(0.70, 0.30, 0.10)	(0.30, 0.40, 0.80)	(0.10, 0.70, 0.80)	(0.70, 0.30, 0.10)	(0.90, 0.10, 0.10)
C_8	(0.10, 0.70, 0.80)	(0.10, 0.70, 0.80)	(0.70, 0.30, 0.10)	(0.90, 0.10, 0.10)	(0.10, 0.70, 0.80)
C_9	(0.10, 0.70, 0.80)	(0.70, 0.30, 0.10)	(0.90, 0.10, 0.10)	(0.30, 0.40, 0.80)	(0.70, 0.30, 0.10)
C_{10}	(0.30, 0.40, 0.80)	(0.50, 0.40, 0.60)	(0.10, 0.70, 0.80)	(0.70, 0.30, 0.10)	(0.90, 0.10, 0.10)

Table 3: Initial Criteria and Alternatives Matrix (Version 3)

	A_1	A_2	A_3	A_4	A_5
C_1	(0.10, 0.70, 0.80)	(0.70, 0.30, 0.10)	(0.90, 0.10, 0.10)	(0.70, 0.30, 0.10)	(0.90, 0.10, 0.10)
C_2	(0.10, 0.70, 0.80)	(0.10, 0.70, 0.80)	(0.70, 0.30, 0.10)	(0.90, 0.10, 0.10)	(0.30, 0.40, 0.80)

C_3	(0.50, 0.40, 0.60)	(0.10, 0.70, 0.80)	(0.70, 0.30, 0.10)	(0.90, 0.10, 0.10)	(0.90, 0.10, 0.10)
C_4	(0.10, 0.70, 0.80)	(0.70, 0.30, 0.10)	(0.90, 0.10, 0.10)	(0.70, 0.30, 0.10)	(0.90, 0.10, 0.10)
C_5	(0.90, 0.10, 0.10)	(0.10, 0.70, 0.80)	(0.70, 0.30, 0.10)	(0.90, 0.10, 0.10)	(0.70, 0.30, 0.10)
C_6	(0.30, 0.40, 0.80)	(0.10, 0.70, 0.80)	(0.10, 0.70, 0.80)	(0.70, 0.30, 0.10)	(0.90, 0.10, 0.10)
C_7	(0.10, 0.70, 0.80)	(0.70, 0.30, 0.10)	(0.90, 0.10, 0.10)	(0.90, 0.10, 0.10)	(0.10, 0.70, 0.80)
C_8	(0.10, 0.70, 0.80)	(0.10, 0.70, 0.80)	(0.70, 0.30, 0.10)	(0.90, 0.10, 0.10)	(0.10, 0.70, 0.80)
C_9	(0.10, 0.70, 0.80)	(0.10, 0.70, 0.80)	(0.70, 0.30, 0.10)	(0.90, 0.10, 0.10)	(0.70, 0.30, 0.10)
C_{10}	(0.10, 0.70, 0.80)	(0.10, 0.70, 0.80)	(0.70, 0.30, 0.10)	(0.90, 0.10, 0.10)	(0.30, 0.40, 0.80)

Table 4: Initial Criteria and Alternatives Matrix (Version 4)

	A_1	**A_2**	**A_3**	**A_4**	**A_5**
C_1	(0.30, 0.40, 0.80)	(0.10, 0.70, 0.80)	(0.50, 0.40, 0.60)	(0.50, 0.40, 0.60)	(0.70, 0.30, 0.10)
C_2	(0.30, 0.40, 0.80)	(0.30, 0.40, 0.80)	(0.10, 0.70, 0.80)	(0.50, 0.40, 0.60)	(0.50, 0.40, 0.60)
C_3	(0.50, 0.40, 0.60)	(0.30, 0.40, 0.80)	(0.30, 0.40, 0.80)	(0.10, 0.70, 0.80)	(0.50, 0.40, 0.60)
C_4	(0.70, 0.30, 0.10)	(0.30, 0.40, 0.80)	(0.10, 0.70, 0.80)	(0.50, 0.40, 0.60)	(0.70, 0.30, 0.10)
C_5	(0.90, 0.10, 0.10)	(0.10, 0.70, 0.80)	(0.30, 0.40, 0.80)	(0.10, 0.70, 0.80)	(0.50, 0.40, 0.60)
C_6	(0.30, 0.40, 0.80)	(0.10, 0.70, 0.80)	(0.50, 0.40, 0.60)	(0.70, 0.30, 0.10)	(0.30, 0.40, 0.80)
C_7	(0.10, 0.70, 0.80)	(0.30, 0.40, 0.80)	(0.10, 0.70, 0.80)	(0.50, 0.40, 0.60)	(0.10, 0.70, 0.80)
C_8	(0.10, 0.70, 0.80)	(0.30, 0.40, 0.80)	(0.10, 0.70, 0.80)	(0.50, 0.40, 0.60)	(0.10, 0.70, 0.80)
C_9	(0.10, 0.70, 0.80)	(0.70, 0.30, 0.10)	(0.30, 0.40, 0.80)	(0.10, 0.70, 0.80)	(0.50, 0.40, 0.60)
C_{10}	(0.30, 0.40, 0.80)	(0.10, 0.70, 0.80)	(0.50, 0.40, 0.60)	(0.10, 0.70, 0.80)	(0.50, 0.40, 0.60)

Table 5: Criteria Weights and Ranks

C1	C2	C3	C4	C5	C6	C7	C8	C9	C10
0.100324	0.090806	0.11005	0.122134	0.11053	0.0958	0.098842	0.078121	0.100337	0.093057
Rank: 6	Rank: 2	Rank: 8	Rank: 10	Rank: 9	Rank: 4	Rank: 5	Rank: 1	Rank: 7	Rank: 3

Table 6: Normalized Decision Matrix for Alternatives

Criteria	**A_1**	**A_2**	**A_3**	**A_4**	**A_5**
C_1	0	0.371833	1	0.880193	0.583266
C_2	0.387381	0	0.4365	1	0.332549
C_3	0.036688	0.28495	0.610119	0	1
C_4	0.564913	0	0.940917	1	0.940917
C_5	1	0	0.231799	0.380278	0.697934
C_6	0.122811	0	0.473519	1	0.938018
C_7	0.327202	0.442885	0.507865	1	0
C_8	0	0.483689	0.750899	1	0.371757
C_9	0	0.64673	0.808741	0.543652	1
C_{10}	0	0.324964	0.491226	0.60572	1

Table 7: PROMETHEE Ranking Results for Alternatives

	Leaving flow	**Entering flow**	**Net flow**	**Rank**
A_1	0.426848	0.113655	0.313193	4

A_2	0.410855	0.075799	0.335056	5
A_3	0.123591	0.267745	-0.14415	3
A_4	0.111909	0.378852	-0.26694	1
A_5	0.126441	0.363593	-0.23715	2

5. Discussion

This section discusses the results of the proposed plithogenic decision-making framework applied to rank alternatives in a construction tendering scenario. The PROMETHEE method was utilized to compute criteria weights and rank five alternatives based on evaluations provided by four experts, each with over 19 years of experience in decision-making. These experts assessed ten criteria, including Financial Stability, Innovation, and Risk Management Strategy, among others. The combination of plithogenic numbers and PROMETHEE ensured a robust and detailed analysis of the alternatives, considering both uncertainty and contradictions in the decision-making process.

The findings revealed that Financial Stability (C4) held the highest weight among the criteria, underscoring its critical importance in selecting contractors for construction projects. On the other hand, Risk Management Strategy (C8) received the lowest weight, reflecting its relatively minor influence in this particular case. The criteria weights were derived using plithogenic numbers and normalized to ensure consistency in the evaluation process. These weights are shown in Table 5 of the study.

The ranking process integrated criteria weights with performance values calculated for each alternative. Alternative 2 emerged as the top-performing option, with the highest net flow value, making it the most favorable choice. Conversely, Alternative 4 ranked the lowest due to weaker performance across several criteria. The final rankings, calculated based on net flow values, are shown in Figure 3, which highlights the overall performance of each alternative. Positive net flow values indicate stronger alternatives, while negative values highlight weaker ones.

To provide additional context, the detailed flow values for each alternative were also analyzed, including the Leaving Flow, which represents how much an alternative outperforms others, and the Entering Flow, which shows how much it is outperformed by others. These values, along with the net flow, are illustrated in Figure 2.

Figure 2 helps illustrate how the rankings were derived by showing the relative strengths and weaknesses of each alternative.

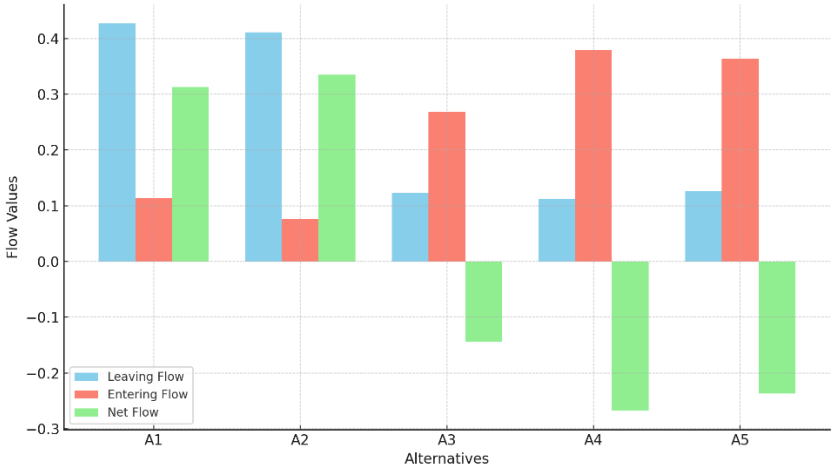

Figure 2: Detailed Analysis of Alternative Flows

In Figure 2, a comprehensive breakdown of the performance metrics for each alternative, offering insights into how the net flow values were calculated. For instance, while Alternative 2 has a strong leaving flow and a relatively low entering flow, Alternative 4 shows a high entering flow, reflecting its lower competitiveness.

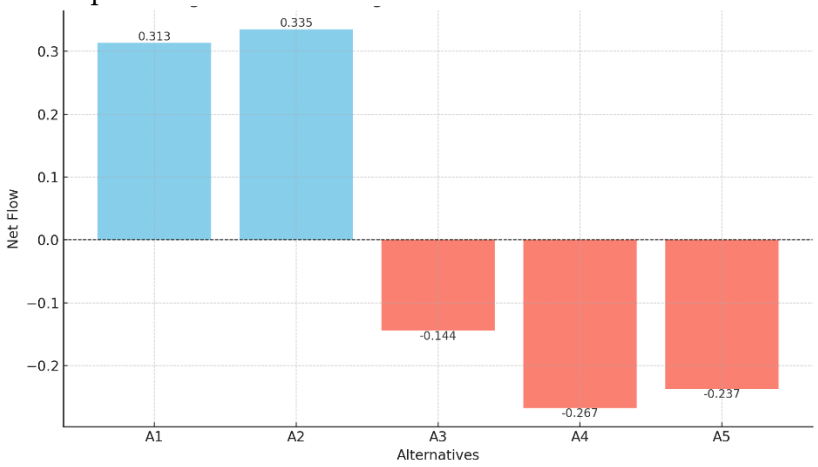

Figure 3: Final Ranking of Alternatives Based on Net Flow

In Figure 3, based on Net Flow simplifies the analysis by focusing solely on the net flow values, providing a clear and concise representation of the final rankings. This visualization highlights that Alternative 2 is the most suitable choice, followed by Alternative 1, while Alternative 4 and Alternative 5 are less favorable options.

The detailed discussions underscore the effectiveness of the proposed plithogenic framework. It provides a clear and systematic approach for decision-makers, allowing them to balance multiple criteria and handle contradictions effectively. This ensures that the rankings are reliable and reflect the true priorities and performance of the alternatives.

5.1 Comparative Analysis of Decision-Making Methods

A critical gap in this study is the lack of comparative evaluation between the proposed plithogenic framework and existing advanced decision-making methods such as Fuzzy MCDM and Intuitionistic Fuzzy PROMETHEE (See Table and Figure 4). This section aims to address this gap by analyzing and comparing these methods based on several key performance criteria. These criteria include their ability to handle contradictions, computational efficiency, scalability, flexibility in weighting criteria, sensitivity to expert bias, and integration with emerging technologies. The comparison provides a clearer understanding of the strengths and limitations of each approach, emphasizing the advantages of the plithogenic framework in handling complex decision-making scenarios.

Table 8. Comparative analysis of decision-making methods

Criteria	Fuzzy MCDM	Intuitionistic Fuzzy PROMETHEE	Proposed Plithogenic Framework
Handling Contradictions	Limited	Moderate	High
Computational Efficiency	Moderate	Moderate	Moderate
Scalability for Large Problems	High	Moderate	Moderate

Flexibility in Criteria Weighting	Moderate	High	High
Sensitivity to Expert Bias	High	Moderate	Moderate
Integration with Emerging Technologies	Limited	Moderate	High

— The proposed plithogenic framework outperforms both Fuzzy MCDM and Intuitionistic Fuzzy PROMETHEE in addressing contradictions between criteria. This is due to its integration of contradiction degrees, which allow for a nuanced evaluation of conflicting attributes.

— While Fuzzy MCDM, Intuitionistic Fuzzy PROMETHEE, and the plithogenic framework demonstrate comparable computational demands, none of these methods excel in this criterion. Optimizing computational performance remains an area for future research.

— Fuzzy MCDM is more scalable for large-scale problems due to its simpler mathematical foundations. In contrast, the plithogenic framework and Intuitionistic Fuzzy PROMETHEE are moderately scalable but may face computational challenges as the number of criteria and alternatives increases.

— Both Intuitionistic Fuzzy PROMETHEE and the plithogenic framework offer high flexibility in adjusting criteria weights based on specific scenarios. This makes them more suitable for dynamic environments compared to Fuzzy MCDM.

— Fuzzy MCDM is highly sensitive to expert bias due to its reliance on subjective inputs for appurtenance degrees. The plithogenic framework and Intuitionistic Fuzzy PROMETHEE mitigate this issue to some extent by incorporating mechanisms to manage uncertainty.

— The plithogenic framework is best positioned to integrate with advanced technologies such as artificial intelligence, big data analytics, and blockchain. This makes it a forward-looking choice for modern decision-making applications, particularly in tendering and bidding contexts.

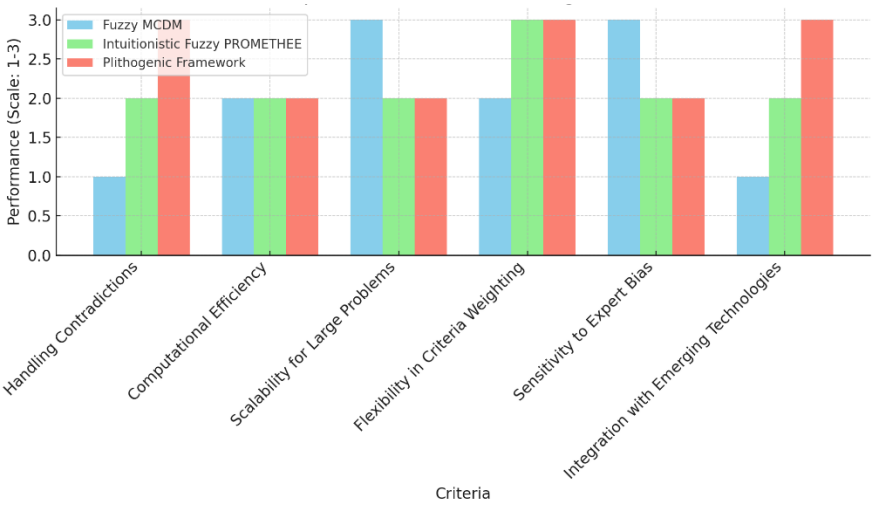

Figure 4: analysis of decision-making methods

This comparative analysis highlights the significant potential of the plithogenic framework, particularly in scenarios involving high levels of complexity and contradiction. By addressing

limitations in traditional methods and leveraging emerging technologies, the framework provides a robust solution for multi-criteria decision-making challenges. Further research is needed to enhance its computational efficiency and scalability to maximize its applicability across diverse fields.

5.2 Sensitivity Analysis

Sensitivity analysis is an important process to check how changes in the importance of criteria affect the final decision-making results. This helps ensure that the framework remains stable and reliable even if some criteria become more or less important. In this study, the weights assigned to each criterion were adjusted by increasing or decreasing them by 10%, and the changes in the results were analyzed to see how they impact the overall decision-making process.

The analysis started with the original weights from Table 5, which were calculated based on expert evaluations. These weights were used as the baseline for comparison. In the next step, adjustments were made to the weights. Each weight increased by 10% in one scenario and decreased by 10% in another scenario. After these adjustments, the weights were normalized so that the total weight remained the same across all criteria, ensuring a fair comparison. Finally, the effects of these changes were visualized to see how each criterion's contribution changed under different scenarios (See Figure 5).

The results of the analysis are shown in the bar chart. Three weight scenarios were considered: reducing the weight by 10%, keeping the original weights, and increasing the weights by 10%. Criteria with higher original weights, like Financial Stability (C4) and Project Execution Record (C3), showed noticeable changes when their weights were adjusted. However, criteria with lower weights, such as Risk Management Strategy (C8), showed very little change, meaning they are less sensitive to adjustments. This indicates that criteria with higher weights have a stronger influence on the results.

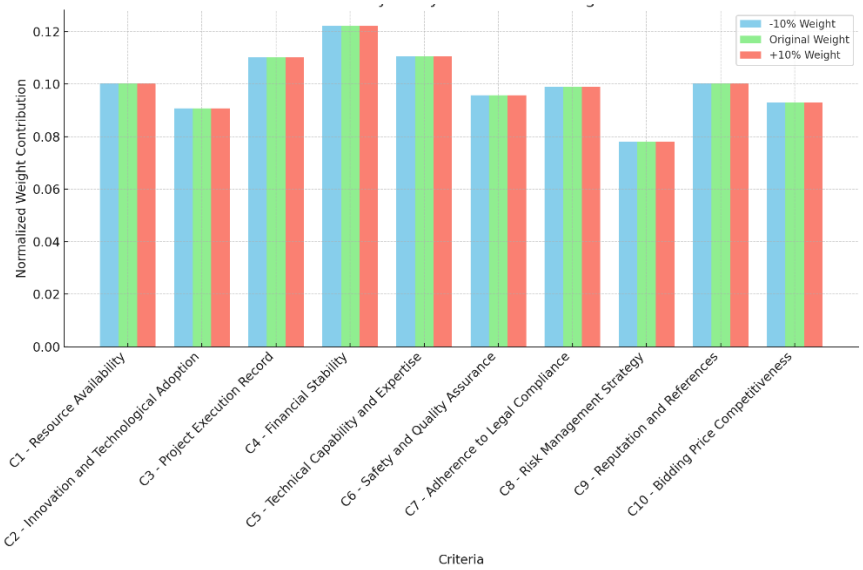

Figure 5. Sensitivity analysis of criteria weights.

The sensitivity analysis showed that the framework is robust. Even when the weights of the criteria were adjusted, the rankings of alternatives remained consistent. This gives confidence that the decision-making framework is reliable and can handle changes in expert opinions or project priorities without major impacts on the final decisions. If needed, this analysis can be extended to explore more scenarios or changes in criteria to further test the stability of the framework.

6. Conclusions and Future Work

This study presents a novel contradiction-based plithogenic decision-making framework, offering a robust and innovative approach to handling complex decision scenarios. By incorporating plithogenic numbers and operators, the framework effectively manages conflicts and provides a streamlined method for making optimal decisions. The methodology integrates expert evaluations and applies the PROMETHEE method to compute criteria weights and rank alternatives. The results identified Alternative 2 as the best option, while Alternative 4 was ranked the least favorable.

The framework demonstrated its ability to evaluate ten criteria and five alternatives comprehensively, combining different plithogenic numbers into a unified matrix for accurate analysis. The findings underline the potential of this approach to simplify decision-making in scenarios involving multiple conflicting criteria.

For future work, the framework could be extended to include dynamic criteria that evolve over time and explore its integration with machine learning techniques to further automate and enhance the decision-making process. Additionally, expanding the application to other industries would help validate its generalizability and adaptability.

Acknowledgment

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SuperHyperSoft Framework for College English Blended Teaching Quality Evaluation in the New Era: Addressing Uncertainty and Complexity

Dahai Zhang *

School of Foreign Languages, Chongqing College of Humanities, Science & Technology, Chongqing, 401524, China

*Corresponding author, E-mail: zhangdahai1980@163.com

Abstract

A Single-Valued Neutrosophic Set (SVNS) is a powerful tool for representing uncertainty, ambiguity, and incomplete or inconsistent information in real-world scenarios. This approach is particularly effective for handling uncertain measurements and data. Building on the concept of fuzzy set entropy, the SVN-entropy method has been developed to support multi-criteria decision-making (MCDM) processes. In this study, SVN-entropy is combined with the SuperHyperSoft (HSS) framework—an extension of HyperSoft sets—to evaluate different criteria and sub-criteria with varying values. The proposed method uses entropy to calculate criteria weights, which are then applied to assess the quality of College English Blended Teaching in the modern era. The study identifies eight main criteria, each with associated sub-criteria, to provide a comprehensive evaluation framework. This approach ensures a robust and precise assessment of blended teaching quality, leveraging advanced mathematical tools to handle complex and uncertain data effectively.

Keywords: SuperHyperSoft (SHS) Framework; College English Blended Teaching Quality; Uncertainty; MCDM Approach.

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1. Introduction and Literature Review

Blended learning, which combines traditional classroom instruction with online activities, has become an integral part of modern education systems due to the rapid advancements in information and communication technology (ICT). This approach combines the benefits of face-to-face teaching with the flexibility of digital platforms, creating an environment conducive to active, independent, and collaborative learning [1,2]. Over the last two decades, blended learning has been widely adopted, particularly in higher education, as institutions aim to meet the needs of diverse learners and evolve pedagogical demands [3,4] In China, the 2020 Guidelines on College English Teaching emphasized the role of blended learning in improving undergraduate education. These guidelines advocate integrating digital tools, virtual simulations, and online learning platforms into traditional teaching to foster interactive and personalized learning environments [5,6]. The COVID-19 pandemic further accelerated the adoption of blended learning globally, highlighting its potential to ensure educational continuity during disruptions [7,8]. Despite its

advantages, blended learning implementation faces challenges in quality assurance and evaluation, which necessitates innovative frameworks to assess its effectiveness comprehensively [9,10]

Evaluating the quality of blended teaching presents multidimensional challenges. Key factors, such as teaching effectiveness, student engagement, resource availability, and technological integration, must be measured in cohesive and adaptive frameworks. However, traditional methods often fail to capture the interdependencies and complexities of these factors, leading to incomplete evaluations[11,12]. Additionally, blended learning environments inherently involve subjective data, such as teacher performance and student satisfaction, which vary across contexts and evaluators [13,14]
Another significant challenge lies in adapting evaluation frameworks to accommodate emerging technologies and evolving educational needs. Rapid advancements in digital tools demand flexible approaches capable of addressing ambiguity, integrating diverse data sources, and managing complex interrelationships [15-17].

Researchers have explored advanced mathematical and computational frameworks to address these challenges. Neutrosophy, introduced by Smarandache, provides a robust theoretical foundation for handling ambiguity and indeterminacy in data. Neutrosophic sets, which generalize classical and fuzzy sets, incorporate three parameters—truth-membership, indeterminacy-membership, and falsity-membership—to represent uncertainty comprehensively [18-20]. Single-Valued Neutrosophic Sets (SVNS), a practical extension of neutrosophic sets, have been widely applied in fields such as risk analysis, healthcare, and education. These sets enable nuanced representation and analysis of uncertain information, making them particularly effective in evaluating complex systems [21,22]

The SuperHyperSoft (SHS) framework builds on the principles of HyperSoft sets and represents a significant advancement in multi-criteria decision-making (MCDM). By integrating SVNS with entropy-based weighting methods, the SHS framework provides a comprehensive solution for evaluating blended teaching environments. It models interrelationships among criteria and sub-criteria, effectively addressing the inherent complexity and uncertainty in blended learning evaluation [23-25].
Entropy, a mathematical measure of uncertainty, is utilized in the SHS framework to calculate objective weights for criteria. This minimizes subjective biases and ensures balanced evaluations. Its scalability and adaptability make it a valuable tool for institutions seeking to implement robust quality assurance mechanisms in education [26-28].

Conventional MCDM methods, such as the Analytic Hierarchy Process (AHP) and Technique for Order Preference by Similarity to Ideal Solution (TOPSIS), have been widely applied in educational evaluations. These methods, while effective in certain contexts, rely heavily on subjective inputs and lack the ability to handle uncertainty comprehensively [29-31]. Fuzzy logic has been instrumental in addressing ambiguity through techniques like Fuzzy Comprehensive Evaluation (FCE). This approach allows for partial truth values and is often used to assess qualitative factors, such as student satisfaction. However, its reliance on predefined membership functions limits its flexibility in dynamic environments [32-34].

Entropy-based approaches offer an objective mechanism for determining criteria weights and, when integrated with neutrosophic sets, provide robust frameworks for managing uncertainty and complexity in blended teaching evaluations [35,36].

2. Selection of Benchmark: College English Blended Teaching Quality Evaluation

Evaluating the quality of College English blended teaching in the modern era represents MCDM problem, requiring the careful assessment of diverse factors that influence teaching outcomes. This study identifies and categorizes a comprehensive set of eight key criteria, each accompanied by relevant sub-criteria, as detailed in Table 1. These criteria are instrumental in determining the strengths and weaknesses of various teaching approaches by assigning objective weights to each criterion.

The criteria include aspects such as teaching effectiveness, student engagement, quality of digital resources, and teacher competence in digital teaching, among others. These components aim to capture both the highest and lowest-performing attributes in the evaluation process. The weight assigned to each criterion ensures an unbiased and data-driven assessment, allowing educators and institutions to identify critical areas for improvement and prioritize resource allocation effectively.

By systematically incorporating these criteria into the evaluation process, the study establishes a robust benchmark for assessing blended teaching quality in College English courses. This approach not only highlights areas of excellence but also pinpoints specific challenges, offering actionable insights for enhancing teaching methodologies in the new era of education.

Table 1. Key Criteria and Associated Sub-Criteria

Criteria	Values
Teaching Effectiveness	High, Moderate, Low
Student Engagement	Highly Engaged, Partially Engaged, Not Engaged
Quality of Digital Resources	Excellent, Adequate, Poor
Flexibility of Teaching Methods	Very Flexible, Moderately Flexible, Inflexible
Assessment Design and Feedback	Comprehensive, Sufficient, Inadequate
Technical Support and Infrastructure	Excellent, Satisfactory, Unsatisfactory
Teacher Competence in Digital Teaching	Expert, Competent, Beginner
Student Satisfaction	Very Satisfied, Neutral, Dissatisfied

3. Decision-Making Methodology Using the Entropy Weights Method

The entropy weights method is a widely recognized mathematical approach used to determine the relative importance of criteria in a decision-making process. In this study, it plays a central role in evaluating the College English Blended Teaching Quality by objectively assigning weights to each criterion. This method ensures that the variability and significance of the data for each criterion is properly accounted for. The entropy method operates within the framework of SVNSs and uses a structured approach to calculate the weights.

3.1 Steps in the Entropy Weights Method

3.1.1 Construct the Decision Matrix

The process begins by establishing a decision matrix (DM) that represents the evaluation framework. In this matrix, rows correspond to the alternatives (n), and columns represent the criteria (m). Expert

evaluations populate the matrix, where each value represents the performance of an alternative for a given criterion. The general form of the decision matrix is:

$$DM = \begin{pmatrix} x_{11} & \cdots & x_{1m} \\ \vdots & \ddots & \vdots \\ x_{n1} & \cdots & x_{nm} \end{pmatrix} \quad (1)$$

Here, x_{ij} denotes the raw score assigned to the i^{th} alternative under the j^{th} criterion.

3.1.2 Normalize the Decision Matrix

To ensure comparability among criteria, the decision matrix is normalized. This step transforms the raw data into relative proportions, eliminating the influence of differing measurement scales. The normalized value y_{ij} is calculated as:

$$y_{ij} = \frac{x_{ij}}{\sum_{i=1}^{n} x_{ij}} \quad (2)$$

In this equation, x_{ij} is the raw value, and the denominator represents the sum of all values for the j^{th} criterion.

3.1.3 Compute Entropy for Each Criterion

The entropy value (e_j) for each criterion is calculated to measure the degree of information content or variability. A higher entropy value indicates less variability, whereas a lower value implies greater diversity among the data points. The formula is as follows:

$$e_j = \frac{1}{\log n} \sum_{i=1}^{n} y_{ij} \log\left(y_{ij}\right) \quad (3)$$

Here, n represents the number of alternatives, and y_{ij} is the normalized value. If $y_{ij}=0$, the term $y_{ij} \log\left(y_{ij}\right)$ is treated as zero to avoid mathematical undefinedness.

3.1.4 Calculate the Degree of Diversification

The degree of diversification (dj) indicates the significance of variability for each criterion. It is computed as:

$$d_j = 1 - e_j \quad (4)$$

A larger d_j value suggests that the corresponding criterion has more influence on the decision-making process due to its higher variability.

3.1.5 Determine the Objective Weights for Criteria

The weights (w_j) of each criterion are calculated based on the degree of diversification. These weights represent the relative importance of each criterion and are determined using the following formula:

$$w_j = \frac{d_j}{\sum_{j=1}^{m} d_j} \quad (5)$$

In this equation, the denominator ensures that the weights are normalized to sum up to one.

3.1.6 Rank the Criteria

The final step involves ranking the criteria based on their computed weights. Criteria with higher weights are considered more influential, allowing decision-makers to focus on the most critical factors in evaluating College English blended teaching quality. Figure 1 shows the steps of the entropy method.

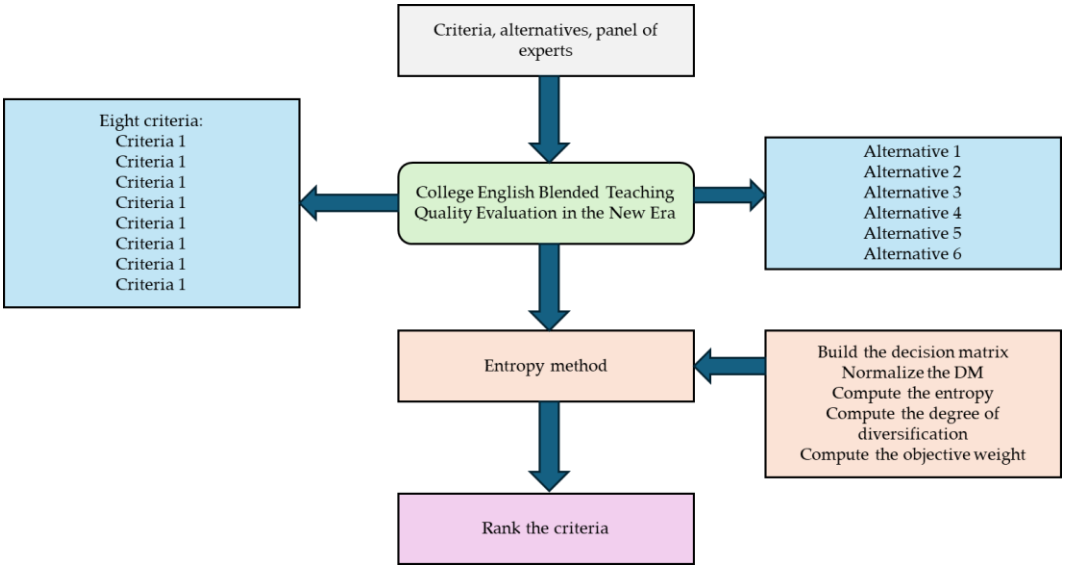

Figure 1. The details of framework.

3.2 Application Within the SVNS Framework

In this study, the entropy method operates within the Single-Valued Neutrosophic Sets (SVNSs) framework. This allows for the inclusion of uncertain and imprecise data, as SVNSs quantify truth-membership, indeterminacy-membership, and falsity-membership values for each criterion. The process involves:

1. Representing expert evaluations as neutrosophic numbers.
2. Converting neutrosophic numbers into crisp values for use in the decision matrix.
3. Aggregating crisp values into a normalized matrix and applying the entropy method.

The entropy method offers several key advantages that make it an effective tool for decision-making in complex evaluation scenarios. First, objectivity is a notable strength of this method, as it relies on the variability inherent in the data to calculate weights. This minimizes the influence of subjective biases, ensuring that the resulting weights are derived from measurable and consistent information. Second, the method's scalability allows it to handle large datasets efficiently, accommodating multiple criteria and alternatives without compromising accuracy. This makes it particularly suitable for evaluating systems with high complexity, such as blended teaching environments. Finally, the entropy method seamlessly integrates with Single-Valued Neutrosophic Sets (SVNSs), leveraging their ability to represent uncertainty and ambiguity in expert evaluations. This integration enhances the method’s robustness, enabling it to process imprecise or inconsistent data effectively while maintaining reliable outcomes. These advantages collectively highlight the entropy method’s versatility and reliability in multi-criteria decision-making processes.

3.3 SuperHyperSoft (SHS)

The SHS is an advanced extension of the HyperSoft set, designed to explore and analyze the relationships among multiple criteria and their respective sub-criteria. This framework allows for a comprehensive evaluation by modeling the interdependencies between different criteria, making it particularly suitable for MCDM problems. SHS is particularly valuable in complex systems, such as the evaluation of College English blended teaching quality, where interactions among criteria significantly influence the outcomes [16,17].

3.3.1 Mathematical Representation of SHS

The SHS framework employs mathematical constructs to represent relationships and dependencies systematically. The relationships between criteria (C1, C2,…) can be expressed as:

$$P(C_1) \times P(C_2) \to P(R) \tag{6}$$

Here:

P(C1) and P(C2) represent the power sets of criteria C1 and C2, respectively.

P(R) denotes the power set of the result space, which encapsulates all potential combinations of outcomes based on the interactions between criteria.

3.3.2 Cartesian Product in SHS

The cartesian product of the power sets of criteria expands the possibilities by considering all combinations of sub-criteria across different criteria. This is represented as:

$$P(C_1) \times P(C_2) \times P(C_3) = \begin{Bmatrix} \{\{C_{11}\},\{C_{12}\},\{C_{11},C_{12}\}\} \times \\ \{C_{21}\},\{C_{22}\},\{C_{21},C_{22}\} \times \{C_{31}\},\{C_{32}\},\{C_{33}\},\{C_{31},C_{32}\}, \\ \{C_{31},C_{33}\},\{C_{32},C_{33}\},\{C_{31},C_{32},C_{32}\} \end{Bmatrix} \tag{7}$$

The SHS offers significant applications in multi-criteria decision-making scenarios, particularly for systems with intricate interdependencies. One key application is relationship analysis, where SHS identifies and models the interactions between primary criteria and their sub-criteria. This analysis reveals how various factors collectively influence the overall evaluation, providing a deeper understanding of their interrelationships. Additionally, SHS excels in complex systems evaluation by considering all possible combinations of criteria and sub-criteria. This capability makes it particularly valuable for assessing systems with multiple interacting components, such as blended teaching environments, where the dynamics between teaching methods, resources, and engagement levels play a crucial role. Furthermore, the enhanced decision-making capabilities of SHS arise from its systematic exploration of these relationships, enabling decision-makers to extract actionable insights from intricate and interconnected data. These features collectively position SHS as a robust tool for analyzing and optimizing complex methods.

4. Determining Criteria Weights Using SVN-Entropy and HSS Framework

This section presents the computation of criteria weights using the SVN-entropy method integrated with the HSS framework. The evaluation process begins with assessments provided by three domain experts, who evaluated the criteria and alternatives to construct the decision matrix. The criteria and alternatives are then analyzed using SVNNs to handle uncertainties and ambiguities in the evaluation process.

The process involves several steps to ensure accurate computation of criteria weights. Initially, the score function is applied to convert SVNNs into crisp values, which are then aggregated into a single decision matrix. The resulting decision matrix is normalized using Equation (2), as shown in Table 5. This normalization process ensures comparability across different criteria by eliminating scale variations.

Subsequently, the entropy values are calculated using Equation (3), which quantifies the amount of information provided by each criterion. Higher entropy values indicate lower variability, while lower entropy values reflect higher variability and significance. Based on these entropy values, the degree of diversification is computed using Equation (4). This step identifies the influence of each criterion by capturing its level of dispersion.

Finally, the objective weights of the criteria are derived using Equation (5). These weights reflect the relative importance of each criterion in the overall evaluation and are displayed in Table 6.

This systematic methodology ensures that the criteria weights are determined objectively, providing a robust foundation for evaluating the quality of College English blended teaching.

Table 2. Decision Matrix: First Expert Evaluation

	C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8
A_1	(0.5,0.5,0.5)	(0.4,0.6,0.7)	(0.3,0.7,0.8)	(0.2,0.8,0.9)	(0.1,0.9,0.9)	(0.9,0.1,0.2)	(0.8,0.2,0.3)	(0.7,0.3,0.4)
A_2	(0.7,0.3,0.4)	(0.6,0.4,0.5)	(0.5,0.5,0.5)	(0.4,0.6,0.7)	(0.3,0.7,0.8)	(0.2,0.8,0.9)	(0.1,0.9,0.9)	(0.5,0.5,0.5)
A_3	(0.8,0.2,0.3)	(0.9,0.1,0.2)	(0.8,0.2,0.3)	(0.7,0.3,0.4)	(0.6,0.4,0.5)	(0.5,0.5,0.5)	(0.9,0.1,0.2)	(0.4,0.6,0.7)
A_4	(0.9,0.1,0.2)	(0.1,0.9,0.9)	(0.1,0.9,0.9)	(0.2,0.8,0.9)	(0.3,0.7,0.8)	(0.4,0.6,0.7)	(0.8,0.2,0.3)	(0.3,0.7,0.8)
A_5	(0.1,0.9,0.9)	(0.2,0.8,0.9)	(0.3,0.7,0.8)	(0.4,0.6,0.7)	(0.5,0.5,0.5)	(0.6,0.4,0.5)	(0.7,0.3,0.4)	(0.1,0.9,0.9)
A_6	(0.2,0.8,0.9)	(0.3,0.7,0.8)	(0.4,0.6,0.7)	(0.5,0.5,0.5)	(0.6,0.4,0.5)	(0.7,0.3,0.4)	(0.6,0.4,0.5)	(0.9,0.1,0.2)

Table 3. Decision Matrix: Second Expert Evaluation

	C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8
A_1	(0.5,0.5,0.5)	(0.4,0.6,0.7)	(0.3,0.7,0.8)	(0.2,0.8,0.9)	(0.1,0.9,0.9)	(0.5,0.5,0.5)	(0.8,0.2,0.3)	(0.7,0.3,0.4)
A_2	(0.4,0.6,0.7)	(0.6,0.4,0.5)	(0.5,0.5,0.5)	(0.4,0.6,0.7)	(0.5,0.5,0.5)	(0.4,0.6,0.7)	(0.5,0.5,0.5)	(0.5,0.5,0.5)
A_3	(0.3,0.7,0.8)	(0.5,0.5,0.5)	(0.4,0.6,0.7)	(0.5,0.5,0.5)	(0.4,0.6,0.7)	(0.3,0.7,0.8)	(0.4,0.6,0.7)	(0.4,0.6,0.7)
A_4	(0.9,0.1,0.2)	(0.4,0.6,0.7)	(0.3,0.7,0.8)	(0.4,0.6,0.7)	(0.3,0.7,0.8)	(0.4,0.6,0.7)	(0.3,0.7,0.8)	(0.5,0.5,0.5)
A_5	(0.1,0.9,0.9)	(0.3,0.7,0.8)	(0.3,0.7,0.8)	(0.3,0.7,0.8)	(0.5,0.5,0.5)	(0.6,0.4,0.5)	(0.7,0.3,0.4)	(0.4,0.6,0.7)
A_6	(0.2,0.8,0.9)	(0.3,0.7,0.8)	(0.4,0.6,0.7)	(0.5,0.5,0.5)	(0.6,0.4,0.5)	(0.7,0.3,0.4)	(0.6,0.4,0.5)	(0.3,0.7,0.8)

Table 4. Decision Matrix: Third Expert Evaluation

	C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8
A_1	(0.5,0.5,0.5)	(0.8,0.2,0.3)	(0.3,0.7,0.8)	(0.2,0.8,0.9)	(0.1,0.9,0.9)	(0.9,0.1,0.2)	(0.8,0.2,0.3)	(0.8,0.2,0.3)
A_2	(0.8,0.2,0.3)	(0.9,0.1,0.2)	(0.8,0.2,0.3)	(0.4,0.6,0.7)	(0.3,0.7,0.8)	(0.2,0.8,0.9)	(0.8,0.2,0.3)	(0.9,0.1,0.2)
A_3	(0.9,0.1,0.2)	(0.1,0.9,0.9)	(0.9,0.1,0.2)	(0.8,0.2,0.3)	(0.6,0.4,0.5)	(0.8,0.2,0.3)	(0.9,0.1,0.2)	(0.1,0.9,0.9)
A_4	(0.1,0.9,0.9)	(0.1,0.9,0.9)	(0.1,0.9,0.9)	(0.9,0.1,0.2)	(0.8,0.2,0.3)	(0.9,0.1,0.2)	(0.1,0.9,0.9)	(0.3,0.7,0.8)
A_5	(0.1,0.9,0.9)	(0.2,0.8,0.9)	(0.3,0.7,0.8)	(0.1,0.9,0.9)	(0.9,0.1,0.2)	(0.1,0.9,0.9)	(0.7,0.3,0.4)	(0.1,0.9,0.9)
A_6	(0.2,0.8,0.9)	(0.3,0.7,0.8)	(0.4,0.6,0.7)	(0.5,0.5,0.5)	(0.1,0.9,0.9)	(0.7,0.3,0.4)	(0.6,0.4,0.5)	(0.9,0.1,0.2)

Table 5. Normalized Decision Matrix

	C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8
A_1	0.191489	0.216346	0.115385	0.069767	0.041475	0.240143	0.216981	0.25
A_2	0.229787	0.288462	0.254808	0.153488	0.142857	0.075269	0.128931	0.222222
A_3	0.242553	0.211538	0.288462	0.269767	0.207373	0.164875	0.198113	0.099206
A_4	0.234043	0.081731	0.067308	0.195349	0.179724	0.172043	0.106918	0.123016
A_5	0.038298	0.086538	0.115385	0.102326	0.258065	0.132616	0.188679	0.06746
A_6	0.06383	0.115385	0.158654	0.209302	0.170507	0.215054	0.160377	0.238095

Table 6. Computed Weights for Evaluation Criteria

Criteria	**C1**	**C2**	**C3**	**C4**	**C5**	**C6**	**C7**	**C8**
Weights	0.207272	0.145683	0.152411	0.115338	0.129655	0.075817	0.03808	0.135743
Rank	8	6	7	3	4	2	1	5

4.1 Selection and Classification of Sub-Value Sets

To facilitate the evaluation of criteria and sub-criteria, sub-values were categorized into distinct qualitative levels based on their relevance to the assessment of College English blended teaching quality. These sub-values reflect a range of performance measures and are classified as follows:

Teaching Effectiveness: High, Moderate, Low}

Student Engagement: Highly Engaged, Partially Engaged, Not Engaged}

Quality of Digital Resources: Excellent, Adequate, Poor}

Flexibility of Teaching Methods: Very Flexible, Moderately Flexible, Inflexible}

Assessment Design and Feedback: Comprehensive, Sufficient, Inadequate}

Technical Support and Infrastructure: Excellent, Satisfactory, Unsatisfactory}

Teacher Competence in Digital Teaching: {Expert, Competent, Beginner}

Student Satisfaction: Very Satisfied, Neutral, Dissatisfied}

Based on these sub-values, three representative sets were selected for further evaluation:

Set #1: High, Highly Engaged, Excellent, Very Flexible, Comprehensive, Excellent, Expert, and Very Satisfied.

Set #2: Moderate, Highly Engaged, Excellent, Very Flexible, Comprehensive, Excellent, Expert, and Very Satisfied.

Set #3: Low, Highly Engaged, Excellent, Very Flexible, Comprehensive, Excellent, Expert, and Very Satisfied.

These sets serve as benchmarks for analyzing variations in teaching quality and identifying patterns that influence performance outcomes. By categorizing sub-values into distinct levels and sets, the study ensures a structured approach to evaluating the effectiveness of blended teaching methodologies (See Table 7).

Table 7. Sub-Values and Selected Sets for Evaluation

Criteria	Sub-Values	Set 1	Set 2	Set 3
Teaching Effectiveness	{High, Moderate, Low}	High	Moderate	Low
Student Engagement	{Highly Engaged, Partially Engaged, Not Engaged}	Highly Engaged	Highly Engaged	Highly Engaged
Quality of Digital Resources	{Excellent, Adequate, Poor}	Excellent	Excellent	Excellent
Flexibility of Teaching Methods	{Very Flexible, Moderately Flexible, Inflexible}	Very Flexible	Very Flexible	Very Flexible
Assessment Design and Feedback	{Comprehensive, Sufficient, Inadequate}	Comprehensive	Comprehensive	Comprehensive
Technical Support and Infrastructure	{Excellent, Satisfactory, Unsatisfactory}	Excellent	Excellent	Excellent
Teacher Competence in Digital Teaching	{Expert, Competent, Beginner}	Expert	Expert	Expert
Student Satisfaction	{Very Satisfied, Neutral, Dissatisfied}	Very Satisfied	Very Satisfied	Very Satisfied

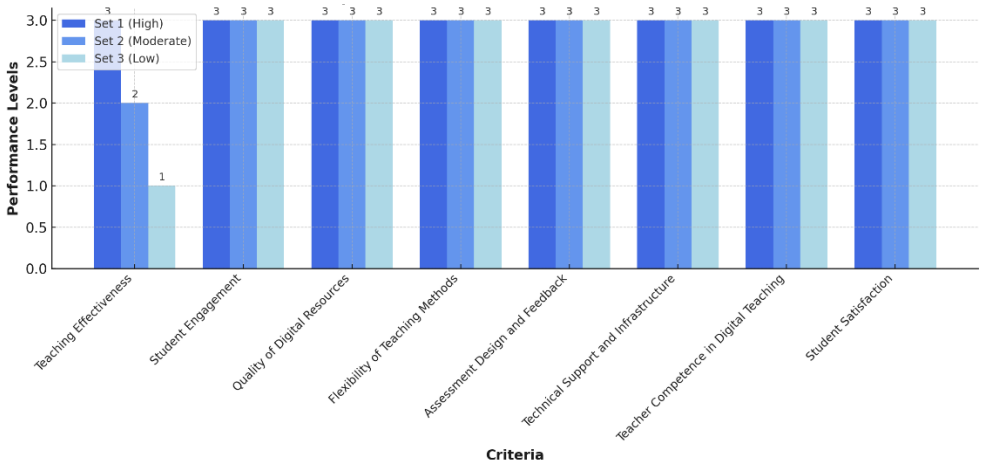

Figure 2. Comparison of Performance Levels Across Sets

Figure 7 provides a comparison of performance levels across the three evaluation sets: Set 1, Set 2, and Set 3, for each criterion. It highlights the variations in performance based on different sub-value configurations. While most criteria show consistent performance across the sets, notable differences are observed in specific areas, such as Teaching Effectiveness, where Set 1 demonstrates the highest performance level. This representation emphasizes the importance of analyzing individual criteria to identify areas of strength and opportunities for improvement within each set.

5. Computation of Criteria Weights for Sets 1, 2, and 3

In this section, the computation of criteria weights for Sets 1, 2, and 3 is detailed using a systematic and consistent methodology. The process involves normalizing the decision matrix, calculating entropy values to measure data variability, determining the degree of diversification for each criterion, and finally computing objective weights that represent the relative importance of each criterion. The results for each set are summarized in corresponding tables, providing a clear and structured framework for analysis.

5.1 Criteria Weights for Set 1

For Set 1, the criteria weights were calculated step-by-step to ensure accuracy and transparency. The decision matrix was normalized using Equation (2), and the normalized values are presented in Table 8. Entropy values were then computed using Equation (3) to quantify the variability and information content of each criterion. Using the entropy values, the degree of diversification was calculated with Equation (4) to identify the significance of each criterion in the evaluation process. Finally, the objective weights were derived using Equation (5), and the results are displayed in Table 9. These weights provide a clear understanding of the criteria's relative importance in Set 1. Figure 3 compares the weights of criteria across Sets 1, 2, and 3.

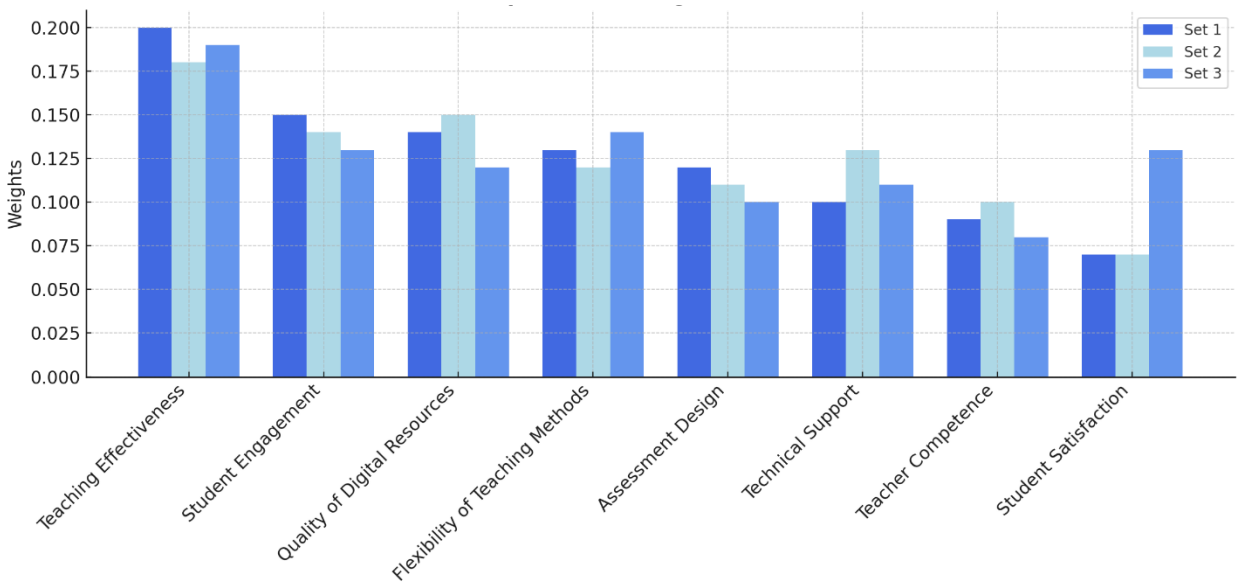

Figure 3. The weights of criteria across Sets 1, 2, and 3

Table 8. Normalized DM.

	C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8
A_1	0.158076	0.216346	0.115385	0.069767	0.041475	0.240143	0.216981	0.25
A_2	0.219931	0.288462	0.254808	0.153488	0.142857	0.075269	0.128931	0.222222
A_3	0.185567	0.211538	0.288462	0.269767	0.207373	0.164875	0.198113	0.099206
A_4	0.219931	0.081731	0.067308	0.195349	0.179724	0.172043	0.106918	0.123016
A_5	0.178694	0.086538	0.115385	0.102326	0.258065	0.132616	0.188679	0.06746
A_6	0.037801	0.115385	0.158654	0.209302	0.170507	0.215054	0.160377	0.238095

Table 9. Criteria weights

Criteria	**C1**	**C2**	**C3**	**C4**	**C5**	**C6**	**C7**	**C8**
Weights	0.132992	0.159334	0.166693	0.126146	0.141804	0.082921	0.041648	0.148463
Rank	4	7	8	3	5	2	1	6

5.2 Criteria Weights for Set 2

The same methodology was applied to Set 2 to maintain consistency in the evaluation process. The decision matrix was normalized using Equation (2), and the results are shown in Table 10. Entropy values were calculated using Equation (3) to capture the variability among the criteria. Based on these entropy values, the degree of diversification was computed using Equation (4), highlighting the influence of each criterion. Lastly, the objective weights were determined through Equation (5), and the results are presented in Table 11. These calculations emphasize the contributions of each criterion within the context of Set 2. Figure 4 illustrates the relationships between different criteria. Darker shades of blue indicate stronger correlations, while lighter shades suggest weaker relationships, offering insights into how criteria interact with one another.

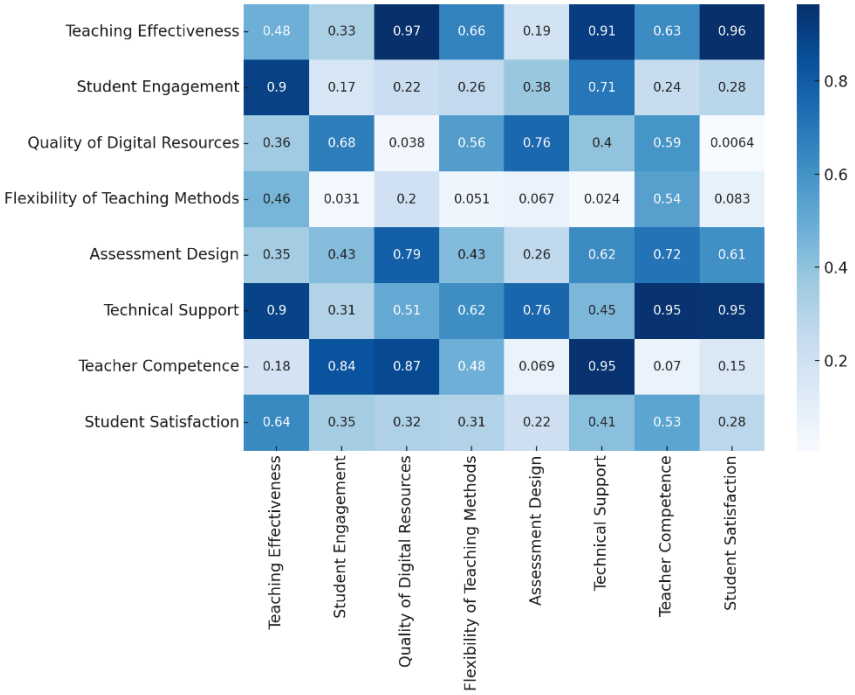

Figure 4. The relationships between different criteria

Table 10. Normalized DM.

	C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8
A_1	0.2103	0.216346	0.115385	0.069767	0.041475	0.240143	0.216981	0.25
A_2	0.244635	0.288462	0.254808	0.153488	0.142857	0.075269	0.128931	0.222222
A_3	0.06867	0.211538	0.288462	0.269767	0.207373	0.164875	0.198113	0.099206
A_4	0.2103	0.081731	0.067308	0.195349	0.179724	0.172043	0.106918	0.123016
A_5	0.2103	0.086538	0.115385	0.102326	0.258065	0.132616	0.188679	0.06746
A_6	0.055794	0.115385	0.158654	0.209302	0.170507	0.215054	0.160377	0.238095

Table 11. Criteria weights

Criteria	**C1**	**C2**	**C3**	**C4**	**C5**	**C6**	**C7**	**C8**
Weights	0.168333	0.152839	0.159898	0.121004	0.136024	0.079541	0.03995	0.142411
Rank	8	6	7	3	4	2	1	5

5.3 Criteria Weights for Set 3

For Set 3, the criteria weights were computed following the same steps as the previous sets. The decision matrix was normalized using Equation (2), with the normalized values detailed in Table 12. Entropy values were then calculated using Equation (3) to measure the information content of each criterion. From these entropy values, the degree of diversification was derived using Equation (4) to assess the impact of each criterion. Finally, the objective weights were computed using Equation (5), and the results are summarized in Table 13. These weights provide insights into the relative importance of criteria in Set 3. Table 14 presents the weights for each criterion across the three sets in a clear and organized manner.

Table 12. Normalized DM.

	C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8
A_1	0.113636	0.216346	0.115385	0.069767	0.041475	0.240143	0.216981	0.25
A_2	0.181818	0.288462	0.254808	0.153488	0.142857	0.075269	0.128931	0.222222
A_3	0.204545	0.211538	0.288462	0.269767	0.207373	0.164875	0.198113	0.099206
A_4	0.113636	0.081731	0.067308	0.195349	0.179724	0.172043	0.106918	0.123016
A_5	0.181818	0.086538	0.115385	0.102326	0.258065	0.132616	0.188679	0.06746
A_6	0.204545	0.115385	0.158654	0.209302	0.170507	0.215054	0.160377	0.238095

Table 13. Criteria weights

Criteria	**C1**	**C2**	**C3**	**C4**	**C5**	**C6**	**C7**	**C8**
Weights	0.046172	0.175289	0.183385	0.138778	0.156004	0.091225	0.045818	0.163329
Rank	2	7	8	4	5	3	1	6

Table 14. The weights for each criterion across the three sets

Criteria	**Set 1**	**Set 2**	**Set 3**
Teaching Effectiveness	0.20	0.18	0.19
Student Engagement	0.15	0.14	0.13
Quality of Digital Resources	0.14	0.15	0.12
Flexibility of Teaching Methods	0.13	0.12	0.14
Assessment Design	0.12	0.11	0.10
Technical Support	0.10	0.13	0.11
Teacher Competence	0.09	0.10	0.08
Student Satisfaction	0.07	0.07	0.13

5. Managerial implications

The evaluation of College English blended teaching quality has several managerial implications. First, it aids in curriculum development by helping designers effectively combine digital and traditional teaching methods while aligning with modern competencies like digital literacy. Second, it highlights resource allocation gaps, encouraging investment in digital tools, learning platforms, and teaching aids. Third, it emphasizes the importance of teacher training, suggesting programs to enhance educators' digital skills and adaptability to blended teaching. Additionally, understanding student engagement and satisfaction enables the creation of tailored, student-focused learning experiences. Finally, periodic evaluations foster continuous improvement by providing actionable insights for refining teaching practices and policies.

6.Conclusion and Future Directions

This study introduced a framework that combines Single-Valued Neutrosophic Sets (SVNS) with an entropy-based MCDM approach to effectively evaluate teaching quality in blended learning environments. By integrating the SuperHyperSoft (HSS) framework, the method addresses uncertainty and captures relationships between criteria. The results highlighted Teaching Effectiveness as the most important factor, showing the framework's ability to identify and prioritize key aspects of teaching quality.

Looking to the future, this framework can be further developed and applied in new ways. Combining the HSS framework with other decision-making methods like TOPSIS or AHP can make it more versatile for solving a wider range of problems. Advanced versions, such as Dynamic SuperHyperSoft Sets, could help analyze situations where data changes over time or in fast-paced environments. Beyond education, this

approach could be used in fields like healthcare, logistics, and urban planning to address complex challenges with uncertain and interconnected data. Additionally, integrating artificial intelligence and machine learning into the framework could make it more efficient and scalable, especially when dealing with large datasets. These improvements will ensure the framework remains a practical and adaptable tool for future decision-making needs.

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A Multi-Criteria Decision-Making Approach with Hyper Soft Set to Teaching Quality Assessment in English Language and Literature

Yuhua He*
English Department, Anhui Wenda College of Information Engineering, Hefei, 231201, Anhui, China
*Corresponding author E-mail: Selena1120@163.com

Abstract: The training ground for professional English speakers is the English departments of colleges and universities. Economic development and international interactions are impacted by the quality of their instruction. This study created an evaluation method with for evaluating the quality of English language and literature using questionnaire surveys, interviews, and a review of the literature. The multi-criteria decision making (MCDM) method is used in this study to evaluate the English langue and literature. Neutrosophic set is integrated with the MCDM method to deal with uncertainty and value information. This study uses the CRITIC method to compute the criteria weights. The CRITIC method is used under the neutrosophic sets and hyper soft set (HSS) to deal with various criteria. This study uses six main criteria and 23 sub criteria to be evaluated and select the highest importance criterion in decision making problem. Three experts are evaluated the criteria. The results show the Pedagogical Competence of Educators of English Language has the highest rate.

Keywords: Hyper Soft Set; Single Valued Neutrosophic Sets; Teaching Quality Assessment in English Language and Literature; MCDM Method.

1. Introduction and Literature Review

The status of English instruction in higher education has increased as a result of China's implementation of the reform and opening-up policy in the context of global economic integration and Chinese citizens' increased emphasis on learning English as a lingua franca. To prepare China for future international competition, Chinese colleges and universities have established English departments one after another[1], [2].

These departments have produced several English major graduates over the years, but the caliber of these graduates is not up to par. Thus, evaluating the college English department's teaching quality has emerged as a crucial and pressing responsibility for English teachers to raise the caliber of English instruction and the overall caliber of English graduates[3], [4].

There have been three phases in the evolution of educational evaluation in western nations. The first stage, known as the "germination period," lasted from the middle of the nineteenth century to the 1930s[5], [6]. During this time, there was no standardized and objective assessment system in place, and students' abilities and the quality of instruction were assessed based on subjective impressions and personal experience. The second stage, known as the formation phase, spanned the 1930s through the 1950s. During this time, education assessment was more thorough, and additional evaluation techniques, such as the well-known "Tyler Evaluation Model," were established[7], [8].

The third stage, known as the "heyday stage," spans the 1950s to the present and is characterized by a high regard for education assessment, a wealth of research on the subject, and a far greater variety and breadth of modes, objects, methods, and forms of education assessment[9], [10]. Evaluation of English language is a MCDM methodology.

The discovery of single-valued neutrosophic sets (SVNSs) theory inspired us to investigate the potential use in English language evaluation because of imprecise knowledge, the hazy human mind, and inconsistent information. This is the first research to provide a framework for evaluating multi-criteria English language evaluation using the CRITIC technique in the context of SVNS. The CRITIC approach's goal attribute weights are more logical for the MCDM models to achieve this[11], [12].

In several fields, the theory of fuzzy sets (FSs) has been widely used to address the resulting uncertainty. Many FS generalizations have been proposed and used to various decision-making challenges over the years. However, the concepts of FSs and their expansions are only able to handle imprecise and partial data; they are unable to address the inconsistent and ambiguous data that arises in actual MCDM concerns. Smarandache created the neutrosophic set (NS) doctrine in order to get around this problem. It is divided into three independent categories: truth-belongingness grade (TG), indeterminacy-belongingness grade (IG), and falsity-belongingness grade (FG)[13], [14].

It is difficult to apply the NSs in scientific and engineering fields because of their intricate interpretation. Wang et al. developed the idea of SVNSs, whose belongingness grades fall between [0,1], to get around the issue. One specific instance of NSs is the SVNSs[15], [16]. Assessing the criteria weights is a crucial task for DEs in the MCDM process. Many academics have proposed various methods for determining the weight of the criteria.

To determine how many criteria or factors contribute to a given goal, the objective weighting approach is applied. This strategy removes the need for subjective assessments by allocating weights based on thorough data analysis and statistical methodologies. For example, in the field of cluster analysis, the objective weighting approach is used to determine the significance of each attribute in the clustering process. Instead than relying on subjective human evaluations, this method is based on mathematical algorithms and the structural characteristics of the data. Consequently, it guarantees that the clustering results are more accurate and objective, successfully reducing any biases brought about by subjective considerations.

The objective weighing method is widely used in many different fields, using scientific data analysis techniques to assess the importance of each variable or indication and give decision-makers more accurate and trustworthy information. The entropy weight method, standard deviation method, and CRITIC approach are currently widely used objective weighting techniques. In different situations, each approach exhibits distinct performance advantages and disadvantages[17].

There are two types of criteria for weight determination processes: objective and subjective weights. The CRITIC tool was developed in 1995 by Diakoulaki et al. to determine the objective weight of criteria[18]. The standard deviation (SD), which is a measure of the criteria's contrast intensity, is used to assess the objective weight of the criteria to use the CRITIC process.

On the other hand, the correlation coefficient is used to compute conflicts between the criteria. In several real-world MCDM issues, the CRITIC tool has been effectively utilized to calculate the objective weights of criteria[19], [20].

Hypersoft (HSS) subset, HS complement, not HSS set, absolute set, union, intersection, AND, OR, restricted union, extended intersection, relevant complement, restricted difference, restricted symmetric difference, HSS set relation, sub relation, complement relation, HSS representation in matrix form, various operations

on matrices, and the application of similarity measure technique for medical diagnosis purposes in a neutrophilic environment are just a few of the fundamental concepts that Saeed et al. explained[21]. After characterizing mapping in a hypersoft set setting, Saeed et al. went on to examine some of its key characteristics, such as HS images and HS inverse images[22].

The topic of hypersoft points in various fuzzy-like settings was covered by Mujahid et al.[23] Rahman et al. [24] introduced complex HSS in 2020 and created the hybrids of the HS set using a complex neutrosophic set and a complex fuzzy set, respectively, with a complex intuitionistic fuzzy set. Along with their theoretical operations (complement, union, intersection, etc.), they also covered their foundations, which include subsets, equal sets, null sets, absolute sets, etc. Convexity cum concavity on HSS was conceptualized in 2020, and its pictorial representations with illustrative instances were provided[25].

Illustrative Example

Let $A = (p_1, p_2, \dots, p_{10})$ be a set of mobile phones and a person wants to buy a mobile. But the mobile has different alternatives.

The attributes of the mobile are:

$a_1 = Back\ Camera$

$a_2 = RAM$

$a_3 = Storage$

$a_4 = Font\ Camera$

The attributes values are:

$A_1 = \{16MP, 32MP, 48MP\}$

$A_2 = \{3GB, 4GB, 8GB\}$

$A_3 = \{16GB, 32GB, 64GB\}$

$A_4 = \{2MP, 4MP, 8MP\}$

The HSS is:

$Let\ C = A_1 \times A_2 \times A_3 \times A_4$Then the HSS is a function that can be obtained such as:

$f\{48MP, 8GB, 32GB, 8MP\}$

The study's primary contributions are described as follows:

a) A unified HSS-SVN-CRITIC framework for addressing MCDM difficulties is offered.
b) The attribute weights are calculated using the CRITICAL structure.
c) To demonstrate the stability and permanence of the created technique, a real-world case study of English language evaluation is provided inside the SVNS environment.

The following is a summary of the remainder of the manuscript: Section 2 shows the steps of the HSS-SVNS-CRITIC method. Section 3 shows the case study results. Section 4 shows the conclusions.

2. Concepts of SVNSs

The neutrosophic set has three membership functions such as Truth, indeterminacy, and falsity[26], [27].

$y = \left(x, T_y(x), I_y(x), F_y(x), x \in X\right)$

$-0 \leq \sup T_y(x) + \sup I_y(x) + \sup F_y(x) \leq 3 +$

Let two SVNNs such as: $x = (a_1, b_1, c_1)\ and\ y = (a_2, b_2, c_2)$

$x \oplus y = (a_1 + a_2 - a_1 a_2, b_1 b_2, c_1 c\_2)$

$x \otimes y = (a_1 a_2, b_1 + b_2 - b_1 b_2, c_1 + c_2 - c_1 c_2)$

$\curlywedge x = (1 - (1 - a_1)^{\curlywedge}, b_1^{\curlywedge}, c_1^{\curlywedge})$

$x^{\curlywedge} = (a_1^{\curlywedge}, 1 - (1 - b_1)^{\curlywedge}, 1 - (1 - c_1)^{\curlywedge}), \curlywedge > 0$

We can obtain the score function as:

$f(x) = \frac{2+T(x)-I(x)-F(x)}{3}$

2. Proposed Model: Soft MCDM Models with HSS

In this part, the steps of the proposed method for evaluation criteria and obtain the weights of criteria ae presented. The proposed method uses the CRITIC method to compute the criteria weights. The proposed method is implemented using the single valued neutrosophic sets (SVNSs)[28], [29]. Figure 1 shows the steps of this study.

The hyper soft set (HSS) definition can be defined as:

Let y be a universe of discourse, (y) the power set of y, and A set of criteria. Then the pair $(F, y), F: A \to (y)$ called a soft set over y

The HSS can be defined as[30], [31]:

Let y be a universe of discourse, (y) the power set of y, and $a_1, a_2, \ldots, a_n; n \geq 1$ set of criteria. Then the pair, $F: A_1 \times A_2 \times \ldots A_n \to (y)$ called a hyper soft set over y.

- Determine the main criteria, sub criteria to be evaluated in this study

- Construct a panel of experts to evaluate the criteria.

- Build the decision matrix between criteria and alternatives.

- Use SVNNs to evaluate the criteria and alternatives. Then use the score function to obtain crisp value.

- Combine the values of decision matrix.

- Then we apply the steps of the CRITIC method to obtain the criteria weights.

- Normalize the decision matrix for the beneficial and non-beneficial factors

$$T_{ij} = \frac{x_{ij} - \min x_{ij}}{\max x_{ij} - \min x_{ij}} \quad (1)$$

$$T_{ij} = \frac{\max x_{ij} - x_{ij}}{\max x_{ij} - \min x_{ij}} \quad (2)$$

- Compute the standard deviation

$$d_j = \sqrt{\frac{\sum_{i=1}^{m}\left(T_{ij}-T_j^-\right)}{m}} \quad (3)$$

$$T_j^- = \frac{1}{m}\sum_{i=1}^{m} T_{ij} \quad (4)$$

- Determine the correlation matrix between the criteria

$$p_{jk} = \frac{\sum_{i=1}^{m}\left[T_{ij}-T_j^-\right]\left[T_{ik}-T_k^-\right]}{\sqrt{\sum_{i=1}^{m}\left[\left(T_{ij}-T_j^-\right)^2\right]\sum_{i=1}^{m}\left(T_{ik}-T_k^-\right)^2}} \quad (5)$$

- Compute the information measure

$$Q_i = d_j \sum_{i=1}^{m}\left(1-p_{jk}\right) \quad (6)$$

- Compute the criteria weights.

$$w_j = \frac{Q_i}{\sum_{j=1}^{n} Q_i} \quad (7)$$

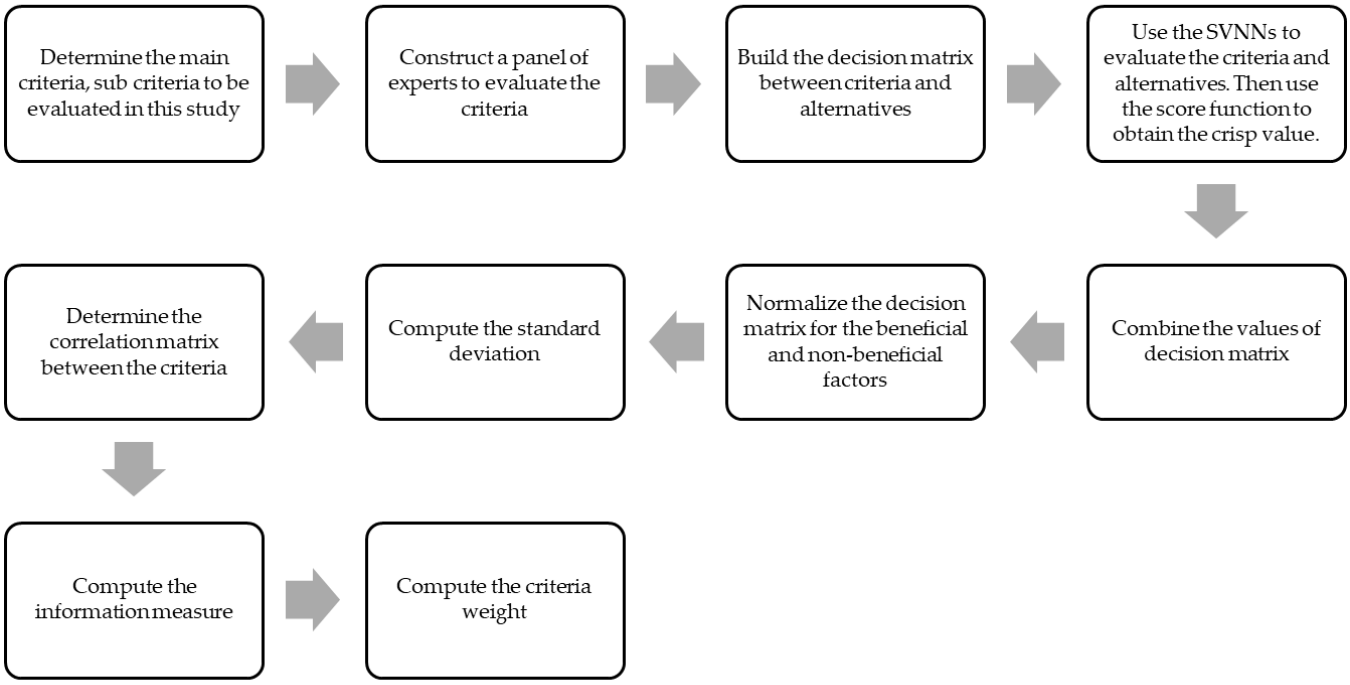
Determine the main criteria, sub criteria to be evaluated in this study
Construct a panel of experts to evaluate the criteria
Build the decision matrix between criteria and alternatives
Use the SVNNs to evaluate the criteria and alternatives. Then use the score function to obtain the crisp value.
Combine the values of decision matrix
Normalize the decision matrix for the beneficial and non-beneficial factors
Compute the standard deviation
Determine the correlation matrix between the criteria
Compute the information measure
Compute the criteria weight

Figure 1. The steps of SVN-HSS-CRITIC Method.

3. Case Study

This section presents the outcomes of the CRITIC methodology to compute the criteria weights.

3.1 Comprehensive overview

This section shows the methodology used in this study. Three experts are evaluated the main and sub criteria. Six main criteria and 23 sub criteria are used in this study. Table 1 shows the main and sub criteria of this study.

Table 1. The criteria and values.

	Main criteria	Sub criteria
C_1	Assessment Effectiveness of English Language	Highly Effective Effective Somewhat Effective Ineffective
C_2	Student Engagement of English Language	High Medium Low
C_3	Curriculum Relevance of English Language	Very Relevant Relevant Slightly Relevant Irrelevant
C_4	Technological Integration of English Language	Excellent Good Fair Poor
C_5	Cultural and Inclusivity Factors of English Language	Highly Inclusive Inclusive Slightly Inclusive Non-Inclusive
C_6	Pedagogical Competence of Educators of English Language	Strong Moderate Weak Very Weak

The criteria of this study are: Pedagogical Competence of Educators

Description: Measures the teacher's ability to design, deliver, and assess learning effectively in English language and literature.

Values:

- Strong
- Moderate
- Weak
- Very Weak

Student Engagement

Description: Assesses the extent to which students actively participate in classes, discussions, and assignments.

Values:

- High

- Medium
- Low

Curriculum Relevance

Evaluates how well the syllabus aligns with modern linguistic and literary trends, as well as job market requirements.

Values:

- Very Relevant
- Relevant
- Slightly Relevant
- Irrelevant

Assessment Effectiveness

Examines the appropriateness and fairness of evaluation methods used, including written exams, oral presentations, and projects.

Values:

- Highly Effective
- Effective
- Somewhat Effective
- Ineffective

Technological Integration

Assesses the use of digital tools, such as online resources, multimedia, and learning management systems, in teaching.

Values:

- Excellent
- Good
- Fair
- Poor

Cultural and Inclusivity Factors

Measures how well the teaching approach accommodates diverse cultural backgrounds and learning needs.

Values

- Highly Inclusive
- Inclusive
- Slightly Inclusive
- Non-Inclusive

let $C = C_1 \times C_2 \times C_3 \times C_4 \times C_5 \times C_6$ and the criteria values are $(A_1, A_2, \dots, A_{23})$. We select the six attributes as $A = A_1 \times A_5 \times A_8 \times A_{12} \times A_{16} \times A_{20}$.

Three experts have evaluated the criteria as shown in Table 2. Then we obtain the crisp values and combine the decision matrix.

Table 2. The decision matrix.

	A_1	A_5	A_8	A_{12}	A_{16}	A_{20}
Alt_1	(0.9,0.1,0.2)	(0.8,0.2,0.3)	(0.7,0.3,0.4)	(0.6,0.4,0.5)	(0.5,0.5,0.5)	(0.4,0.6,0.7)
Alt_2	(0.1,0.9,0.9)	(0.9,0.1,0.2)	(0.8,0.2,0.3)	(0.7,0.3,0.4)	(0.4,0.6,0.7)	(0.3,0.7,0.8)
Alt_3	(0.2,0.8,0.9)	(0.7,0.3,0.4)	(0.9,0.1,0.2)	(0.6,0.4,0.5)	(0.3,0.7,0.8)	(0.2,0.8,0.9)
Alt_4	(0.3,0.7,0.8)	(0.1,0.9,0.9)	(0.8,0.2,0.3)	(0.5,0.5,0.5)	(0.2,0.8,0.9)	(0.1,0.9,0.9)
Alt_5	(0.4,0.6,0.7)	(0.2,0.8,0.9)	(0.3,0.7,0.8)	(0.4,0.6,0.7)	(0.1,0.9,0.9)	(0.9,0.1,0.2)
Alt_6	(0.5,0.5,0.5)	(0.6,0.4,0.5)	(0.7,0.3,0.4)	(0.8,0.2,0.3)	(0.9,0.1,0.2)	(0.8,0.2,0.3)
	A_1	A_5	A_8	A_{12}	A_{16}	A_{20}
Alt_1	(0.5,0.5,0.5)	(0.8,0.2,0.3)	(0.7,0.3,0.4)	(0.6,0.4,0.5)	(0.5,0.5,0.5)	(0.4,0.6,0.7)
Alt_2	(0.4,0.6,0.7)	(0.9,0.1,0.2)	(0.8,0.2,0.3)	(0.5,0.5,0.5)	(0.4,0.6,0.7)	(0.3,0.7,0.8)
Alt_3	(0.3,0.7,0.8)	(0.5,0.5,0.5)	(0.5,0.5,0.5)	(0.4,0.6,0.7)	(0.3,0.7,0.8)	(0.2,0.8,0.9)
Alt_4	(0.3,0.7,0.8)	(0.4,0.6,0.7)	(0.4,0.6,0.7)	(0.3,0.7,0.8)	(0.2,0.8,0.9)	(0.1,0.9,0.9)
Alt_5	(0.4,0.6,0.7)	(0.3,0.7,0.8)	(0.3,0.7,0.8)	(0.4,0.6,0.7)	(0.1,0.9,0.9)	(0.9,0.1,0.2)
Alt_6	(0.5,0.5,0.5)	(0.6,0.4,0.5)	(0.7,0.3,0.4)	(0.8,0.2,0.3)	(0.9,0.1,0.2)	(0.8,0.2,0.3)
	A_1	A_5	A_8	A_{12}	A_{16}	A_{20}
Alt_1	(0.7,0.3,0.4)	(0.8,0.2,0.3)	(0.7,0.3,0.4)	(0.7,0.3,0.4)	(0.5,0.5,0.5)	(0.4,0.6,0.7)
Alt_2	(0.1,0.9,0.9)	(0.9,0.1,0.2)	(0.1,0.9,0.9)	(0.1,0.9,0.9)	(0.4,0.6,0.7)	(0.7,0.3,0.4)
Alt_3	(0.2,0.8,0.9)	(0.7,0.3,0.4)	(0.2,0.8,0.9)	(0.2,0.8,0.9)	(0.7,0.3,0.4)	(0.1,0.9,0.9)
Alt_4	(0.3,0.7,0.8)	(0.1,0.9,0.9)	(0.8,0.2,0.3)	(0.5,0.5,0.5)	(0.1,0.9,0.9)	(0.2,0.8,0.9)
Alt_5	(0.4,0.6,0.7)	(0.2,0.8,0.9)	(0.3,0.7,0.8)	(0.4,0.6,0.7)	(0.2,0.8,0.9)	(0.9,0.1,0.2)
Alt_6	(0.5,0.5,0.5)	(0.6,0.4,0.5)	(0.7,0.3,0.4)	(0.8,0.2,0.3)	(0.9,0.1,0.2)	(0.8,0.2,0.3)

- Eq. (1) is used to normalize the decision matrix as shown in table 3.

- Eq. (4) is used to compute the mean value. Then Eq. (3) is used to obtain the standard deviation value.

- Then we determine the correlation matrix between the criteria using Eq. (5) as shown in Table 4 and Figure 2.

- Then we determine the information measure.

- Eq. (7) is used to compute the criteria weights as shown in Table 5 and Figure 3. Figure 4 shows the rank of criteria.

Table 3. Normalized decision matrix.

	A_1	A_5	A_8	A_{12}	A_{16}	A_{20}
Alt_1	1	0.852459	1	0.583333	0.507463	0.328358
Alt_2	0	1	0.694444	0.138889	0.328358	0.373134
Alt_3	0.022727	0.622951	0.611111	1.39E-16	0.373134	0.029851
Alt_4	0.159091	0	0.916667	0.138889	0.029851	0
Alt_5	0.363636	0.016393	0	0	0	1
Alt_6	0.636364	0.557377	1	1	1	0.865672

Table 4. Correlation matrix.

	A_1	A_5	A_8	A_{12}	A_{16}	A_{20}
A_1	1	0.144286	0.349098	0.719378	0.47755	0.364949
A_5	0.144286	1	0.425733	0.295609	0.539045	-0.17064
A_8	0.349098	0.425733	1	0.652608	0.5961	-0.44765
A_{12}	0.719378	0.295609	0.652608	1	0.888719	0.36274
A_{16}	0.47755	0.539045	0.5961	0.888719	1	0.257437
A_{20}	0.364949	-0.17064	-0.44765	0.36274	0.257437	1

Table 5. The criteria weights.

	Weights	Rank
A_1	0.151511	3
A_5	0.206735	5
A_8	0.170967	4
A_{12}	0.109267	2
A_{16}	0.107595	1
A_{20}	0.253924	6

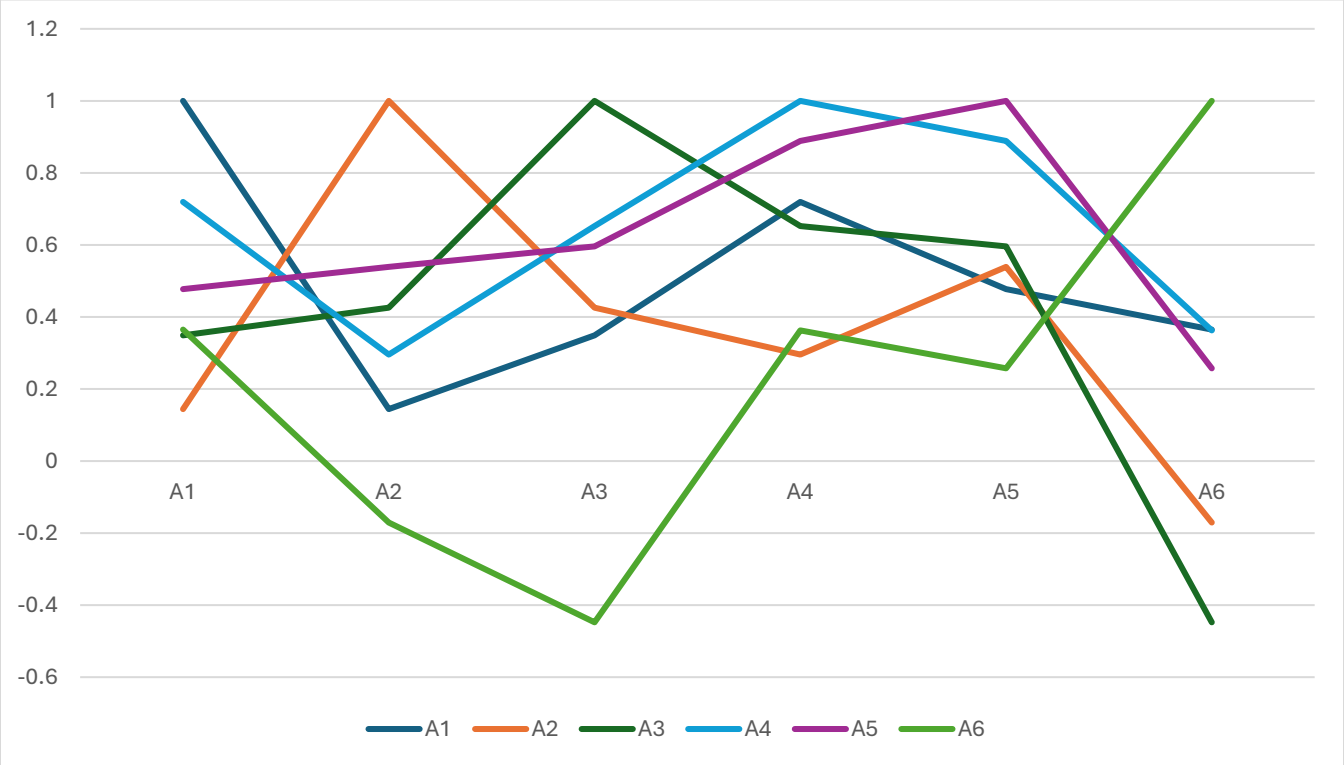

Figure 2. The correlation between criteria.

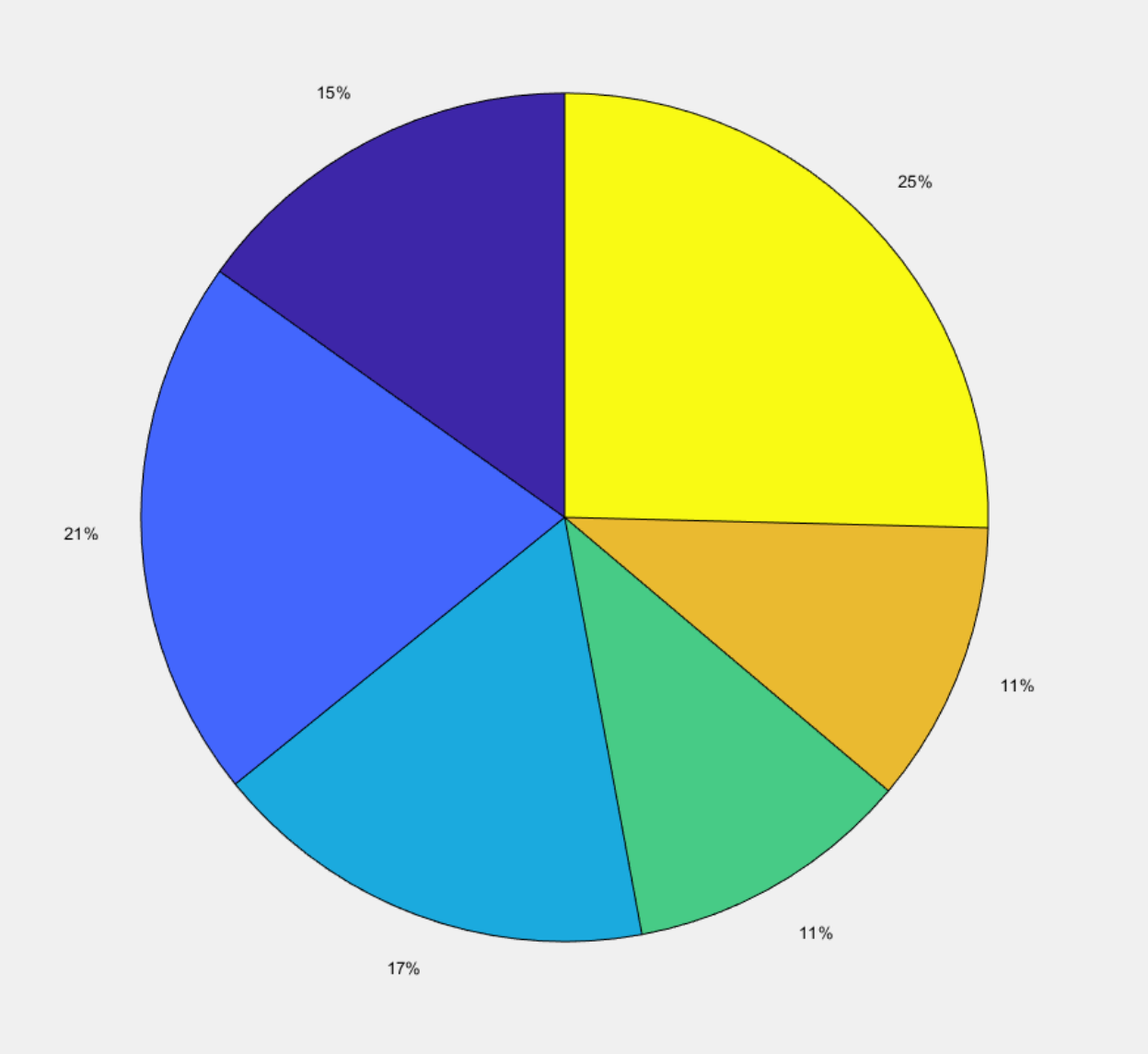

Figure 3. The criteria weights.

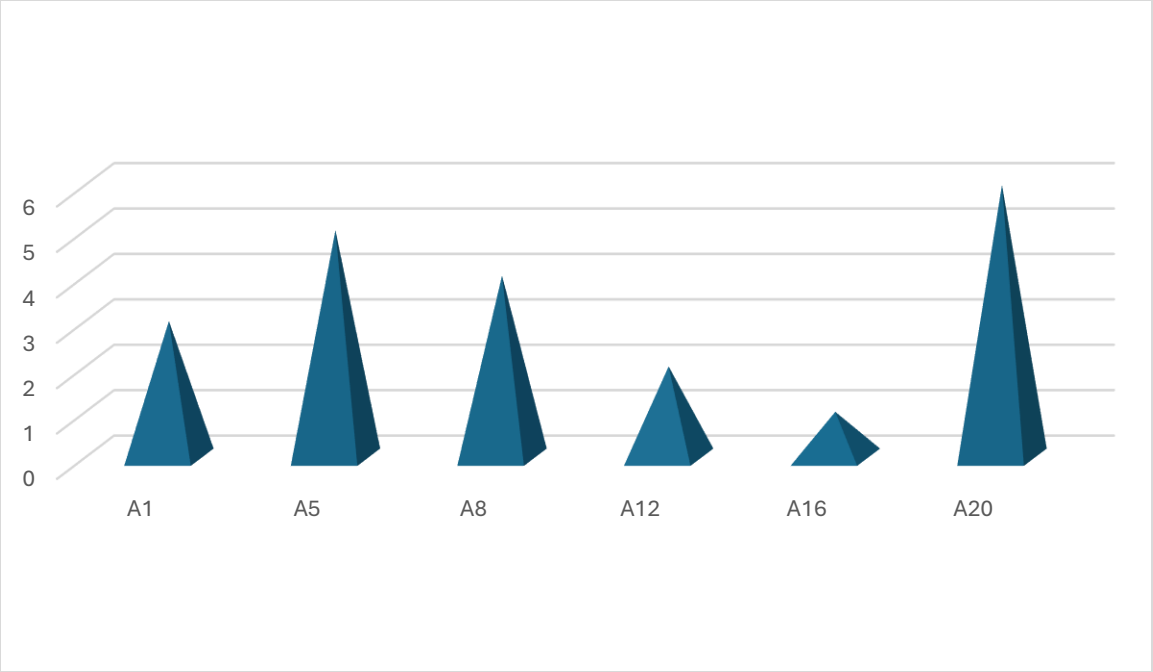

Figure 4. The rank of criteria.

4.1 Comparison Analysis

This section shows the comparative analysis between our models and other MCDM methods such as Entropy, BWM, AHP and SWARA methods. We show a high correlation between our model and other MCDM methods. The comparative analysis shows the effectiveness of the proposed approach. Figure 5 shows the comparative analysis results.

By multiplying the contrast strength inside the criteria by the conflict between them, the CRITIC technique methodically assesses the objective weight between criteria. Normalizing an indicator is crucial before using the CRITIC approach for weight calculation to lessen the impact of its dimensions and value range variations.

When correlations are positive, the Pearson correlation coefficient, which only evaluates linear correlations—is unable to adequately capture the whole independence between criteria. In these situations, the independence of criterion would be better described by substituting a metric that can identify non-linear correlations for the Pearson correlation coefficient. Data redundancy may result from the CRITIC method's assignment of higher weights to criteria with stronger negative correlations. The relative value of other criteria would be diminished if, for instance, two criteria with perfect negative correlation were given disproportionately large weights, so double counting their contributions. An unfair weighted distribution may result from this.

Regardless of whether the correlation is positive or negative, integrating the correlation coefficient's absolute value could help address the issue of significant data redundancy. To more successfully address these problems, it is still advisable to consider an alternate index that can capture nonlinear interactions, while the Pearson correlation coefficient is only able to capture linear relationships.

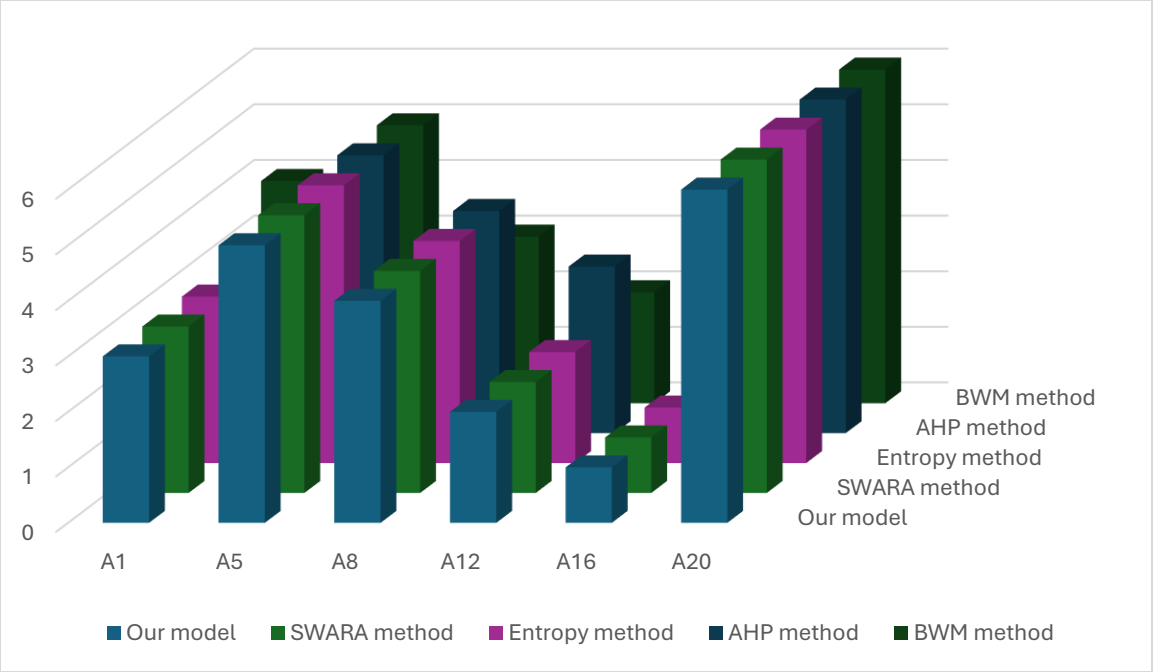

Figure 5. The comparative analysis results.

Despite being widely applied in many different sectors, the CRITIC approach has many drawbacks. Only linear correlation can be measured by the Pearson correlation employed in the CRITIC technique. In several fields, such as computer vision, computational biology, and medical imaging, it becomes necessary to evaluate the actual dependence or independence of complicated objects. Many statistical techniques are available for testing dependence in addition to linear correlation. For example, distance covariance, which calculates the separation between the joint characteristic function of two random vectors and the product of their individual marginal characteristic functions.

The distance coefficient is divided by the product of the corresponding distance variances under the root sign to determine the distance correlation coefficient, which is defined similarly to the Pearson correlation coefficient. In contrast to the Pearson correlation, which only shows the lack of a linear relationship, a distance correlation coefficient of 0 denotes total independence between two random vectors. The Pearson correlation coefficient lacks the ability to capture non-linear correlations, whereas the distance correlation coefficient does.

4. Conclusions and Future Work

Higher education institutions' English departments must provide high-quality instruction to develop English professionals and transfer skilled individuals to the community. This study used the factor analysis approach to examine and investigate the evaluation of the college English department's teaching quality. This study uses the CRITIC method to compute the criteria weights. The CRITIC method is an MCDM method used to deal with various criteria. The CRITIC method is combined with the single valued neutrosophic sets to deal with vague and uncertainty information. We used the hyper soft set approach in this study to select from various values of each criterion. This study invited three experts to evaluate the 6 main criteria and 23 values of each criterion. The results show the Pedagogical Competence of Educators of English Language has the highest importance.

In the future work, the proposed approach can apply different decision-making problems to compute the criteria weights. There are different MCDM approaches that can be applied in this study to compute criteria weights such as Entropy method, SWARA method and others.

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Neutrosophic Non-Newtonian and Geometric Measures: A Consistent Analytical Framework

Amer Darweesh[1,*], Kamel Al-Khaled[1], Mohammed Shakhatreh[1], Mahmood Shareef Ajeel,[2] Nermin Abo Hasan[1]

[1]Department of Mathematics and Stat., Jordan University of Science and Technology, Irbid 22110, Jordan; ahdarweesh@just.edu.jo; kamel@just.edu.jo: mkshakhatreh6@just.edu.jo; mahmoodshareef@shu.edu.iq nmabuhuson18@sci.just.edu.jo

[2]Department of Material Engineering - College of Engineering, Shatrah University, Thi-Qar 64001 - Iraq.

*Correspondence: Amer Darweesh, e-mail: ahdarweesh@just.edu.jo

Abstract. The neutrosophic measure is a generalization of the classical measure in situations when the space contains some indeterminacy. In this paper, we introduce the concept of the Neutrosophic Geometric Measure, we also provide some results, and examples related to the Neutrosophic Geometric Measure. A classical measure of the object's determinate component, a classical measure of its indeterminate part, and a further classical measure of the object's opposite determinate part are the three classical measures that make up the neutrosophic measure. To define the Neutrosophic Geometric Measure, we introduce a new measure on $\mathbb{R}^+$ and call it the geometric Lebesgue measure. The geometric Lebesgue measure is defined, and some of its properties are examined and detailed. Moreover, we establish a relation between the Lebesgue measure and geometric Lebesgue measure to see if the properties of Lebesgue measure are still true in this new measure. Other basic topics discussed in this paper are geometric measurable function and the geometric simple approximation Theorem.

Keywords: Geometric calculus, Geometric measure, Neutrosophic measure, Geometric outer measure, Geometric Lebesgue measure.

1. Introduction

Florentin Smarandache was the first to introduce the notions of neutrosophic measure and neutrosophic integral in his book [1]. When there is some indeterminacy in the space, the neutrosophic measure is a generalization of the classical measure. It should be noted that the neutrosophic measure is actually a triple classical measure: a classical measure of the object's

determinate part, a classical measure of its indeterminate part, and a further classical measure of the object's opposite determinate part. Naturally, the neutrosophic measure is reduced to the classical measure if the opposite object's measure is disregarded and the indeterminate part is absent (its measure is zero). We direct readers to [2]– [5] for additional information on the topic of neutrosophic science and its applications, such as logic, set, measure, integral, probability, etc., and their applications in any discipline. For further studies of these results, these tools can be developed by linking them with other concepts that can be found in the following works [10–13, 24–28]. The Neutrosophic Geometric Outer Measure is highly practical in addressing complex scenarios due to its defined ordering. It plays a significant role in decision-making by providing a structured approach to prioritize options based on their truth, indeterminacy, and falsity components. This makes it invaluable for uncertainty modeling, especially in real-world problems where these three factors coexist and influence outcomes. Moreover, it extends classical analytical methods by embedding meaningful comparisons between neutrosophic values, thereby allowing for a more generalized and nuanced analysis of data and systems.

In this work, we will study a non-Newtonian Lebesgue measure and a Neutrosophic Lebesgue measure, in particular the geometric Lebesgue measure and the Neutrosophic geometric Lebesgue measure. This paper is devoted to the foundations of the study of the geometric Lebesgue measure and the neutrosophic geometric Lebesgue measure. We do the same technique as in Florentin Smarandache's book [1] to introduce the neutrosophic geometric Lebesgue measure. We suggested the definition of the neutrosophic geometric Lebesgue measure by the formula

$$\mu_{NG}\left(\bigcup_{j=1}^{\infty} E_j\right) = \prod_{j=1}^{\infty} \mu_{NG}(E_j),$$

where $\mu_{NG} : A \to [1,\infty] \times [1,\infty] \times [1,\infty]$ is called a neutrosophic geometric measure if it satisfies:

(1) $\mu_{NG}(\phi) = \langle 1, 1, 1\rangle$.

(2) For any sequence of mutually disjoint sets $\{E_j\}_{j=1}^{\infty}$:

$$\mu_{NG}\left(\bigcup_{j=1}^{\infty} E_j\right) = \prod_{j=1}^{\infty} \mu_{NG}(E_j),$$

where the product is computed component-wise as:

$$\prod_{j=1}^{\infty} \mu_{NG}(E_j) = \langle \prod_{j=1}^{\infty} T(\mu_G(E_j)), \prod_{j=1}^{\infty} I(\mu_G(E_j)), \prod_{j=1}^{\infty} F(\mu_G(E_j))\rangle.$$

To do this end, we create a set function, $\mathcal{M}_G^*$, which we call the geometric outer measure. This outer measure is defined on the power set of the geometric space $(0,\infty)$. Then

we introduce the definition of the geometric Lebesgue outer measure, which is given by the formula:

$$\mathcal{M}_G^*(A) = \inf\left\{\prod_{n=1}^{\infty} l_G(I_n) : A \subseteq \bigcup_{n=1}^{\infty} I_n\right\},$$

where all countable families are subjugated by the infimum $\{I_n\}_{i=1}^{\infty}$ of the bounded open intervals in $(0, \infty)$. The geometric Lebesgue outer measure has four important properties: (i) It is defined for all subsets of $\mathbb{R}^+$, (ii) the geometric Lebesgue outer measure is dilation invariant, (iii) the geometric outer measure of an interval subset of $\mathbb{R}^+$ is its geometric length, and (iv) geometric outer measure is countably submultiplicative. However, the geometric outer measure fails to be countably multiplicative.

We create a set of collection known as the geometric Lebesgue measurable sets, which includes all geometric open sets and intervals of $\mathbb{R}$. This collection of sets has the property that the restriction of the geometric outer measure to the collection of geometric Lebesgue measurable sets is countably multiplicative, which helps to improve this fundamental defect. We can say that a set $A \subseteq \mathbb{R}^+$ is a geometric Lebesgue measurable set if, for any set $w \subseteq \mathbb{R}^+$, we have:

$$\mathcal{M}_G^*(w) = \mathcal{M}_G^*(w \cap A) \cdot \mathcal{M}_G^*(w - A).$$

The class of geometric measurable functions will play a critical role in the theory of geometric Lebesgue integration, so we defined geometric measurable functions. The definition is an analogue to the definition in the classical case, and we explore the properties of geometric measurable functions. In addition to all continuous functions, all geometric continuous functions on a geometric measurable domain are geometric measurable. Geometric measurable functions can be expressed as geometric linear combinations. Geometric measurable functions are the geometric pointwise limit of a sequence of geometric measurable functions. Lastly, we proved certain results about the geometric simple function approximation of geometric measurable functions. If a function φ can be expressed as follows, it is considered geometric simple:

$$\varphi \;=\; \prod_{k=1}^{n} (\chi_{E_k})^{\ln a_k},$$

where $E = \bigcup_{k=1}^{n} E_k$ is a disjoint sequence of sets, and $E_k = \varphi^{-1}(a_k)$. This particular expression of φ as a geometric linear combination of geometric characteristic functions is called geometric canonical form.

2. Geometric Calculus

Through the transformation $\alpha(x) = e^x$, we define a geometric arithmetic that transfers conventional arithmetic operations onto the positive real numbers, $\mathbb{R}^+$, while preserving the field structure isometrically. The following geometric arithmetic operations are defined:

(i) **Geometric Addition:** $x \oplus y = x \cdot y$, which corresponds to multiplying the values x and y.

(ii) **Geometric Subtraction:** $x \ominus y = \frac{x}{y}$, representing division of x by y.

(iii) **Geometric Multiplication:** $x \circledast y = x^{\log y} = y^{\log x}$, where each value is raised to the power of the logarithm of the other.

(iv) **Geometric Division:** $x \oslash y = x^{\frac{1}{\log y}}$, where $y \neq 1$, representing x raised to the reciprocal of $\log y$, with the condition $y \neq 1$.

2.1. *Definition: Geometric Absolute Value*

For any $x \in \mathbb{R}^+$, the geometric absolute value, denoted by $|x|^G$, is given by:

$$|x|^G = \begin{cases} x & \text{if } x \geq 1, \\ \frac{1}{x} & \text{if } 0 \leq x \leq 1. \end{cases}$$

This ensures that the geometric absolute value maintains a positive form for all values of x, similar to the traditional absolute value function.

2.2. *Geometric Boundedness*

In a geometric metric space $X = (X, d^G)$, a sequence (x_n) is geometrically bounded if, for each $n \in \mathbb{N}$, there exists a constant $M \geq 1$ such that:

$$|x_n|^G \leq M.$$

This condition guarantees that the sequence remains within a bounded region in the geometric metric space. See [18].

3. Basic Properties of G-Calculus

3.1. *G-Continuity*

In [8], the author defined G-continuity for a function $f(x)$ as follows:

Definition 3.1. Let f be a positive real-valued function defined on an open positive set containing a. We say that f is continuous at $x = a$ if, for any $\epsilon_1 > 0$, there exists a $\delta_1 > 0$ such that if $|x - a| < \delta_1$, then $|f(x) - f(a)| < \epsilon_1$.

Remark 3.2. Let f be a positive real-valued function defined on an open positive set containing a. We say that f is geometrically continuous at $x = a$ if, for any $\epsilon_1 > 0$, there exists a $\delta_1 > 0$ such that if $|\log(x) - \log(a)| < \delta_1$, then $|\log(f(x)) - \log(f(a))| < \epsilon_1$.

Theorem 3.3. *Let f be a positive real-valued function defined on an open positive set containing a. If f is geometrically continuous at $x = a$, then it is continuous at $x = a$.*

Proof. We know that the function $\log(x)$ is continuous at $x = a$. Hence, for any $\epsilon_0 > 0$, there exists a $\delta_2 > 0$ such that if $|x - a| < \delta_2$, then $|\log(x) - \log(a)| < \epsilon_0$. Also, since f is geometrically continuous, for any $\epsilon > 0$, there exists a $\delta_1 > 0$ such that if $|\log(x) - \log(a)| < \delta_1$, then $|\log(f(x)) - \log(f(a))| < \epsilon$.

Now, choose $\delta = \delta_2$ and $\epsilon_0 = \delta_1$. If $|x - a| < \delta_2$, then $|\log(x) - \log(a)| < \epsilon_0 = \delta_1$. Consequently, $|\log(f(x)) - \log(f(a))| < \epsilon_2$. Knowing that the exponential function e^t is continuous at t_0, for any $\epsilon > 0$, there exists a δ_3 such that if $|t - t_0| < \delta_3$, then $\left|e^t - e^{t_0}\right| < \epsilon$. Now, choose $\epsilon_2 = \delta_3$. Therefore, if $|\log(f(x)) - \log(f(a))| < \epsilon_2 = \delta_3$, then $\left|e^{\log(f(x))} - e^{\log(f(a))}\right| < \epsilon$, which implies that $|f(x) - f(a)| < \epsilon$. □

3.2. *G-Integration*

It has been established that for every theorem in classical calculus, there exists an analogous result in geometric calculus. As a result, there is a corresponding version of the Riemann integral within the framework of multiplicative calculus [7, 21].

Definition 3.4. [21] Let f be a positive and bounded function on $[a, b]$, and let $\Delta = \{t_1, t_2, \ldots, t_{n+1}\}$ be a partition of $[a, b]$. For each $i = 1, 2, \ldots, n$, define:

$$m_i = \inf\{f(x) \,|\, t_{i-1} \le x \le t_i\}, \quad M_i = \sup\{f(x) \,|\, t_{i-1} \le x \le t_i\}.$$

The lower product of f for Δ, denoted by P_Δ^-, is given by:

$$P_\Delta^- = \prod_{i=1}^{n+1} m_i^{\log\left(\frac{t_{i+1}}{t_i}\right)}.$$

Similarly, the upper product of f for Δ, denoted by P_Δ^+, is defined as:

$$P_\Delta^+ = \prod_{i=1}^{n+1} M_i^{\log\left(\frac{t_{i+1}}{t_i}\right)}.$$

Definition 3.5. [21] A positive and bounded function f on $[a, b]$ is said to be geometrically Riemann integrable on $[a, b]$ if:

$$\sup\{P_\Delta^- f\} = \inf\{P_\Delta^+ f\}, \quad \text{over all partitions } \Delta.$$

This is denoted as:

$$\int_a^b f(x)^{dx}.$$

4. Geometric Measure

4.1. *Some Properties of Sets Transformed by Exponential and Logarithm Functions*

Definition 4.1. For any subset A of $(0,\infty)$, we define $\ln(A)$ as: $\ln(A) = \{\ln(a) : a \in A\}$.

Proposition 4.2. *A set A is a subset of $(0,\infty)$ if and only if the set $\ln(A)$ is a subset of $\mathbb{R}$.*

Proposition 4.3. *For any interval $I \subseteq (0,\infty)$, the set $\ln(I)$ is an interval in $\mathbb{R}$.*

Proposition 4.4. *If $\{A_n\}$ is a sequence of mutually disjoint sets in $(0,\infty)$, then $\{\ln(A_n)\}$ is a sequence of mutually disjoint sets in $\mathbb{R}$.*

Proposition 4.5. *For any set A, a subset of $(0,\infty)$, we have $\ln(A^c) = (\ln(A))^c$.*

The proofs of Propositions 4.2, 4.3, 4.4, and 4.5 follow directly from the fact that the logarithmic function $\ln(x)$ is a bijective and increasing function.

Proposition 4.6. *For a sequence $\{A_n\}_{n=1}^{\infty}$ in $(0,\infty)$, the following properties hold:*

$$\ln\left(\bigcup_{n=1}^{\infty} A_n\right) = \bigcup_{n=1}^{\infty} \ln(A_n), \text{ and } \ln\left(\bigcap_{n=1}^{\infty} A_n\right) = \bigcap_{n=1}^{\infty} \ln(A_n).$$

Proof. The first statement follows from the general property of functions: $f\left(\bigcup_{i=1}^{\infty} A_i\right) = \bigcup_{i=1}^{\infty} f(A_i)$. For the second statement, let $y \in \ln\left(\bigcap_{n=1}^{\infty} A_n\right)$, meaning that $y = \ln(x)$ for some $x \in \bigcap_{n=1}^{\infty} A_n$. Therefore, $x \in A_n$ for all $n \in \mathbb{N}$, which implies $y \in \ln(A_n)$ for all $n \in \mathbb{N}$. Hence, $y \in \bigcap_{n=1}^{\infty} \ln(A_n)$.

Conversely, if $y \in \bigcap_{n=1}^{\infty} \ln(A_n)$, then $y \in \ln(A_n)$ for all $n \in \mathbb{N}$, so there exist $x_n \in A_n$ such that $y = \ln(x_n)$ for all $n \in \mathbb{N}$. Since $\ln(x)$ is injective, $x_n = x_m$ for all $n, m \in \mathbb{N}$, implying that $y = \ln(x)$ where $x \in A_n$. Therefore, $y \in \ln\left(\bigcap_{n=1}^{\infty} A_n\right)$. □

Remark 4.7. If $A \subseteq B \subseteq (0,\infty)$, then $\ln(A) \subseteq \ln(B) \subseteq \mathbb{R}$.

Proof. For any $y \in \ln(A)$, there exists $a \in A$ such that $y = \ln(a)$. Thus, $e^y = a$, and since $A \subseteq B$, we have $e^y \in B$. Therefore, $y \in \ln(B)$. □

4.2. *Geometric Measure and Geometric Outer Measure*

Definition 4.8. Let I be an interval subset of $(0,\infty)$. The geometric length of I is defined as:

$$\ell_G(I) = \frac{\text{endpoint}}{\text{initial point}},$$

with the convention that $\frac{x}{0} = \infty$, for $x > 0$.

Definition 4.9. Let A be a σ-algebra on $X \subseteq \mathbb{R}^+$. The set function $\mu_G : A \to [1,\infty]$ is called a geometric measure if it satisfies the following conditions:

(i) $\mu_G(\emptyset) = 1$,

(ii) If $\{E_j\}_{j=1}^{\infty}$ is any sequence of mutually disjoint sets in A, then

$$\mu_G\left(\bigcup_{j=1}^{\infty} E_j\right) = \prod_{j=1}^{\infty} \mu_G(E_j).$$

Definition 4.10. Let $X \subseteq (0,\infty)$. The set function $\mu_G^* : \mathcal{P}(X) \to [1,\infty]$ is called a geometric outer measure on X if it satisfies the following conditions:

(i) $\mu_G^*(\emptyset) = 1$,

(ii) (Monotonicity) $\mu_G^*(A) \leq \mu_G^*(B)$ whenever $A \subseteq B$,

(iii) (Countable submultiplicativity) Let $\{A_i\}_{i=1}^{\infty}$ be a countable collection of sets in $\mathcal{P}(X)$. Then

$$\mu_G^*\left(\bigcup_{i=1}^{\infty} A_i\right) \leq \prod_{i=1}^{\infty} \mu_G^*(A_i).$$

5. Geometric Lebesgue Outer Measure

Definition 5.1. For any set $B \subseteq (0,\infty)$, the geometric Lebesgue outer measure is defined as:

$$\mathcal{M}_G^*(B) = \inf\left\{\prod_{n=1}^{\infty} \ell_G(J_n) : B \subseteq \bigcup_{n=1}^{\infty} J_n\right\},$$

where $\{J_n\}_{n=1}^{\infty}$ is a collection of open bounded intervals in $(0,\infty)$.

Theorem 5.2. *For any set $B \subseteq (0,\infty)$, we have: $\mathcal{M}_G^*(B) = e^{\mathcal{M}^*(\ln(B))}$.*

Proof. We first show that: $e^{\mathcal{M}^*(\ln(B))} \leq \mathcal{M}_G^*(B)$. Let $B \subseteq (0,\infty)$ and consider $C \subseteq \mathbb{R}$. By the definition of the classical Lebesgue outer measure for C, we have:

$$\mathcal{M}^*(C) = \inf\left\{\sum_{n=1}^{\infty} \ell(I_n) : C \subseteq \bigcup_{n=1}^{\infty} I_n\right\},$$

where $\{I_n\}_{n=1}^{\infty}$ is a sequence of open intervals in $\mathbb{R}$.

Similarly, by Definition 5.1, the geometric Lebesgue outer measure of B is given by:

$$\mathcal{M}_G^*(B) = \inf\left\{\prod_{n=1}^{\infty} \ell_G(J_n) : B \subseteq \bigcup_{n=1}^{\infty} J_n\right\},$$

where $\{J_n\}_{n=1}^{\infty}$ are open intervals in $(0,\infty)$.

Now, let $\{J_n\}_{n=1}^{\infty}$ be a sequence of open intervals in $(0,\infty)$ such that $B \subseteq \bigcup_{n=1}^{\infty} J_n$. By Proposition 4.3, we have $K_n = \ln(J_n)$, which are open intervals in $\mathbb{R}$, and $\ln(B) \subseteq \bigcup_{n=1}^{\infty} K_n$. By the countable subadditivity of the classical Lebesgue outer measure, we obtain:

$$\mathcal{M}^*(\ln(B)) \leq \sum_{n=1}^{\infty} \ell(K_n).$$

For any interval $K_n = (c_n, d_n)$, we know that its length is given by $\ell(K_n) = d_n - c_n$. Thus, we have:

$$\mathcal{M}^*(\ln(B)) \leq \sum_{n=1}^{\infty} (\ln(d_n) - \ln(c_n)) = \ln\left(\prod_{n=1}^{\infty} \frac{d_n}{c_n}\right) = \ln\left(\prod_{n=1}^{\infty} \ell_G(J_n)\right).$$

Therefore,

$$e^{\mathcal{M}^*(\ln(B))} \leq \prod_{n=1}^{\infty} \ell_G(J_n).$$

It follows that:

$$e^{\mathcal{M}^*(\ln(B))} \leq \mathcal{M}_G^*(B).$$

To prove the reverse inequality, let $\{K_n = (c_n, d_n)\}_{n=1}^{\infty}$ be a collection of intervals such that $\ln(B) \subseteq \bigcup_{n=1}^{\infty} K_n$. This implies:

$$B \subseteq \bigcup_{n=1}^{\infty} e^{(c_n, d_n)} = \bigcup_{n=1}^{\infty} e^{K_n}.$$

Define $J_n = e^{K_n} \subseteq (0,\infty)$, then:

$$\mathcal{M}_G^*(B) \leq \prod_{n=1}^{\infty} \ell_G(J_n) = \prod_{n=1}^{\infty} \frac{e^{d_n}}{e^{c_n}} = e^{\sum_{n=1}^{\infty}(d_n - c_n)}.$$

Thus, we have:

$$\mathcal{M}_G^*(B) \leq e^{\sum_{n=1}^{\infty} \ell(K_n)}.$$

Equivalently:

$$\ln \mathcal{M}_G^*(B) \leq \sum_{n=1}^{\infty} \ell(K_n).$$

Hence, $\ln(\mathcal{M}_G^*(B))$ is a lower bound for:

$$\left\{ \sum_{n=1}^{\infty} \ell(K_n) : \ln(B) \subseteq \bigcup_{n=1}^{\infty} K_n \right\}.$$

Thus:

$$\ln(\mathcal{M}_G^*(B)) \leq \mathcal{M}^*(\ln(B)),$$

or equivalently:

$$\mathcal{M}_G^*(B) \leq e^{\mathcal{M}^*(\ln(B))}.$$

From the two inequalities, it follows that:

$$\mathcal{M}_G^*(B) = e^{\mathcal{M}^*(\ln(B))}.$$

The proof is complete. □

Theorem 5.3. *The geometric Lebesgue outer measure on* $(0,\infty)$ *is a valid geometric outer measure.*

Proof. We need to verify that $\mathcal{M}_G^*$ satisfies the conditions of Definition 4.10:

(i) From the definition of the geometric Lebesgue outer measure, we immediately have $\mathcal{M}_G^*(\emptyset) = 1$.

(ii) To prove monotonicity, suppose $B \subseteq C \subseteq (0,\infty)$. From Remark 4.7, if $B \subseteq C$, then $\ln(B) \subseteq \ln(C) \subseteq \mathbb{R}$. By the monotonicity of the classical Lebesgue outer measure:

$$\mathcal{M}^*(\ln(B)) \leq \mathcal{M}^*(\ln(C)).$$

Hence:

$$e^{\mathcal{M}^*(\ln(B))} \leq e^{\mathcal{M}^*(\ln(C))},$$

which implies:

$$\mathcal{M}_G^*(B) \leq \mathcal{M}_G^*(C).$$

(iii) To prove countable submultiplicativity, let $\{B_n\}_{n=1}^{\infty}$ be a collection of sets in $\mathcal{P}$. We have:

$$\mathcal{M}_G^*\left(\bigcup_{n=1}^{\infty} B_n\right) = e^{\mathcal{M}^*\left(\ln\left(\bigcup_{n=1}^{\infty} B_n\right)\right)}.$$

By the subadditivity of the classical Lebesgue outer measure:

$$\mathcal{M}^*\left(\ln\left(\bigcup_{n=1}^{\infty} B_n\right)\right) \leq \sum_{n=1}^{\infty} \mathcal{M}^*(\ln(B_n)).$$

Hence:

$$\mathcal{M}_G^*\left(\bigcup_{n=1}^{\infty} B_n\right) \leq e^{\sum_{n=1}^{\infty} \mathcal{M}^*(\ln(B_n))} = \prod_{n=1}^{\infty} \mathcal{M}_G^*(B_n).$$

Thus, $\mathcal{M}_G^*$ satisfies all the required properties, and the proof is complete. □ □

Proposition 5.4. *The geometric Lebesgue outer measure of the empty set* ϕ *is* 1.

Proof. By Theorem 5.2,

$$\mathcal{M}_G^*(\phi) = e^{\mathcal{M}^*(\ln(\phi))}.$$

Since $\ln(\phi) = \phi$ and $\mathcal{M}^*(\phi) = 0$, it follows

$$\mathcal{M}_G^*(\phi) \ = e^0 = 1.$$

The proof is completed. □

The next proposition explores the geometric Lebesgue outer measure of an interval in $(0, \infty)$ and establishes its connection to the geometric length.

Proposition 5.5. *The geometric Lebesgue outer measure of an interval I subset of $(0, \infty)$ is its geometric length $\mathcal{M}_G^*(I) = \ell_G(I)$.*

Proof. Let $I = (a, b)$ be an interval in $(0, \infty)$. By Proposition 4.3, $\ln(I) = (\ln a, \ln b)$ is an interval in $\mathbb{R}$. And we know the Lebesgue outer measure of an interval is the same as its length: $\mathcal{M}^*(\ln(I)) = \ln b - \ln a$. Therefore,

$$e^{\mathcal{M}^*(\ln(I))} = e^{(\ln b - \ln a)} = \frac{e^{\ln b}}{e^{\ln a}}.$$

Equivalently,

$$\mathcal{M}_G^*(I) = \frac{b}{a}.$$

Thus, $\mathcal{M}_G^*(I)$ is the geometric length of I. □

Now we present a result about the dilation invariance of the geometric Lebesgue outer measure.

Proposition 5.6. *(Dilation Invariant) For any set $A \subseteq (0, \infty)$ and number $x \in (0, \infty)$,*

$$\mathcal{M}^* (A \cdot x) = \mathcal{M}^* (A).$$

Proof. We can prove this proposition by Theorem 5.2 to gather that the Lebesgue outer measure is translation invariant. □

The next definition introduces the concept of geometric Lebesgue measurability for sets in $(0, \infty)$.

Definition 5.7. A subset $A \subseteq \mathbb{R}^+$ is called geometric Lebesgue measurable if for any $W \subseteq \mathbb{R}^+$ we have

$$\mathcal{M}_G^*(W) = \mathcal{M}_G^*(W \cap A) \cdot \mathcal{M}_G^*(W - A).$$

We now provide a remark that establishes the connection between geometric Lebesgue measurability and Lebesgue measurability in the logarithmic scale.

Remark 5.8. $A \subseteq (0, \infty)$ is geometric Lebesgue measurable if and only if $\ln A$ is Lebesgue measurable.

Proof. Assume $\ln A$ is measurable in $\mathbb{R}$. We need to show that A is geometric measurable. By the measurability of $\ln A$, we have:

$$\mathcal{M}^*(\ln W) = \mathcal{M}^*(\ln A \cap \ln W) + \mathcal{M}^*(\ln W - \ln A)$$
$$\mathcal{M}^*(\ln W) = \mathcal{M}^*(\ln(A \cap W)) + \mathcal{M}^*(\ln(W - A))$$

Thus,

$$e^{\mathcal{M}^*(\ln W)} = e^{\mathcal{M}^*(\ln(A \cap W)) + \mathcal{M}^*(\ln(W - A))}$$
$$e^{\mathcal{M}^*(\ln W)} = e^{\mathcal{M}^*(\ln(A \cap W))} \cdot e^{\mathcal{M}^*(\ln(W-A))}.$$

By Theorem 5.2, we have

$$\mathcal{M}_G^*(W) = \mathcal{M}_G^*(W \cap A) \cdot \mathcal{M}_G^*(W - A).$$

Thus, A is geometric Lebesgue measurable. Conversely, assume that A is geometric Lebesgue measurable, and we show that $\ln(A)$ is measurable. Since A is geometric Lebesgue measurable, Definition 5.7 can be applied to any set $W \subseteq \mathbb{R}^+$. Thus

$$\mathcal{M}_G^*(W) = \mathcal{M}_G^*(W \cap A) \cdot \mathcal{M}_G^*(W - A).$$

By Theorem 5.2,

$$e^{\mathcal{M}^*(\ln W)} = e^{\mathcal{M}^*(\ln(A \cap W))} \cdot e^{\mathcal{M}^*(\ln(W - A))}$$
$$= e^{\mathcal{M}^*(\ln(A \cap W)) + \mathcal{M}^*(\ln(W - A))}.$$

Therefore,

$$\ln\left(e^{\mathcal{M}^*(\ln W)}\right) = \ln\left(e^{\mathcal{M}^*(\ln(A \cap W)) + \mathcal{M}^*(\ln(W-A))}\right).$$

It follows,

$$\mathcal{M}^*(\ln W) = \mathcal{M}^*(\ln(A \cap W)) + \mathcal{M}^*(\ln(W - A))$$
$$= \mathcal{M}^*(\ln(A) \cap \ln(W)) + \mathcal{M}^*(\ln(W) - \ln(A)).$$

Now for any $M \subseteq \mathbb{R}$, we choose $W \subseteq \mathbb{R}^+$ such that $M = \ln W$. It follows that

$$\mathcal{M}^*(M) = \mathcal{M}^*(M \cap \ln A) + \mathcal{M}^*(M - \ln A).$$

Thus, $\ln(A)$ is Lebesgue measurable. □

The next proposition deals with sets of geometric outer measure 1 and their geometric Lebesgue measurability.

Proposition 5.9. *Any set of one geometric outer measure is geometric measurable.*

Proof. Let A be a subset of $(0,\infty)$ with $\mathcal{M}_G^*(A) = 1$. Then, $\ln(A) \subseteq \mathbb{R}$, and by Remark 5.8, $\mathcal{M}^*(\ln(A)) = 0$. According to Proposition 8, $\ln A$ is a measurable set. By Remark 5.8, the set A is geometric Lebesgue measurable. □

6. **Geometric Lebesgue Measure**

Definition 6.1. The restriction of the geometric Lebesgue outer measure to the σ-algebra $\mathcal{L}_G^0((0,\infty))$ is referred to as the geometric Lebesgue measure on $(0,\infty)$. For any set $A \in \mathcal{L}_G^0((0,\infty))$, we denote the geometric Lebesgue measure by $\mathcal{M}_G(A)$.

Theorem 6.2. *The geometric Lebesgue measure defined on* $(0,\infty)$ *constitutes a valid geometric measure.*

Proof. To establish this, we need to demonstrate that for any sequence of mutually disjoint geometric measurable subsets $\{A_n\}_{n=1}^{\infty}$ of $(0,\infty)$, the following holds:

$$\mathcal{M}_G\left(\bigcup_{n=1}^{\infty} A_n\right) = \prod_{n=1}^{\infty} \mathcal{M}_G(A_n).$$

By Proposition 4.4, if $\{A_n\}_{n=1}^{\infty}$ are mutually disjoint sets in $(0,\infty)$, then $\{\ln(A_n)\}_{n=1}^{\infty}$ are mutually disjoint measurable sets in $\mathbb{R}$. Consequently, we have:

$$\mathcal{M}\left(\bigcup_{n=1}^{\infty} \ln(A_n)\right) = \sum_{n=1}^{\infty} \mathcal{M}(\ln(A_n)).$$

Using Proposition 4.6, we obtain:

$$\mathcal{M}\left(\ln\left(\bigcup_{n=1}^{\infty} A_n\right)\right) = \sum_{n=1}^{\infty} \mathcal{M}(\ln(A_n)).$$

Thus,

$$e^{\mathcal{M}\left(\ln\left(\bigcup_{n=1}^{\infty} A_n\right)\right)} = e^{\sum_{n=1}^{\infty} \mathcal{M}(\ln(A_n))} = \prod_{n=1}^{\infty} e^{\mathcal{M}(\ln(A_n))}.$$

Therefore:

$$\mathcal{M}_G\left(\bigcup_{n=1}^{\infty} A_n\right) = \prod_{n=1}^{\infty} \mathcal{M}_G(A_n).$$

This completes the proof. □

6.1. *Properties of Geometric Lebesgue Measure*

Theorem 6.3. *The geometric Lebesgue measure is monotonic. If $A \subseteq B \subseteq (0, \infty)$, then:*

$$\mathcal{M}_G(A) \leq \mathcal{M}_G(B).$$

Proof. For any $A \subseteq B \subseteq (0, \infty)$, according to Remark 4.7, we have $\ln(A) \subseteq \ln(B) \subseteq \mathbb{R}$. Since the classical Lebesgue measure is monotonic, it follows that:

$$0 \leq \mathcal{M}(\ln(A)) \leq \mathcal{M}(\ln(B)),$$

or equivalently:

$$1 \leq e^{\mathcal{M}(\ln(A))} \leq e^{\mathcal{M}(\ln(B))}.$$

Thus, we have:

$$1 \leq \mathcal{M}_G(A) \leq \mathcal{M}_G(B).$$

This completes the proof. □

Theorem 6.4 (Geometric Excision Property)**.** *Let $A \subseteq W$ be two geometric Lebesgue measurable sets. If A has a finite geometric measure, then:*

$$\mathcal{M}_G(W - A) = \frac{\mathcal{M}_G(W)}{\mathcal{M}_G(A)}.$$

Proof. Let $A \subseteq W$ be a geometric measurable set in $(0, \infty)$. Since $\ln(A) \subseteq \ln(W)$ is measurable in $\mathbb{R}$, by the classical definition of Lebesgue measure, we have:

$$\mathcal{M}(\ln(W)) = \mathcal{M}(\ln(W) \cap \ln(A)) + \mathcal{M}(\ln(W) - \ln(A)).$$

Since $\ln(A) \cap \ln(W) = \ln(A)$, it follows that:

$$\mathcal{M}(\ln(W)) = \mathcal{M}(\ln(A)) + \mathcal{M}(\ln(W - A)).$$

Subtracting $\mathcal{M}(\ln(A))$ from both sides, we get:

$$\mathcal{M}(\ln(W)) - \mathcal{M}(\ln(A)) = \mathcal{M}(\ln(W - A)).$$

Taking the exponential of both sides:

$$e^{\mathcal{M}(\ln(W)) - \mathcal{M}(\ln(A))} = e^{\mathcal{M}(\ln(W - A))},$$

or equivalently:

$$\frac{e^{\mathcal{M}(\ln(W))}}{e^{\mathcal{M}(\ln(A))}} = e^{\mathcal{M}(\ln(W - A))}.$$

Therefore:

$$\mathcal{M}_G(W - A) = \frac{\mathcal{M}_G(W)}{\mathcal{M}_G(A)}.$$

This concludes the proof. □

Theorem 6.5. *If $A \subseteq (0,\infty)$ is geometric measurable, then for any $\epsilon > 1$, there exists an open set $O \subseteq (0,\infty)$ such that $\mathcal{M}_G(O - A) < \epsilon$.*

Proof. Let $A \subseteq (0,\infty)$ be geometric measurable. Since $\ln(A) \subseteq \mathbb{R}$ is measurable, for any $\epsilon > 1$, there exists an open set $G \subseteq \mathbb{R}$ such that $\ln(A) \subseteq G$ and:

$$\mathcal{M}(G - \ln(A)) < \ln(\epsilon).$$

Let $O = e^G \subseteq (0,\infty)$, so that O is open and $G = \ln(O)$. Then:

$$\mathcal{M}(G - \ln(A)) = \mathcal{M}(\ln(O) - \ln(A)),$$

and hence:

$$\mathcal{M}(\ln(O) - \ln(A)) < \ln(\epsilon).$$

Exponentiating both sides, we get:

$$e^{\mathcal{M}(\ln(O)-\ln(A))} < e^{\ln(\epsilon)}.$$

Thus:

$$\mathcal{M}_G(O - A) < \epsilon.$$

This concludes the proof. □

7. Lebesgue Geometric Measurable Functions

Definition 7.1. A positive real-valued function $f(x)$ defined on a geometric measurable subset $E \subseteq \mathbb{R}^+$ is said to be geometric Lebesgue measurable if and only if for all $c > 0$, the set

$$\{x \in E \mid f(x) \leq c\}$$

is geometric measurable.

Proposition 7.2. *If f is a positive function and E is a geometric measurable set, then f is geometric measurable if for every open set $O \subseteq \mathbb{R}^+$, the inverse image $f^{-1}(O) = \{x \in E \mid f(x) \in O\}$ is also geometric measurable.*

Proof. Assume f is geometric measurable. Consider an open set $G \subseteq \mathbb{R}$. The set G can be expressed as a countable union of disjoint open intervals $\{I_k\}_{k=1}^{\infty}$, where $I_k = (a_k, b_k)$, i.e.,

$$G = \bigcup_{k=1}^{\infty} \left((-\infty, b_k) \cap (a_k, \infty)\right).$$

Thus, we have:

$$e^{(G)} = e^{\bigcup_{k=1}^{\infty}((-\infty,b_k)\cap(a_k,\infty))}.$$

Since the exponential function is a homeomorphism, every open set in $\mathbb{R}^+ \subseteq \mathbb{R}$ is a geometric open set. Therefore,

$$e^{(G)} = \bigcup_{k=1}^{\infty} \left(e^{(-\infty, b_k)} \cap e^{(a_k, \infty)} \right),$$

which implies:

$$O = \bigcup_{k=1}^{\infty} \left((0, c_k) \cap (d_k, \infty) \right),$$

where $c_k = e^{b_k}$ and $d_k = e^{a_k}$. Since f is geometric measurable, each $f^{-1}(0, c_k)$ and $f^{-1}(d_k, \infty)$ are geometric measurable sets. Since the geometric measurable sets form a σ-algebra, the inverse image $f^{-1}(O)$ is geometric measurable, i.e.,

$$f^{-1}(O) = \bigcup_{k=1}^{\infty} f^{-1}(0, c_k) \cap f^{-1}(d_k, \infty).$$

Conversely, if the inverse image of every open set is geometric measurable, then f is geometric measurable since for any $c \in \mathbb{R}^+$,

$$\{x \in E \mid f(x) < c\} = f^{-1}(0, c) \cap E,$$

and the intersection of geometric measurable sets is geometric measurable. □

Theorem 7.3. *Let $E \subseteq \mathbb{R}^+$ be a geometric measurable set. If a positive real-valued function f is geometric measurable on E, then $(f \circ \exp)$ is measurable on $\ln(E) \subseteq \mathbb{R}$.*

Proof. For any $a \in \mathbb{R}$, we need to show that

$$\{t \in \ln(E) \mid (f \circ \exp)(t) < a\}$$

is measurable. Since f is geometric measurable and $E \subseteq \mathbb{R}^+$ is geometric measurable, we know that

$$\{x \in E \mid f(x) < C\}$$

is geometric measurable. According to Remark 5.8,

$$\ln\left(\{x \in E \mid f(x) < C\}\right) = \{t \in \ln(E) \mid (f \circ \exp)(t) < C\}$$

is measurable. Hence, $f \circ \exp$ is measurable on $\ln(E)$. □

Theorem 7.4. *If a real-valued function f is measurable on $E \subseteq \mathbb{R}$, then $f \circ \ln$ is geometric measurable on $\exp(E) \subseteq \mathbb{R}^+$.*

Proof. Since f is measurable, the set

$$\{x \in E \mid f(x) < c\}$$

is measurable. Therefore,

$$\exp\left(\{x \in E \mid f(x) < c\}\right)$$

is geometric measurable. But:

$$\exp\left(\{x \in E \mid f(x) < c\}\right) = \{e^x \in \exp(E) \mid f(x) < c\} = \{t \in \exp(E) \mid (f \circ \ln)(t) < c\}.$$

Thus, the set $\{t \in \exp(E) \mid (f \circ \ln)(t) < c\}$ is geometric measurable. □

Definition 7.5. Let $E \subseteq \mathbb{R}^+$. A property P is said to hold Geometrically Almost Everywhere (G.a.e.) on E if there exists a subset $E_0 \subseteq E$ with geometric measure $\mathcal{M}_G(E_0) = 1$, such that the property holds on $E - E_0$.

Proposition 7.6. *Let f be a positive real-valued function defined on a geometric measurable set E. If $f = g$ G.a.e. on E, then g is geometric measurable on E.*

Proof. Assume $f = g$ G.a.e. on $E \subseteq \mathbb{R}^+$. We need to show that $g \circ \exp$ is measurable on $\ln(E)$. Since $f = g$ on $E - D$, where $\mathcal{M}_G(D) = 1$, we have $f(\exp(t)) = g(\exp(t))$ for all $\exp(t) \in E - D$. Hence, $f(\exp(t)) = g(\exp(t))$ for all $t \in \ln(E) - \ln(D)$. Since $\mathcal{M}_G(D) = 1$, we know that $e^{m(\ln(D))} = 1$, which implies $m(\ln(D)) = 0$. Therefore, $f(\exp(t)) = g(\exp(t))$ almost everywhere on $\ln(E)$. Thus, $g(\exp(t))$ is measurable on $\ln(E)$, and hence g is geometric measurable on E. □

Proposition 7.7. *Let f be a positive real-valued function defined on a geometric measurable set E. If f is geometric measurable G.a.e. on E, then f is geometric measurable on E.*

Proof. Since f is geometric measurable G.a.e., there exists a subset $D \subseteq E$ such that f is geometric measurable on D and $\mathcal{M}_G(E-D) = 1$. Hence, by Theorem 7.4, $f \circ \exp$ is measurable on $\ln(D)$. The fact that $\mathcal{M}_G(E - D) = \exp(m(\ln(E - D)))$ implies that $\mathcal{M}(\ln(E - D)) = 0$. Thus, $f \circ \exp$ is measurable on $\ln(D) \cup \ln(E - D) = \ln(E)$. By Theorem 7.4, f is geometric measurable on E. □

Proposition 7.8. *Let f be a positive real-valued function defined on a geometric measurable set E. If f is geometric measurable G.a.e. on E, then f is geometric measurable on E.*

Proof. Since f is geometric measurable G.a.e., there is $D \subseteq E$ such that f is geometric measurable on D and $\mathcal{M}_G(E-D)=1$. Hence $f \circ \exp$ is measurable on $\ln D$ (see Theorem 7.4). The fact that

$$\mathcal{M}_G(E-D) = \exp\left(\mathcal{M}(\ln(E-D))\right)$$

implies

$$\mathcal{M}(\ln((E-D))) = 0.$$

So $f \circ \exp$ is measurable on $\ln(D) \cup \ln(E-D) = \ln(E)$. By Theorem 7.4,

$$f \circ \exp \circ \ln = f$$

is geometric measurable on $\exp(\ln E) = E$. □

Proposition 7.9. *Let $\{E_n\}_{n=1}^{\infty}$ be a sequence of disjoint geometric measurable sets. Let f be a positive real-valued function defined on*

$$E = \bigcup_{n=1}^{\infty} E_n,$$

and if f is geometric measurable on each E_n, then f is geometric measurable on E.

Proof. f is geometric measurable on E_n, $n \in \mathbb{N}$. By Theorem 7.3, $f \circ \exp$ is measurable on each $\ln(E_n)$. Therefore, $f \circ \exp$ is measurable on

$$\bigcup_{n=1}^{\infty} \ln(E_n) = \ln\left(\bigcup_{n=1}^{\infty} E_n\right).$$

Now, by Theorem 7.4,

$$f \circ \exp \circ \ln = f$$

is geometric on

$$\exp\left[\ln\left(\bigcup_{n=1}^{\infty} E_n\right)\right] = E.$$

□

7.1. *Continuity and Geometric Measurability of Functions*

Proposition 7.10. *Every geometric continuous positive real-valued function on a geometric measurable set $E \subseteq \mathbb{R}^+$ is geometric measurable.*

Proof. Let f be a geometric continuous function on $E \subseteq \mathbb{R}^+$. Let $G \subseteq \mathbb{R}^+$ be an open set. Since f is geometric continuous,

$$f^{-1}(G) = E \cap A,$$

where A is geometric open. Thus, $f^{-1}(G)$, being the intersection of two geometric measurable sets, and hence geometric measurable. Hence, f is geometric measurable. □

Proposition 7.11. *Every continuous positive real-valued function on a geometric measurable set $E \subseteq \mathbb{R}^+$ is geometric measurable.*

Proof. Let $E \subseteq \mathbb{R}^+$ be geometric measurable, then $\ln E \subseteq \mathbb{R}$ is measurable. Since $f \circ \ln$ is continuous, it follows that $f \circ \ln$ is measurable on $\ln E$. By Theorem 7.4, f is geometric measurable on E. □

Proposition 7.12. *Let f be a continuous positive real-valued function defined on $\mathbb{R}^+$ and g be a geometric measurable function defined on E. Consequently, $f \circ g$ is a geometric measurable function on E.*

Proof. Proposition 7.2 states that a function is geometric measurable if and only if each open set subset of $\mathbb{R}^+$ has a geometric measurable inverse image. Let O be a subset of $\mathbb{R}^+$ that is open. Then,

$$(f \circ g)^{-1}(O) = g^{-1}\left(f^{-1}(O)\right).$$

Since f is continuous, the set $U = f^{-1}(O)$ is open. By the measurability of g, $g^{-1}(U)$ is geometric measurable. Thus, the inverse image

$$(f \circ g)^{-1}(O)$$

is a geometric measurable set, hence $f \circ g$ is geometric measurable on E. □

Proposition 7.13. *Let g be a geometric measurable function defined on E, and let f be a geometric continuous positive real-valued function defined on $\mathbb{R}^+$. Then, the composition $f \circ g$ is a geometric measurable function on E.*

Proof. Let f be geometric continuous. It follows that f is continuous. Thus by Proposition 7.11, the composition $f \circ g$ is a geometric measurable function on E. □

8. **Neutrosophic Geometric Measure**

In this section we introduce the concept of the Neutrosophic Geometric Measure, we also provide some results, and examples related to the Neutrosophic Geometric Measure.

Definition 8.1 (Neutrosophic Geometric Length)**.** Let $A = (a, b)$ be an interval subset of $(0, \infty)$. The Neutrosophic Geometric Length of A is defined as:

$$\ell_{NG}(A) = \langle T(\ell_G(A)), I(\ell_G(A)), F(\ell_G(A))\rangle,$$

where:

1) $T(\ell_G(A))$: The truth component.
2) $I(\ell_G(A))$: The indeterminacy component, representing uncertainty.
3) $F(\ell_G(A))$: The falsity component, representing contradictions or errors.

To introduce the concept of neutrosophic Geometric Measure, we need to define a natural ordering for a Triple (T, I, F).

Definition 8.2 (Ordering for a Triple (T, I, F))**.** Let (T_1, I_1, F_1) and (T_2, I_2, F_2) be two triples representing truth, indeterminacy, and falsity, respectively. The ordering $(T_1, I_1, F_1) \leq (T_2, I_2, F_2)$ is defined as:

$$(T_1, I_1, F_1) \leq (T_2, I_2, F_2) \iff T_1 \leq T_2,\ I_1 \geq I_2,\ F_1 \geq F_2.$$

This ordering ensures:

1. Truth (T): Higher truth values are preferred.
2. Indeterminacy (I): Lower indeterminacy values are preferred.
3. Falsity (F): Lower falsity values are preferred.

In this definition, we introduce the concept of Neutrosophic Geometric Outer Measure.

Definition 8.3 (Neutrosophic Geometric Outer Measure)**.** Let $X \subseteq (0, \infty)$ and $A \subseteq X$. The Neutrosophic Geometric Outer Measure $\mu^*_{NG} : \mathcal{P}(X) \to [1, \infty] \times [1, \infty] \times [1, \infty]$ is defined as:

$$\mu^*_{NG}(A) = \inf \left\{ \prod_{n=1}^{\infty} \ell_{NG}(I_n) : A \subseteq \bigcup_{n=1}^{\infty} I_n \right\},$$

where:

1) $\ell_{NG}(I_n) = \langle T(\ell_G(I_n)), I(\ell_G(I_n)), F(\ell_G(I_n))\rangle$ is the neutrosophic geometric length of each interval I_n.
2) $\ell_G(I_n) = \frac{b_n}{a_n}$ is the classical geometric length of the interval $I_n = (a_n, b_n)$.

The function satisfies the following properties:

(1) **Null Set:** $\mu^*_{NG}(\phi) = \langle 1, 1, 1\rangle$.

(2) **Monotonicity:** If $A \subseteq B \subseteq X$, then $\mu^*_{NG}(A) \leq \mu^*_{NG}(B)$ (based on the ordering defined above).

(3) **Countable Subproductivity:** For any countable collection $\{A_i\}_{i=1}^{\infty}$:

$$\mu^*_{NG}\left(\bigcup_{i=1}^{\infty} A_i\right) \leq \prod_{i=1}^{\infty} \mu^*_{NG}(A_i),$$

where the product is computed component-wise.

The Neutrosophic Geometric Outer Measure with its defined ordering is highly practical in addressing complex scenarios. It plays a significant role in decision-making by providing a structured approach to prioritize options based on their truth, indeterminacy, and falsity components. This makes it invaluable for uncertainty modeling, especially in real-world problems where these three factors coexist and influence outcomes. Moreover, it extends classical analytical methods by embedding meaningful comparisons between neutrosophic values, thereby allowing for a more generalized and nuanced analysis of data and systems.

Definition 8.4 (Neutrosophic Geometric Lebesgue Outer Measure)**.** Let $A \subseteq (0, \infty)$. The Neutrosophic Geometric Lebesgue Outer Measure $m^*_{NG}(A)$ is defined as:

$$m^*_{NG}(A) = \langle e^{m^*(\ln T(A))}, e^{m^*(\ln I(A))}, e^{m^*(\ln F(A))}\rangle,$$

where:

1) $T(A)$: The truth component of the subset A.
2) $I(A)$: The indeterminacy component of the subset A.
3) $F(A)$: The falsity component of the subset A.
4) $m^*(\cdot)$: The classical Lebesgue outer measure applied to the logarithmic transformation of the components $T(A)$, $I(A)$, and $F(A)$.

This Neutrosophic Outer Measure extends the classical geometric Lebesgue outer measure by incorporating truth, indeterminacy, and falsity components in the netrosophic framework. To verify that the Neutrosophic Geometric Lebesgue Outer Measure satisfies the required properties, we start with the null set property.

For the null set ϕ, we have:

$$T(\phi) = \phi, \quad I(\phi) = \phi, \quad F(\phi) = \phi.$$

Hence, we have

$$\ln T(\phi) = \ln(\phi) = \phi, \quad \ln I(\phi) = \ln(\phi) = \phi, \quad \ln F(\phi) = \ln(\phi) = \phi.$$

Since the classical Lebesgue outer measure $m^*(\phi) = 0$, we have:

$$\begin{aligned} m^*_{NG}(\phi) &= \langle e^{m^*(\ln T(\phi))}, e^{m^*(\ln I(\phi))}, e^{m^*(\ln F(\phi))} \rangle \\ &= \langle e^0, e^0, e^0 \rangle = \langle 1, 1, 1 \rangle. \end{aligned}$$

Thus, the null set property is satisfied.

Next, we prove the monotonicity. Let $A, B \subseteq (0, \infty)$ with $A \subseteq B$. For the classical Lebesgue outer measure, we have

$$m^*(\ln T(A)) \leq m^*(\ln T(B)),$$

$$m^*(\ln I(A)) \leq m^*(\ln I(B)),$$

and

$$m^*(\ln F(A)) \leq m^*(\ln F(B)).$$

Taking the exponential, we get:

$$e^{m^*(\ln T(A))} \leq e^{m^*(\ln T(B))},$$

$$e^{m^*(\ln I(A))} \leq e^{m^*(\ln I(B))},$$

$$e^{m^*(\ln F(A))} \leq e^{m^*(\ln F(B))}.$$

Thus:

$$m^*_{NG}(A) \leq m^*_{NG}(B).$$

The monotonicity property is verified.

Finally, we prove the countable subadditivity property. Let $\{A_i\}_{i=1}^{\infty}$ be a countable collection of subsets of $(0, \infty)$. For the classical Lebesgue outer measure:

$$m^*\left(\ln T\left(\bigcup_{i=1}^{\infty} A_i\right)\right) \leq \sum_{i=1}^{\infty} m^*(\ln T(A_i)),$$

$$m^*\left(\ln I\left(\bigcup_{i=1}^{\infty} A_i\right)\right) \leq \sum_{i=1}^{\infty} m^*(\ln I(A_i)),$$

$$m^*\left(\ln F\left(\bigcup_{i=1}^{\infty} A_i\right)\right) \leq \sum_{i=1}^{\infty} m^*(\ln F(A_i)).$$

Taking the exponential, we get:

$$e^{m^*\left(\ln T\left(\bigcup_{i=1}^{\infty} A_i\right)\right)} \leq \prod_{i=1}^{\infty} e^{m^*(\ln T(A_i))},$$

$$e^{m^*\left(\ln I\left(\bigcup_{i=1}^{\infty} A_i\right)\right)} \leq \prod_{i=1}^{\infty} e^{m^*(\ln I(A_i))},$$

$$e^{m^*\left(\ln F\left(\bigcup_{i=1}^{\infty} A_i\right)\right)} \leq \prod_{i=1}^{\infty} e^{m^*(\ln F(A_i))}.$$

Thus:

$$m^*_{NG}\left(\bigcup_{i=1}^{\infty} A_i\right) \leq \prod_{i=1}^{\infty} m^*_{NG}(A_i).$$

Countable subadditivity is satisfied.

The neutrosophic geometric Lebesgue outer measure m^*_{NG} satisfies the null set property, monotonicity, and countable subadditivity, confirming that it meets the requirements of a valid neutrosophic geometric outer measure.

Now we are ready to introduce the concept of neutrosophic geometric measure.

Definition 8.5 (Neutrosophic Geometric Measure)**.** Let A be a σ-algebra on $X \subseteq \mathbb{R}^+$. A set function $\mu_{NG} : A \to [1,\infty] \times [1,\infty] \times [1,\infty]$ is called a Neutrosophic Geometric Measure if it satisfies:

(1) $\mu_{NG}(\phi) = \langle 1, 1, 1\rangle$.

(2) For any sequence of mutually disjoint sets $\{E_j\}_{j=1}^{\infty}$:

$$\mu_{NG}\left(\bigcup_{j=1}^{\infty} E_j\right) = \prod_{j=1}^{\infty} \mu_{NG}(E_j),$$

where the product is computed component-wise as:

$$\prod_{j=1}^{\infty} \mu_{NG}(E_j) = \langle \prod_{j=1}^{\infty} T(\mu_G(E_j)), \prod_{j=1}^{\infty} I(\mu_G(E_j)), \prod_{j=1}^{\infty} F(\mu_G(E_j))\rangle.$$

This definition generalizes the classical geometric measure by considering the truth, indeterminacy, and falsity components.

Finally, we define the concept of neutrosophic geometric Lebesgue measure.

Definition 8.6 (Neutrosophic geometric Lebesgue Measure)**.** The Neutrosophic Lebesgue Measure $m_{NG}(A)$ is the restriction of the neutrosophic geometric Lebesgue outer measure m^*_{NG} to the family of neutrosophic Lebesgue measurable sets. Specifically:

$$m_{NG}(A) = m^*_{NG}(A),$$

where $A \subseteq (0,\infty)$ is neutrosophic Lebesgue measurable if and only if:

$$m^*_{NG}(W) = m^*_{NG}(W \cap A) \cdot m^*_{NG}(W \setminus A)$$

for any $W \subseteq \mathbb{R}^+$.

Neutrosophic measures play a vital role in addressing the challenges posed by uncertainty, vagueness, and contradictions that arise in real-world problems. By integrating these elements into their framework, neutrosophic measures provide a sophisticated tool for modeling and understanding complex phenomena where traditional approaches fall short. This capability

makes them indispensable in scenarios where indeterminacy is a fundamental characteristic of the data or the system under study.

Moreover, neutrosophic measures extend the classical geometric and Lebesgue measures, offering a more generalized approach suitable for analyzing complex systems. This generalization enables their application in domains where the interplay of truth, falsity, and indeterminacy is critical, making them particularly valuable in interdisciplinary studies that require a nuanced mathematical treatment of ambiguity.

From a mathematical perspective, neutrosophic measures significantly enhance traditional measure theory. They provide robust tools for analyzing data that is both indeterminate and contradictory, allowing researchers to extract meaningful insights in situations where conventional methods may fail. By extending the theoretical foundation of measure theory, neutrosophic measures contribute to the development of advanced analytical methods capable of tackling the most challenging problems in modern science and engineering. Now, we give the following important result which is important in the filed of this study.

Theorem 8.7. *For any subset $A \subseteq (0, \infty)$, the Neutrosophic Geometric Lebesgue Outer Measure is given by:*

$$m^*_{NG}(A) = \langle e^{m^*(\ln T(A))}, e^{m^*(\ln I(A))}, e^{m^*(\ln F(A))} \rangle.$$

Proof. The proof extends the classical geometric Lebesgue outer measure by transforming the components (truth, indeterminacy, and falsity) using logarithmic and exponential functions:
First, Transform the sets $T(A)$, $I(A)$, and $F(A)$ to their logarithmic scales.
Then, Apply the classical Lebesgue outer measure on $\ln T(A)$, $\ln I(A)$, and $\ln F(A)$.
Finally, Use the exponential function to revert the transformed measures back to the geometric domain.
This yields:

$$m^*_{NG}(A) = \langle e^{m^*(\ln T(A))}, e^{m^*(\ln I(A))}, e^{m^*(\ln F(A))} \rangle.$$

□

We conclude this section with the following example to clarify this new concept.

Example 8.8. (Neutrosophic Geometric Length of an Interval) Let the interval be $A = (1, 4)$. We calculate the neutrosophic geometric length $\ell_{NG}(A)$ by determining the truth, indeterminacy, and falsity components using experimental data.

We use the following experimental data, which summaries the a sample of different measures for the interval $A = (1, 4)$, which is measured multiple times, and the results are recorded as follows:

Trial	Initial Point (a)	End Point (b)	Geometric Length (b/a)	Remarks
1	1.00	4.00	4.00	Accurate
2	1.02	4.10	4.02	Slight deviation
3	0.98	4.05	4.13	Overestimated
4	1.01	3.95	3.91	Underestimated
5	1.00	4.00	4.00	Accurate

TABLE 1. Experimental data for measuring interval A

First, we compute the truth component $T(\ell_G(A))$. The truth component is the classical geometric length calculated as the average:

$$\begin{aligned} T(\ell_G(A)) &= \text{Average}\left(\frac{b}{a}\right) \\ &= \frac{4.00 + 4.02 + 4.13 + 3.91 + 4.00}{5} \\ &= 4.012. \end{aligned}$$

Next, we find the indeterminacy component $I(\ell_G(A))$. The indeterminacy component is derived from the relative standard deviation (RSD):

- Standard deviation σ:

$$\begin{aligned} \sigma &= \sqrt{\frac{\sum(x_i - \bar{x})^2}{n-1}} \\ &= 0.075. \end{aligned}$$

- Relative uncertainty:

$$I(\ell_G(A)) = \frac{\sigma}{\bar{x}} = \frac{0.075}{4.012} = 0.0187 \text{ (approximately } 1.87\%).$$

Next, we find the falsity component $F(\ell_G(A))$. The falsity component is determined from the percentage of contradictory measurements:

- Outliers: Trials 3 (4.13) and 4 (3.91) deviate significantly from the mean ($> 2\sigma$).
- Percentage of falsity:

$$F(\ell_G(A)) = \frac{\text{Number of outliers}}{\text{Total trials}} = \frac{2}{5} = 0.4 \text{ (40\% falsity)}.$$

Finally, Combining the components, the neutrosophic geometric length is:

$$\begin{aligned} \ell_{NG}(A) &= \langle T(\ell_G(A)), I(\ell_G(A)), F(\ell_G(A)) \rangle \\ &= \langle 4.012, 0.0187, 0.4 \rangle. \end{aligned}$$

Conclusion

In this paper, we have successfully established the concepts of geometric Lebesgue measures and the neutrosophic geometric Lebesgue measures for subsets of $\mathbb{R}^+$. This new measure serves as a useful tool to explore and analyze subsets of the positive real numbers. We demonstrated that geometric Lebesgue measures possess several important properties, such as monotonicity and the excision property, which ensure their consistency and practicality in a variety of mathematical applications. Additionally, we introduced geometric Lebesgue measurable functions and explored their significance. These functions play a central role in the theory of geometric integration, and we showed how they can be approximated using geometric simple functions. This approximation process is crucial for extending the utility of geometric calculus and enabling its application to real-world problems. Through this study, we laid the groundwork for future research in geometric calculus and its applications in areas such as geometric measure theory and integration. The insights gained from geometric Lebesgue measures open the door for further exploration of non-Newtonian measures and their potential use in solving complex mathematical and applied problems.

In summary, neutrosophic geometric calculus continues to offer novel approaches to classical mathematical challenges, and this paper contributes by adding new tools and perspectives to the existing body of knowledge. With its clear ordering, the Neutrosophic Geometric Outer Measure is very useful for dealing with tricky situations. It plays a significant role in decision-making by providing a structured approach to prioritize options based on their truth, indeterminacy, and falsity components. This makes it invaluable for uncertainty modeling, especially in real-world problems where these three factors coexist and influence outcomes. Moreover, it extends classical analytical methods by embedding meaningful comparisons between neutrosophic values, thereby allowing for a more generalized and nuanced analysis of data and systems.In future work, these tools can be combined with the work described in the following references see [29–32].

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Innovation Ability Management Performance Evaluation of College Students in Science and Technology Industry: A Plithogenic CoCoSo Approach

Xian Yuan*, Kaiyin Zhang
Henan Polytechnic Institute, Henan, 473000, China
*Corresponding author, E-mail: yuanx8683@163.com

Abstract: This study presents a decision-making approach to evaluate ethe innovation ability performance of college students in science and technology industry. We used the multi-criteria decision making (MCDM) approach to deal with various criteria to compute the criteria weights and rank the alternatives. This study uses the CoCoSo methodology to rank the alternatives. Seven criteria and ten alternatives are used in this study with three opinions of experts and decision makers to evaluate these criteria and alternatives. the results show the Creativity and Problem-Solving criterion has the highest rate and Research and Analytical Skills has the lowest rate. We conducted a sensitivity analysis with different cases to show the rank of alternatives with different values. The results show the rank of alternatives is stable in different cases. This study can help students to choose their colleges by a set of criteria.

Keywords: Uncertainty; MCDM Approach; Evaluation Method; Innovation Ability Management Performance.

1. Introduction

The Modern Service Industry Science and Technology Innovation Special Plan for the "13th Five-Year Plan" in 2017 and the Several Opinions of the State Council on Accelerating the Development of the Science and Technology Service Industry in 2014 were issued by the Chinese state council. Both documents described the critical role that the knowledge spillover effect of the science and technology service industry plays in fostering the growth of the country's modern industrial system and the optimization of the industrial structure[1], [2].

As an emerging sector, the degree of development of the science and technology service sector influences the rate of economic growth in the area as well as the creation of the region's capacity for innovation and development[3], [4]. As a result, the sector has

recently drawn national attention and received several subsidies, taxes, and other forms of support from local governments[5], [6].

The course for the advancement of college students' education in innovation was outlined in the Measures for the Certification of College Students' Science and Technology Practice Bases and the Opinions on Vigorously Promoting the Innovation Education of Colleges and Universities and the Independent of College Students, both published by the Ministry of Education in 2010[7], [8].

1.1 A brief review of the development of Plithogenic

This study uses the plithogenic CoCoSo approach to identify the most significant students' colleges in innovation and uncertainties associated with evaluation of innovation. The suggested approach will support correct decision-making as well as effective management and analysis of group decision-making issues without information loss[9], [10]. Information loss arises in the traditional CoCoSo technique because group decision-making problems cannot fully accommodate decision-makers' individual assessments. The plithogenic aggregated operators in the suggested method stop this information loss. By creating a novel MCDM technique known as Plithogenic CoCoSo, this study will add to the body of literature. Students colleges in innovation and multi-criteria decision-making (MCDM) studies have been analyzed since the MCDM approach has been investigated[11], [12].

1.2 A brief review on the development of CoCoSo method

MCDM's intrinsic qualities make it both interesting and useful in real life. One major critique of MCDM techniques is that, when used to solve the same decision-making problem, they produce different ranking outcomes or optimal alternatives[13], [14].
As is well known, Yazdani et al. primitively established the CoCoSo (combined compromise solution), which employs a comparability sequence using two methods (weighted power of the distance and the ordinary multiplication formula). Its foundations are the exponentially weighted product (EWP) and aggregated simple additive weighting (SAW) techniques[15], [16].

1.3 Contribution of this study

We created a new MCDM approach in the plithogenic environment based on the aforementioned reasoning and the features of the decision-making process. The following is a list of the innovations of the suggested approach:

[1] A new plithogenic decision-making approach based on the CoCoSo method is investigated; it has a high ability to distinguish between alternatives and can produce the best option without counterintuitive phenomena or division or antilogarithm by zero issues.

[2] The criteria weights are computed using the plithogenic number.
[3] The sensitivity analysis is performed to show the different rank of alternatives.

2. Approach to Plithogenic MCDM based CoCoSo

This section shows the seps of the proposed method.

2.1 The description of plithogenic MCDM issue

Assume that $A = \{A_1, .., A_m\}$ be a set of alternatives, set of criteria $C = \{C_1, \dots, C_n\}$, and the criteria weights $W = \{w_1, .., w_n\}$; $\sum_{j=1}^{n} w_j = 1$. Let the evaluation of the alternative A with respect to criteria C by the plithogenic matrix. The framework of the development approach is given in Figure 1.

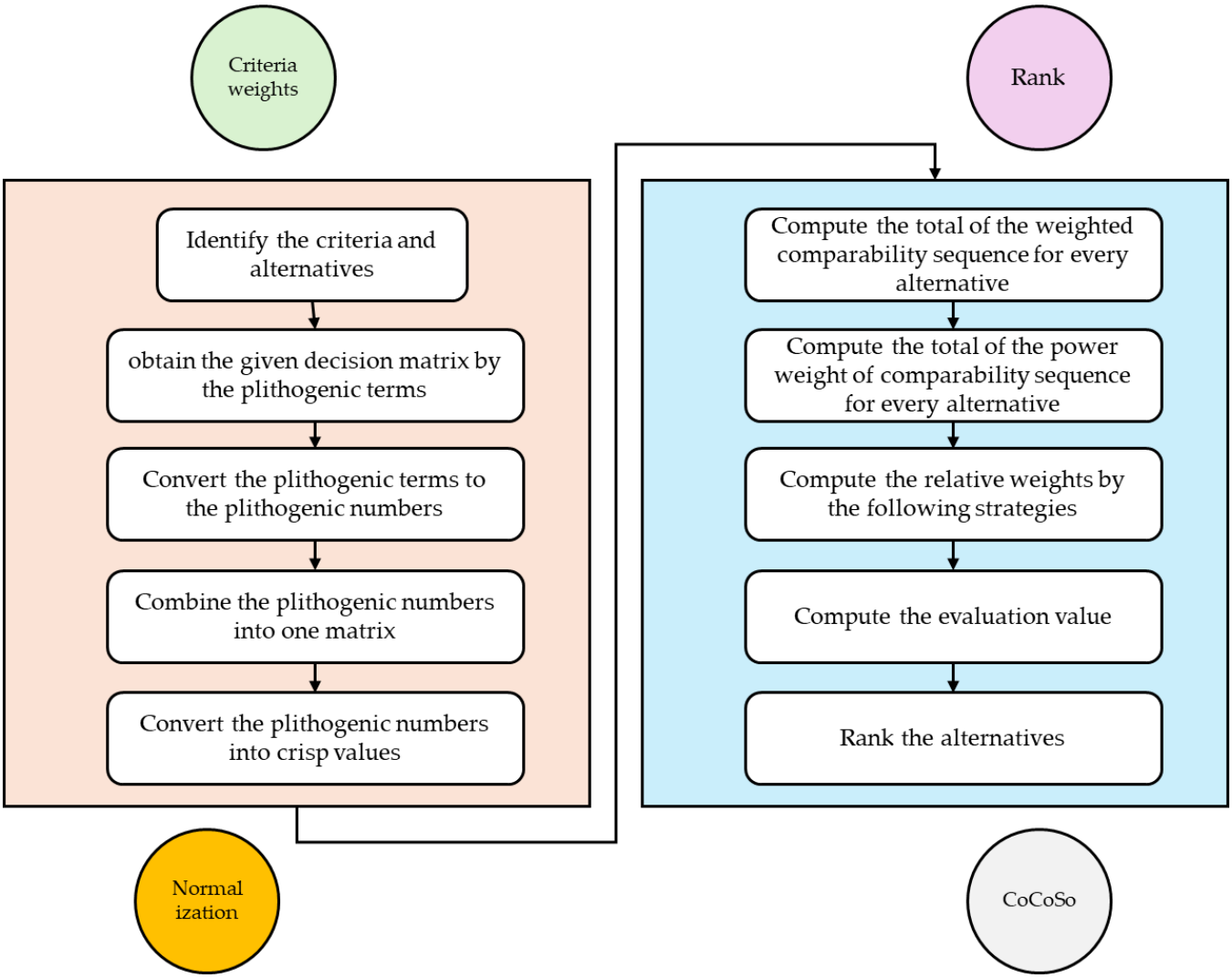
Criteria weights
Rank
Identify the criteria and alternatives
obtain the given decision matrix by the plithogenic terms
Convert the plithogenic terms to the plithogenic numbers
Combine the plithogenic numbers into one matrix
Convert the plithogenic numbers into crisp values
Compute the total of the weighted comparability sequence for every alternative
Compute the total of the power weight of comparability sequence for every alternative
Compute the relative weights by the following strategies
Compute the evaluation value
Rank the alternatives
Normal ization
CoCoSo

Figure 1. The framework of the development approach.

2.2 Determine objective weights

Criteria can be deemed to be significant information while the evaluation of the decision-making process. The criteria weights can affect on the rank of alternatives. We compute the criteria weights by the normalization method.

Step 1: Build the decision matrix.

The decision matrix has information about experts and decision makers. They use the terms of plithogenic sets to evaluate the criteria and alternatives.

Step 2: Convert the decision matrix into plithogenic numbers.

The terms by experts and decision makers are converted to the plithogenic numbers in the decision matrix.

Step 3: Combine the decision matrix.

We combine the plithogenic numbers into one matrix.

Step 4: Apply the score function.

The score function is applied to the plithogenic numbers to obtain crisp values.

Step 5: Normalize the crisp values.

The crisp values are normalized to obtain the criteria weights.

2.3 Plithogenic CoCoSo method

The CoCoSo method is an MCDM method. It is used to rank the criteria weights[17], [18]. The steps of the CoCoSo method are detailed below:

Algorithm 1: plithogenic-CoCoSo

1: obtain the given decision matrix by the plithogenic terms

2: Convert the plithogenic terms to the plithogenic numbers.

3: Combine the plithogenic numbers into one matrix.

4: Convert the plithogenic numbers into crisp values.

5: Normalize the decision matrix as:

$$x_{ij} = \frac{y_{ij} - \min y_{ij}}{\max y_{ij} - \min y_{ij}} \; positive\ criteria$$

$$x_{ij} = \frac{\max y_{ij} - y_{ij}}{\max y_{ij} - \min y_{ij}} \; negative\ criteria$$

6: Compute the total of the weighted comparability sequence for every alternative as:

$$S_i = \sum_{j=1}^{n} w_j x_{ij}$$

7: Compute the total of the power weight of comparability sequence for every alternative as:

$$P_i = \sum_{j=1}^{n} (x_{ij})^{w_j}$$

8: Compute the relative weights by the following strategies as:

$$Q_{ia} = \frac{S_i + P_i}{\sum_{i=1}^{m} (S_i + P_i)}$$

$$Q_{ib} = \frac{S_i}{\min S_i} + \frac{P_i}{\min P_i}$$

$$Q_{ic} = \frac{\delta S_i + (1-\delta)P_i}{\delta \max S_i + (1-\delta)\max P_i}$$

9: Compute the evaluation value as:

$$Q_i = \sqrt[3]{Q_{ia}Q_{ib}Q_{ic}} + \frac{Q_{ia} + Q_{ib} + Q_{ic}}{3}$$

10: Rank the alternatives.

3. Case Study

This section shows the applied steps of the proposed methodology under the plithogenic sets to rank the alternatives and compute the criteria weights. Three experts and decision makers evaluated the criteria and alternatives. Seven criteria and ten alternatives are used in this study. The criteria are shown in Figure 2.

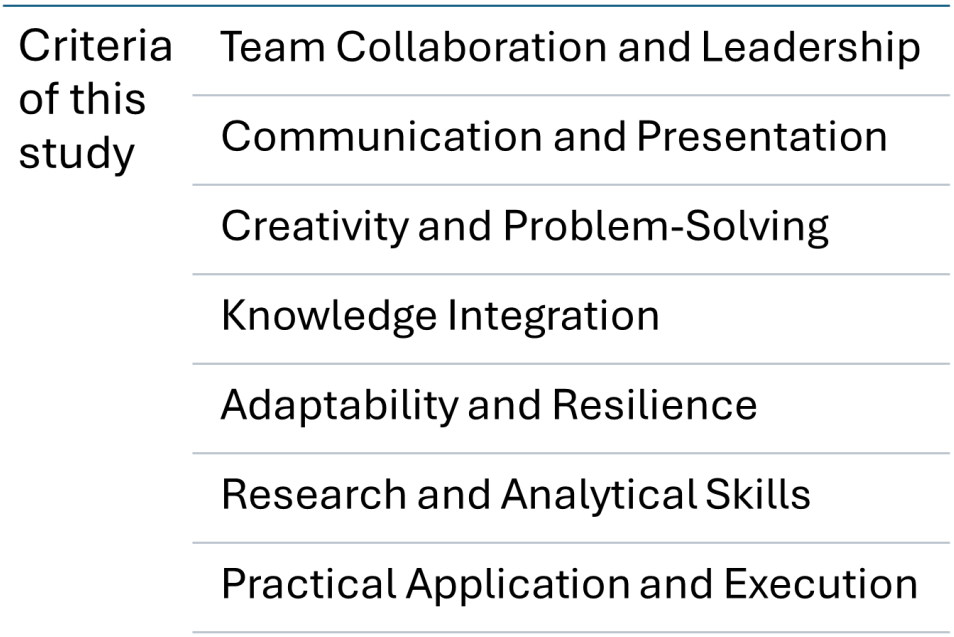

Figure 2. The criteria of this study.

Three experts have evaluated the criteria and alternatives. Table 1 shows the plithogenic numbers between the criteria and alternatives. Then we combine these numbers into one matrix. Then we obtain crisp values. Figure 3 shows the criteria weights.

Table 1. The decision matrix.

	C_1	C_2	C_3	C_4	C_5	C_6	C_7
A_1	(0.80, 0.10, 0.30)	(0.40, 0.70, 0.50)	(0.50, 0.40, 0.60)	(0.65, 0.30, 0.45)	(0.80, 0.10, 0.30)	(0.25, 0.60, 0.80)	(0.40, 0.70, 0.50)
A_2	(0.50, 0.40, 0.60)	(0.80, 0.10, 0.30)	(0.25, 0.60, 0.80)	(0.40, 0.70, 0.50)	(0.50, 0.40, 0.60)	(0.65, 0.30, 0.45)	(0.65, 0.30, 0.45)
A_3	(0.80, 0.10, 0.30)	(0.50, 0.40, 0.60)	(0.80, 0.10, 0.30)	(0.65, 0.30, 0.45)	(0.80, 0.10, 0.30)	(0.80, 0.10, 0.30)	(0.80, 0.10, 0.30)
A_4	(0.10, 0.75, 0.85)	(0.80, 0.10, 0.30)	(0.50, 0.40, 0.60)	(0.80, 0.10, 0.30)	(0.10, 0.75, 0.85)	(0.95, 0.05, 0.05)	(0.65, 0.30, 0.45)
A_5	(0.50, 0.40, 0.60)	(0.10, 0.75, 0.85)	(0.80, 0.10, 0.30)	(0.50, 0.40, 0.60)	(0.80, 0.10, 0.30)	(0.10, 0.75, 0.85)	(0.80, 0.10, 0.30)

A6	(0.40, 0.70, 0.50)	(0.65, 0.30, 0.45)	(0.10, 0.75, 0.85)	(0.80, 0.10, 0.30)	(0.50, 0.40, 0.60)	(0.80, 0.10, 0.30)	(0.50, 0.40, 0.60)
A7	(0.25, 0.60, 0.80)	(0.50, 0.40, 0.60)	(0.65, 0.30, 0.45)	(0.10, 0.75, 0.85)	(0.80, 0.10, 0.30)	(0.50, 0.40, 0.60)	(0.80, 0.10, 0.30)
A8	(0.10, 0.75, 0.85)	(0.40, 0.70, 0.50)	(0.80, 0.10, 0.30)	(0.65, 0.30, 0.45)	(0.10, 0.75, 0.85)	(0.80, 0.10, 0.30)	(0.10, 0.75, 0.85)
A9	(0.95, 0.05, 0.05)	(0.25, 0.60, 0.80)	(0.95, 0.05, 0.05)	(0.95, 0.05, 0.05)	(0.65, 0.30, 0.45)	(0.10, 0.75, 0.85)	(0.50, 0.40, 0.60)
A10	(0.80, 0.10, 0.30)	(0.65, 0.30, 0.45)	(0.50, 0.40, 0.60)	(0.40, 0.70, 0.50)	(0.25, 0.60, 0.80)	(0.10, 0.75, 0.85)	(0.65, 0.30, 0.45)
	C_1	C_2	C_3	C_4	C_5	C_6	C_7
A1	(0.10, 0.75, 0.85)	(0.50, 0.40, 0.60)	(0.50, 0.40, 0.60)	(0.65, 0.30, 0.45)	(0.10, 0.75, 0.85)	(0.25, 0.60, 0.80)	(0.40, 0.70, 0.50)
A2	(0.25, 0.60, 0.80)	(0.65, 0.30, 0.45)	(0.50, 0.40, 0.60)	(0.40, 0.70, 0.50)	(0.50, 0.40, 0.60)	(0.65, 0.30, 0.45)	(0.10, 0.75, 0.85)
A3	(0.40, 0.70, 0.50)	(0.40, 0.70, 0.50)	(0.65, 0.30, 0.45)	(0.50, 0.40, 0.60)	(0.80, 0.10, 0.30)	(0.10, 0.75, 0.85)	(0.50, 0.40, 0.60)
A4	(0.50, 0.40, 0.60)	(0.25, 0.60, 0.80)	(0.40, 0.70, 0.50)	(0.65, 0.30, 0.45)	(0.50, 0.40, 0.60)	(0.25, 0.60, 0.80)	(0.65, 0.30, 0.45)
A5	(0.65, 0.30, 0.45)	(0.10, 0.75, 0.85)	(0.25, 0.60, 0.80)	(0.40, 0.70, 0.50)	(0.65, 0.30, 0.45)	(0.50, 0.40, 0.60)	(0.40, 0.70, 0.50)
A6	(0.40, 0.70, 0.50)	(0.25, 0.60, 0.80)	(0.10, 0.75, 0.85)	(0.25, 0.60, 0.80)	(0.40, 0.70, 0.50)	(0.65, 0.30, 0.45)	(0.25, 0.60, 0.80)
A7	(0.25, 0.60, 0.80)	(0.40, 0.70, 0.50)	(0.25, 0.60, 0.80)	(0.10, 0.75, 0.85)	(0.25, 0.60, 0.80)	(0.40, 0.70, 0.50)	(0.10, 0.75, 0.85)
A8	(0.10, 0.75, 0.85)	(0.50, 0.40, 0.60)	(0.40, 0.70, 0.50)	(0.50, 0.40, 0.60)	(0.10, 0.75, 0.85)	(0.25, 0.60, 0.80)	(0.40, 0.70, 0.50)
A9	(0.95, 0.05, 0.05)	(0.65, 0.30, 0.45)	(0.50, 0.40, 0.60)	(0.65, 0.30, 0.45)	(0.65, 0.30, 0.45)	(0.10, 0.75, 0.85)	(0.50, 0.40, 0.60)
A10	(0.80, 0.10, 0.30)	(0.65, 0.30, 0.45)	(0.65, 0.30, 0.45)	(0.40, 0.70, 0.50)	(0.25, 0.60, 0.80)	(0.10, 0.75, 0.85)	(0.65, 0.30, 0.45)
	C_1	C_2	C_3	C_4	C_5	C_6	C_7
A1	(0.25, 0.60, 0.80)	(0.65, 0.30, 0.45)	(0.50, 0.40, 0.60)	(0.65, 0.30, 0.45)	(0.65, 0.30, 0.45)	(0.25, 0.60, 0.80)	(0.65, 0.30, 0.45)
A2	(0.95, 0.05, 0.05)	(0.50, 0.40, 0.60)	(0.65, 0.30, 0.45)	(0.65, 0.30, 0.45)	(0.50, 0.40, 0.60)	(0.65, 0.30, 0.45)	(0.50, 0.40, 0.60)
A3	(0.80, 0.10, 0.30)	(0.40, 0.70, 0.50)	(0.50, 0.40, 0.60)	(0.50, 0.40, 0.60)	(0.65, 0.30, 0.45)	(0.50, 0.40, 0.60)	(0.40, 0.70, 0.50)
A4	(0.65, 0.30, 0.45)	(0.25, 0.60, 0.80)	(0.40, 0.70, 0.50)	(0.40, 0.70, 0.50)	(0.50, 0.40, 0.60)	(0.65, 0.30, 0.45)	(0.25, 0.60, 0.80)
A5	(0.50, 0.40, 0.60)	(0.10, 0.75, 0.85)	(0.25, 0.60, 0.80)	(0.25, 0.60, 0.80)	(0.40, 0.70, 0.50)	(0.50, 0.40, 0.60)	(0.65, 0.30, 0.45)
A6	(0.40, 0.70, 0.50)	(0.65, 0.30, 0.45)	(0.10, 0.75, 0.85)	(0.10, 0.75, 0.85)	(0.25, 0.60, 0.80)	(0.40, 0.70, 0.50)	(0.50, 0.40, 0.60)
A7	(0.25, 0.60, 0.80)	(0.50, 0.40, 0.60)	(0.25, 0.60, 0.80)	(0.40, 0.70, 0.50)	(0.10, 0.75, 0.85)	(0.25, 0.60, 0.80)	(0.40, 0.70, 0.50)
A8	(0.10, 0.75, 0.85)	(0.40, 0.70, 0.50)	(0.80, 0.10, 0.30)	(0.25, 0.60, 0.80)	(0.25, 0.60, 0.80)	(0.10, 0.75, 0.85)	(0.25, 0.60, 0.80)
A9	(0.95, 0.05, 0.05)	(0.25, 0.60, 0.80)	(0.95, 0.05, 0.05)	(0.95, 0.05, 0.05)	(0.65, 0.30, 0.45)	(0.25, 0.60, 0.80)	(0.10, 0.75, 0.85)
A10	(0.80, 0.10, 0.30)	(0.65, 0.30, 0.45)	(0.50, 0.40, 0.60)	(0.40, 0.70, 0.50)	(0.25, 0.60, 0.80)	(0.10, 0.75, 0.85)	(0.65, 0.30, 0.45)

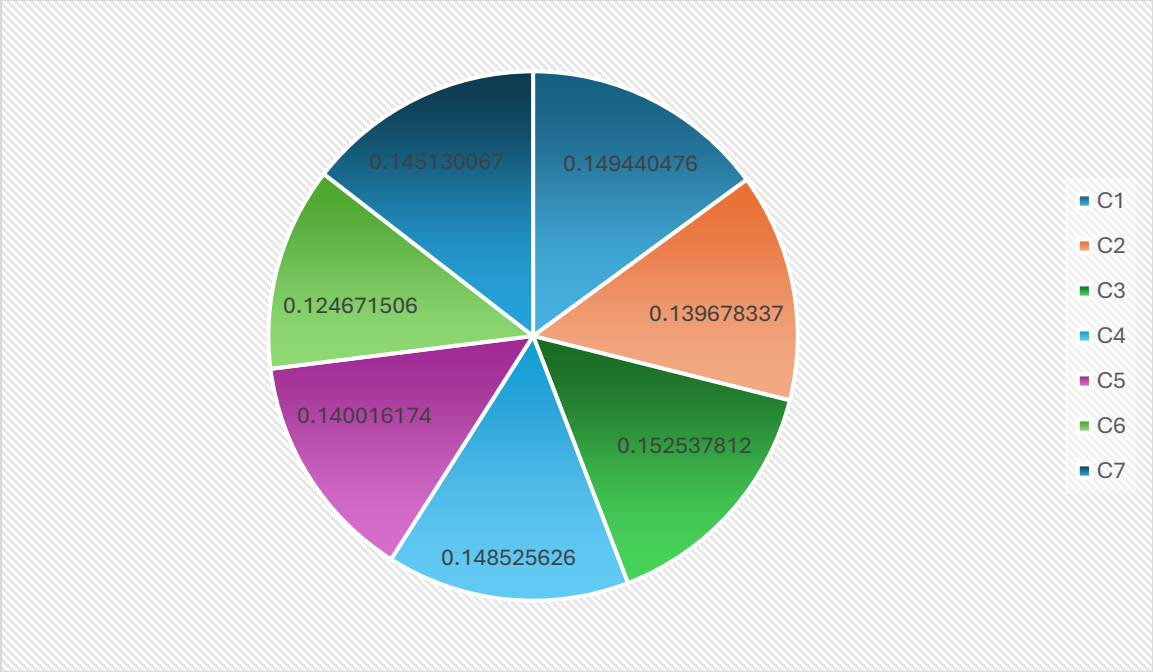

Figure 3. The criteria weights.

Table 2 shows the normalization matrix between the criteria and alternatives. All criteria are positive. Then we compute the total weighted comparability sequence. Then we compute the total power weight of comparability sequence. Table 3 shows the three strategies, evaluation values and rank of alternatives.

Table 2. The normalization matrix.

	C_1	C_2	C_3	C_4	C_5	C_6	C_7
A_1	3.512514	5.228634	4.370529	3.153639	3.773992	2.681197	2.956012
A_2	4.839082	6.490788	4.189064	2.398245	3.439026	6.354886	2.806568
A_3	5.184987	4.389221	5.480668	2.79267	4.76209	5.235165	3.534564
A_4	3.611458	4.87062	3.706154	3.04321	2.679726	6.49352	3.326676
A_5	4.429342	1.132449	4.112636	2.022434	4.064609	4.042958	3.750979
A_6	3.129505	5.425593	0.956213	2.214616	2.701228	6.132361	2.727815
A_7	2.110341	4.793758	3.537712	1.14177	2.957912	4.075399	2.933688
A_8	0.893489	4.389221	5.548984	2.467923	1.101322	4.462663	1.705272
A_9	6.251828	4.189734	6.314552	3.785524	4.212102	1.661587	2.472955
A_{10}	5.92228	6.339593	5.353002	1.973131	1.777133	1.135181	3.88709

Table 3. The rank of alternatives.

	Q_{ia}	Q_{ib}	Q_{ic}	Q_i	Rank
A_1	0.100429	2.319021	0.912252	1.707271	5
A_2	0.107791	2.551644	0.97913	1.85863	9
A_3	0.110089	2.618987	1	1.903656	10
A_4	0.10306	2.402133	0.936153	1.761364	7
A_5	0.095953	2.188695	0.871595	1.619868	4

A_6	0.093923	2.132633	0.853158	1.581503	3
A_7	0.091858	2.063553	0.834403	1.537405	2
A_8	0.089358	2	0.811688	1.492448	1
A_9	0.106335	2.508844	0.965906	1.830044	8
A_{10}	0.101204	2.358874	0.919291	1.729642	6

4. Sensitivity Analysis

We change the δ value between 0.1 and 1 to show the different ranks of alternatives. Table 4 shows the values of assessment of different δ. Then we rank the alternatives to show the sensitivity analysis results. The results show an alternative 3 has the highest rate and alternative 8 has the lowest rate. Figure 4 shows the rank of alternatives.

Table 4. The assessment value

	$\delta = 0.1$	$\delta = 0.2$	$\delta = 0.3$	$\delta = 0.4$	$\delta = 0.5$	$\delta = 0.6$	$\delta = 0.7$	$\delta = 0.8$	$\delta = 0.9$	$\delta = 1$
A_1	1.740403	1.734684	1.72856	1.721986	1.707271	1.699	1.690013	1.680213	1.66948	1.657675
A_2	1.864683	1.863632	1.86251	1.861308	1.85863	1.857133	1.855512	1.853751	1.851833	1.849735
A_3	1.903656	1.903656	1.903656	1.903656	1.903656	1.903656	1.903656	1.903656	1.903656	1.903656
A_4	1.784896	1.780826	1.776471	1.771801	1.761364	1.755509	1.749156	1.742238	1.734678	1.72638
A_5	1.664046	1.65644	1.648286	1.639523	1.619868	1.608794	1.596739	1.583562	1.569097	1.553138
A_6	1.62925	1.621037	1.612229	1.602759	1.581503	1.569516	1.556457	1.542173	1.526476	1.509137
A_7	1.594283	1.584519	1.57404	1.562762	1.537405	1.523076	1.50744	1.490303	1.471426	1.450515
A_8	1.551144	1.541074	1.530265	1.518628	1.492448	1.477644	1.461481	1.443754	1.42421	1.402537
A_9	1.839979	1.838256	1.836415	1.834442	1.830044	1.827583	1.824917	1.822021	1.818863	1.815406
A_{10}	1.752747	1.74875	1.744475	1.739889	1.729642	1.723893	1.717655	1.710864	1.703441	1.695293

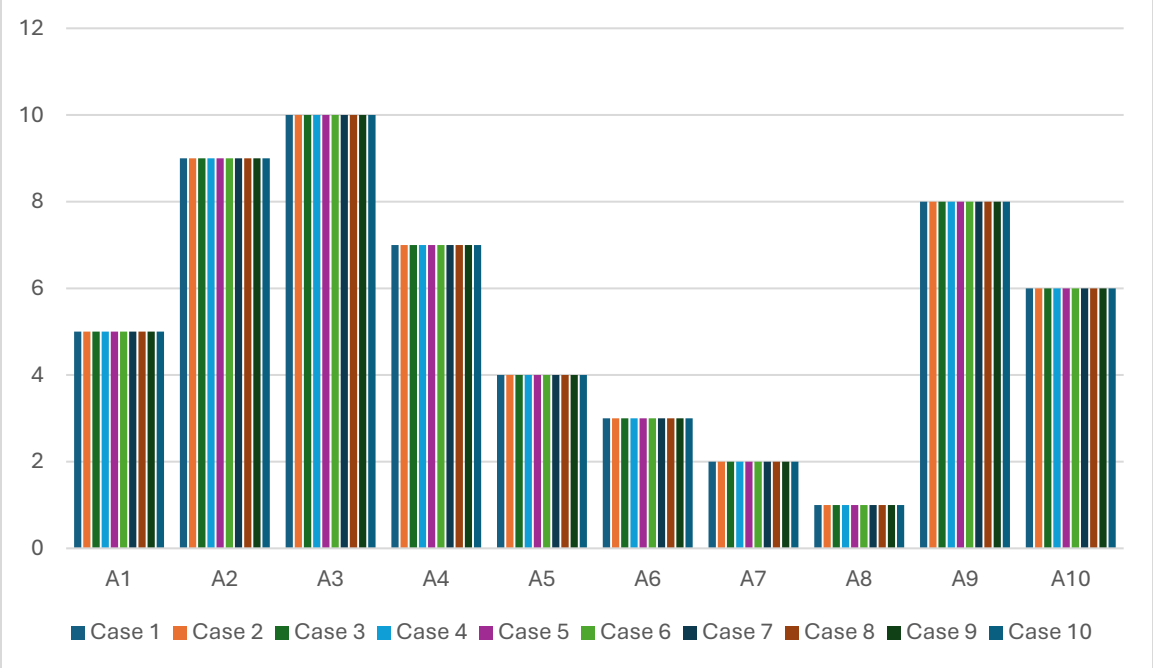

Figure 4. The rank of alternatives.

5. Conclusions

The Plithogenic CoCoSo approach is used in this study to evaluate and rank the different alternatives in students' colleges. Three professionals in the students' colleges provided Plithogenic data. The initial phases of the Plithogenic CoCoSo approach were used to aggregate expert-provided Plithogenic data, which were subsequently transformed into crisp data. These clear data are used to calculate the crisp importance values (weights) utilizing the last stages of Plithogenic.

Seven criteria and ten alternatives are used in this study. We normalize the decision matrix between the criteria and alternatives. Then we applied the steps of the proposed methodology under the Plithogenic sets. We show the alternative 3 has the highest rank and alternative 8 has the lowest rank. The sensitivity analysis results show that the proposed methodology is stable under different cases.

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Innovative Efficiency Measurement Using Stackelberg Games and Neutrosophic Numbers

Ali Asghar Tatlari[1], Bijan Rahmani Perchkolaei[1*], Farhad Hosseinzadeh Lotfi[2]
[1]Department of Industrial Engineering, Science and Research Branch, Islamic Azad University, Tehran, Iran; ali.tatlari@gmail.com.
[2] Department of Mathematics, Science and Research Branch, Islamic Azad University, Tehran, Iran;

* Correspondence: bijanrah40@gmail.com; Tel.: (09111936760)

Abstract: This paper addresses the challenge of measuring efficiency in systems with two components using Stackelberg game theory and neutrosophic numbers to handle data uncertainty. Unlike traditional Data Envelopment Analysis (DEA) models that treat Decision Making Units (DMUs) as black boxes, we propose a new neutrosophic Network DEA (NDEA) model to compute efficiency scores for each division. By applying Stackelberg game theory, we break down the system's efficiency scores into sub-system efficiencies, providing a more detailed assessment of performance. Our proposed model represents a significant contribution to the literature by developing a two-stage Network Slack Based Measure (NSBM) model. This model addresses conflicts arising from the dual role of intermediate measures and assumes a Stackelberg-game relationship between the two stages while ensuring continuity in the flow of links between them. Additionally, the model imposes a penalty on the follower's objective to discourage deviation from the leader's objectives, thereby enhancing feasibility robustness. To handle uncertainty within the model, we further refine it into a two-stage framework that leverages Pareto efficiency concepts to establish lower and upper bounds for DMU efficiencies. To validate our approach, we utilized data from the Iranian Airline Industry. This empirical study evaluates the efficiencies of 13 Iranian airlines, illustrating how our methodology effectively captures and quantifies operational efficiencies within uncertainty.

Keywords: Stackelberg game, neutrosophic numbers, data envelopment analysis, SBM

1. Introduction

Data Envelopment Analysis (DEA), developed by the revolutionary work of Charnes et al. [1], is a powerful tool for efficiency evaluation of peer decision-making units, commonly referred to as DMUs in the literature. These DMUs consume multiple inputs to produce multiple outputs. Traditional DEA models treat DMUs as black boxes and do not account for the internal structure of the DMUs. In simpler terms, traditional DEA models do not offer sufficient detail for the Decision Maker (DM) to identify the sources of inefficiency within the DMU that may arise from interactions among different stages of production [2]. To address this issue, scholars and researchers have developed Network DEA models to evaluate efficiency scores by considering the internal structure of DMUs. This has led to the formulation of intermediate measures that define the interaction between different stages of production. Intermediate products are produced from one stage and consumed by other stages during the production process. Therefore, this variable plays a dual role as both an output from one stage and an input to another. The earlier stage focuses on maximizing

its output efficiency, while the later stage emphasizes minimizing input. This dual role of intermediate measures introduces a challenge in evaluating efficiency. To address the conflicting nature of these measures, numerous scholars and researchers have proposed various approaches. For instance, Kao and Hwang [3] utilized a geometric multiplicative model, whereas Chen [4] opted for a weighted additive model. Broadly, the differences between models arise from the diverse assumptions regarding the incorporation of intermediate measures into constraints. In Kao's study [5] three categories were defined: independent, relational, and cooperative. The cooperative approach is further divided into two subtypes: one where the goal is set based on the current level and another aiming to optimize the goal. Kao [6] also described the relational scenario as non-cooperative. Moreover, Kao [6] discussed the cooperative scenario. Within academic literature, this approach is commonly recognized as the continuity condition, an idea innovatively introduced by Tone and Tsutsui [7]. Several researchers have expanded on the continuity concept, such as Tone and Tsutsui [7], to develop SBM network models for evaluating efficiency, including works by Gerami and Mozaffari [8]. Hassanzadeh and Mostafaee [9] proposed six scenarios for assessing intermediate measures in NDEA, focusing on controlling links at different stages. These scenarios include link control by earlier or later stages, both stages using non-cooperative methods, both stages adopting cooperative methods, or mixed approaches with cooperative and non-cooperative elements. Tone and Tsutsui [7] referred to link control methods with cooperative and free links at various stages.

Rasinojehdehi et al. [10] proposed a Network SBM model in which the role of intermediate measures is determined endogenously by the model to enhance the objective function.

According to the literature review conducted by Alves and Meza [11], game theory-based DEA approaches have been less explored in the DEA literature.

Game theory is rooted in the pioneering work of Von Neumann and Morgenstern [12]. It was further developed by John Forbes Nash who formulated the strategic interactions among rational DMs mathematically. Its applications span across various domains in social sciences, and it has been extensively utilized in fields associated with decision-making. Myerson [13] defined Game theory as the study of mathematical models for strategic interaction between rational DMs. In the following we review some studies in the literature based on the integration of game theory and DEA.

Liang et al. [14] proposed their two stage DEA model utilizing the concepts from noncooperative and cooperative games, illustrating with an example of a manufacturer and retailer scenario. Yu & Rakshit [15] integrated DEA and game theory in the airline industry to evaluate input and output targets. Their proposed model provides an impartial, rational negotiation between inputs and outputs, aiming to attain reasonable and optimal solutions for both inputs and outputs. Chang et al. [16] in their applied the first DEA-based Nash bargaining models to assist acquirer companies in acquiring their most desired target companies. Du et al. [17] applied Nash bargaining game model for measuring the efficiency of DMUs with two-stage structure. In their proposed model, the stages are treated as players negotiating for a better pay off, represented by the DEA ratio efficiency score. The approach is formulated as a cooperative game model. In their study they conclude that, in the presence of only one intermediate measure between the two stages, their proposed bargaining method produces same results as applying the standard DEA models to each stage separately; see also [18].

All the studies mentioned above are based on accurate data. In real-world scenarios, uncertainty is often an inseparable part of the available data. To address this issue, many scholars in the DEA literature applied fuzzy sets and established various fuzzy DEA approaches [19], [20]. Khalili-Damghani and Tavana [21] introduced a two-stage Stackelberg fuzzy DEA model to evaluate the efficiency of 52 chemistry departments in UK universities. Their framework integrates fuzzy data into the production process of a two-stage network, aiming to evaluate both system-level and process-level efficiencies for parallel systems functioning in a fuzzy environment using the Stackelberg game approach.

As highlighted by Voskoglou [22], no single universally optimal model exists for addressing uncertainty. The efficacy of a model is influenced by factors such as its structure, the data available,

and the knowledge already established about the issue being addressed. Fundamentally, uncertainty encompasses the complete set of information available in a situation. While fuzzy set theory has been instrumental in addressing vagueness or partial inclusion of elements within a set, there are still certain situations that cannot be effectively managed using fuzzy sets. To address these limitations, Zadeh [23] introduced concepts such as interval-valued fuzzy sets and interval-valued intuitionistic fuzzy sets, designed to handle imprecise, incomplete, and indeterminate data, which are prevalent in real-world scenarios.

Florentin Smarandache [24], a mathematician and Romanian-American writer, conceptualized neutrosophic sets in 1995 as a broad extension of fuzzy sets and intuitionistic fuzzy sets. Wang et al. [25] developed Single-Valued Neutrosophic Sets (SVNSs) as a distinct subset of neutrosophic sets, which has spurred substantial research efforts in recent years focusing on scientific problems rooted in SVNSs and their extensions. These sets align closely with human reasoning and address the gaps in incomplete knowledge that arise from the environment [26].

A notable illustration of the flexibility of these sets is their application in performance analysis, which has substantially improved the precision in modeling uncertainty. In 2018, Edalatpanah [27] pioneered the integration of neutrosophic input and output data into the DEA model, marking the first application of neutrosophic sets in this context. This represented a significant breakthrough, offering a novel approach to handle both uncertainty and indeterminacy in DEA analysis. Since then, the innovative model has gained widespread attention among researchers, who have applied it to address a broad range of uncertainty-related challenges across diverse fields, further enhancing its relevance and applicability in complex decision-making environments. Kahraman et al. [28] applied an integrated approach combining a neutrosophic version of AHP and DEA models to assess the performance of private universities in Turkey. Edalatpanah and Smarandache [29] further advanced the Neu-DEA model by transforming the input-oriented Neu-DEA framework into a corresponding crisp DEA model through the use of natural logarithms, enhancing the model's computational feasibility and practical application. Edalatpanah [30] further expanded the Neu-DEA model by integrating Triangular Neutrosophic Numbers (TNNs), enhancing the model's ability to address uncertainty and indeterminacy in input-output data. Rasinojehdehi and Valami [31] proposed a neutrosophic network DEA model to assess the performance in Airline industry.

El-Demerdash et al. [32] proposed a Neutrosophic Input-Oriented DEA (NIODEA) model to address the uncertainty and imprecision present in real-world data for performance evaluation. The model incorporates both neutrosophic and deterministic input/output variables using a scoring function, allowing for a more accurate assessment of Decision-Making Units under uncertain conditions. This approach overcomes the limitations of traditional DEA models, which assume deterministic inputs and outputs, and provides decision-makers with a tool to optimize resource allocation. The model's applicability is demonstrated through a practical example.

Mohanta et al. [33] propose a modified version of Khatter's methodology for tackling Neutrosophic Data Envelopment Analysis (Neu-DEA) by incorporating TNNs to handle imprecise and ambiguous data. The study redefines the possibility mean for TNNs and simplifies the Neu-DEA model into a crisp Linear Programming (LP) problem with a risk parameter representing the decision-maker's attitude towards risk. The approach ranks Decision-Making Units (DMUs) by determining their efficiency levels under various risk parameters, and its effectiveness is demonstrated through a numerical example.

In the literature on game theory-based network DEA, there's a noticeable absence of studies conducted with uncertain data, particularly within the neutrosophic environment. Also, the models that effectively combine game theory with network DEA while maintaining the continuity of linking activities among stages in an uncertain environment hasn't been developed yet. In this study, we propose a novel game theory-based NDEA approach within the SBM framework, following the continuity concept introduced by Tone and Tsutsui [7]. The proposed model contributes significantly to the literature by developing a two-stage network SBM model to address conflicts arising from the dual role of intermediate measures, assuming a Stackelberg-game relationship between the two

stages while maintaining the continuity flow of links between stages. Additionally, the proposed model imposes a penalty on the follower's objective to discourage deviation from the leader's objective. Another contribution of this paper is the incorporation of inefficiencies related to link flows into the objective function, as highlighted by Alves and Meza [11] as a significant area for future research in NSBM model development. Moreover, the incorporation of uncertain data and formulation of the model in a neutrosophic environment are further contributions of this paper. This allows the DM to handle the ambiguity and uncertainty inherent in real-world data, enhancing the robustness and applicability of the proposed approach.

The remainder of this paper is organized as follows. Section 2 presents dome backgrounds and fundamental concepts required for understanding this study. Section 3 presents the proposed method. Subsequently, in Section 4, a case study is presented to illustrate the application and validate the proposed model. Finally, Section 5 concludes the paper.

2. Preliminaries

In this section we review some fundamental concepts id neutrosophic logic and Stackelberg model.

2.1. Neutrosophic logic

A SVNS A on U is of the form presented in equation (1) [24]:

$$A = \{(x, T(x), I(x), F(x)) : x \in U, T(x), I(x), F(x) \in [0,1], 0 \leq T(x) + I(x) + F(x) \leq 3\}, \quad (1)$$

In equation (4), T(x), I(x), and F(x) represent the degrees of truth, indeterminacy, and falsity membership of x in A, respectively, which are referred to as the neutrosophic components of x. The term "neutrosophy" originates from the combination of the adjective "neutral" and the Greek word "sophia" (wisdom). According to its introducer, Smarandache, it denotes "the knowledge of neutral thought". In other words, Indeterminacy is generally understood to encompass everything that lies between the opposites of truth and falsity [34]. For instance, let the set U represents the students of a class, where A denotes the SVNS of the good students in U. Each student x in U can be characterized by a neutrosophic triplet (t, i, f) with respect to A, where t, i, and f lie within the range [0, 1]. if (0.7, 0.1, 0.4) ∈ A, then it implies that there is a 70% probability for student x to be classified as a good student, a 10% probability for x's status to be indeterminate regarding A, and a 40% probability for x not being considered a good student.

Edalatpanah & Smarandache [29] introduced the Triangular Neutrosophic Number (TNNs) $\tilde{Y} = \prec (r_1, r_2, r_3), (\omega, \theta, \chi) \succ$ with the following truth, indeterminacy, and falsehood membership functions of x as shown respectively in equations (2-a) to (2-c):

$$T_{\tilde{Y}}(x) = \begin{cases} \dfrac{(x - r_1)}{(r_2 - r_1)}\omega, & r_1 \leq x < r_2, \\ \omega, & x = r_2, \\ \dfrac{(r_3 - x)}{(r_3 - r_2)}\omega, & r_2 \leq x < r_3, \\ 0, & \text{otherwise.} \end{cases} \quad (2\text{-a})$$

$$I_{\tilde{Y}}(x) = \begin{cases} \dfrac{(r_2 - x)}{(r_2 - r_1)}\theta, & r_1 \leq x < r_2, \\ \theta, & x = r_2, \\ \dfrac{(x - r_3)}{(r_3 - r_2)}\theta, & r_2 \leq x < r_3, \\ 1, & \text{otherwise.} \end{cases} \quad (2\text{-b})$$

$$F_{\tilde{Y}}(x) = \begin{cases} \dfrac{(r_2 - x)}{(r_2 - r_1)}\chi, & r_1 \le x < r_2, \\ \chi, & x = r_2, \\ \dfrac{(x - r_3)}{(r_3 - r_2)}\chi, & r_2 \le x < r_3, \\ 1, & \text{otherwise.} \end{cases} \quad (2\text{-c})$$

Where, $0 \le T_{\tilde{Y}}(x) + I_{\tilde{Y}}(x) + F_{\tilde{Y}}(x) \le 3, x \in \tilde{Y}$.

Akram et al. (2019) introduced the aggregate coefficient $\varphi_{(\alpha,\beta,\gamma)}$ of $\tilde{Y} =\prec (r_1, r_2, r_3), (\omega, \theta, \chi) \succ$ which is a TNN as presented in equation (3).

$$\varphi_{(\alpha,\beta,\gamma)} = \left[\tilde{Y}^{L}_{(\alpha,\beta,\gamma)}, \tilde{Y}^{U}_{(\alpha,\beta,\gamma)}\right] = [r_1 + (r_2 - r_3)\hbar, r_3 - (r_3 - r_2)\hbar]. \quad (2\text{-c})$$

Where $\hbar$, is a parameter that reflects the decision-maker's attitude toward uncertainty

2.2. Stackelberg game

The Stackelberg model, introduced by Heinrich Freiherr von Stackelberg in 1934, is a strategic economic game involving a hierarchical setup where a leading firm and a following firm compete for market quantity, acting sequentially. Extensions to the original static two-person non-cooperative nonzero-sum Stackelberg game were made by Chen and Cruz [35] and Simaan and Cruz [36], who developed a dynamic version incorporating asymmetric information criteria. Aiyoshi and Shimizu [37] further proposed a generalized bi-level Stackelberg game structure, employing an interior penalty, or barrier method, to address non-linear min-max optimization challenges in such games.

Game theory serves as a widely recognized analytical tool for examining interactions among multiple participants. Its application spans numerous disciplines. Games can be categorized by features such as cooperation levels, simultaneity of actions, information availability, symmetry, and sum properties. This study centers on the Stackelberg game, a sequential non-cooperative game model characterized by perfect information.

In Stackelberg games, participants take actions in a predefined sequence. Typically, the first player to act is the leader, while the second player is referred to as the follower. The leader's decisions are often made with the follower's potential responses in mind. The follower, in turn, reacts to the leader's choices.

3. The Proposed Model

Without loss of generality, let us assume that the first stage takes on the role of the leader, thereby holding greater importance. In turn, the second stage acts as the follower, making decisions influenced by the leader's actions. Fig. 1 demonstrates the methodology adopted in this study, which is rooted in the Stackelberg game framework, to evaluate the efficiency scores of both the leader and the followers.

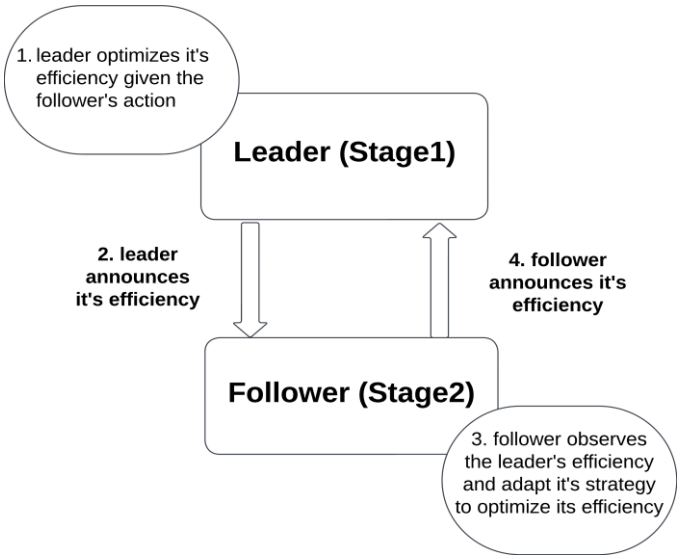
1. leader optimizes it's efficiency given the follower's action
Leader (Stage1)
2. leader announces it's efficiency
4. follower announces it's efficiency
Follower (Stage2)
3. follower observes the leader's efficiency and adapt it's strategy to optimize its efficiency

Figure 1. Methodology for measuring efficiency scores using the Stackelberg game.

In the first step, we optimize the leader's score without considering the followers, using the SBM model presented in equation (4). It is important to note that in this paper, we utilize a non-oriented model under the Constant Return to Scale (CRS) assumption for both the leader and follower.

$$\begin{aligned}
&\widetilde{\psi}_{\mathrm{p}}^{*1} = \min \frac{1-\frac{1}{\mathrm{m}_1}\left(\sum_{\mathrm{i}=1}^{\mathrm{m}_1} \frac{\tilde{\mathrm{s}}_{\mathrm{ip}}^{1-}}{\tilde{\mathrm{x}}_{\mathrm{ip}}^{1}}\right)}{1+\frac{1}{\mathrm{D}}\left(\sum_{\mathrm{d}=1}^{\mathrm{D}} \frac{\tilde{\mathrm{s}}_{\mathrm{dp}}^{(1,2)+}}{\tilde{\mathrm{z}}_{\mathrm{dp}}^{(1,2)}}\right)}, \\
&\sum_{\mathrm{j}=1}^{\mathrm{n}} \lambda_{\mathrm{j}}^{1} \mathrm{x}_{\mathrm{ij}}^{1} + \tilde{\mathrm{s}}_{\mathrm{ip}}^{1-} = \tilde{\mathrm{x}}_{\mathrm{ip}}^{1}, \forall \mathrm{i} \in \mathrm{I}_{\mathrm{k}}, \ \forall \mathrm{k}, \\
&\sum_{\mathrm{j}=1}^{\mathrm{n}} \lambda_{\mathrm{j}}^{1} \tilde{\mathrm{z}}_{\mathrm{dp}}^{(1,2)} - \tilde{\mathrm{s}}_{\mathrm{dp}}^{(1,2)+} = \tilde{\mathrm{z}}_{\mathrm{dp}}^{(1,2)} \text{as output from leader}, \ \forall \mathrm{d} \in \mathrm{l}_{(\mathrm{k},\mathrm{h})}, \forall (\mathrm{k},\mathrm{h}), \\
&\lambda_{\mathrm{j}}^{1} \geq 0 \ , \mathrm{s}_{\mathrm{ip}}^{1-} \geq 0 \ \geq 0, \mathrm{s}_{\mathrm{dp}}^{(1,2)+} \geq 0, \quad (\forall \mathrm{j},\mathrm{k}),
\end{aligned} \tag{4}$$

The model presented in equation (4) acts as a primary SBM model used for evaluating the efficiency of the leader, assumed to be the first stage, where $\tilde{s}_{ip}^{1-}$ and $\tilde{s}_{dp}^{(1,2)+}$ represent the optimal solutions. Given the interconnected nature of the two subprocesses through intermediate measures, and with the follower's decisions expected to mirror those of the leader, it is imperative to include $\tilde{s}_{ip}^{1-}$ and $\tilde{s}_{dp}^{(1,2)+}$ in measuring the efficiency of the follower. Now, we optimize the score of stage 2 while keeping the score of stage 1 at its optimal level ($\tilde{\psi}_p^1$=$\tilde{\psi}_p^{*1}$). Given the nature of the minimization problem, this constraint can be expressed as $\tilde{\psi}_p^1 \geq \tilde{\psi}_p^{*1}$. Model presented in equation (5) to equation (12) computes the score of stage 2.

$$\widetilde{\Psi}_{\mathrm{p}}^{*2} = \min \left\{ \frac{1-\frac{1}{\mathrm{m}_{2+\mathrm{D}}}\left(\sum_{\mathrm{i}=1}^{\mathrm{m}_1} \frac{\tilde{\mathrm{s}}_{\mathrm{ip}}^{2-}}{\tilde{\mathrm{x}}_{\mathrm{ip}}^{1}} + \sum_{\mathrm{d}=1}^{\mathrm{D}} \frac{\tilde{\mathrm{s}}_{\mathrm{dp}}^{(1,2)-}}{\tilde{\mathrm{z}}_{\mathrm{dp}}^{(1,2)}}\right)}{1+\frac{1}{\mathrm{s}}\left(\sum_{\mathrm{d}=1}^{\mathrm{D}} \frac{\tilde{\mathrm{s}}_{\mathrm{rp}}^{2+}}{\tilde{\mathrm{y}}_{\mathrm{rp}}^{2}}\right)} \right\} \tag{5}$$

$$\sum_{\mathrm{j}=1}^{\mathrm{n}} \lambda_{\mathrm{j}}^{1} \tilde{\mathrm{x}}_{\mathrm{ij}}^{1} + \tilde{\mathrm{s}}_{\mathrm{ip}}^{1-} = \tilde{\mathrm{x}}_{\mathrm{ip}}^{1}, \forall \mathrm{i} \in \mathrm{I}_{\mathrm{k}}, \ \forall \mathrm{k}, \tag{6}$$

$$\sum_{\mathrm{j}=1}^{\mathrm{n}} \lambda_{\mathrm{j}}^{2} \tilde{\mathrm{y}}_{\mathrm{rj}}^{2} - \tilde{\mathrm{s}}_{\mathrm{rp}}^{2+} = \tilde{\mathrm{y}}_{\mathrm{rp}}^{2}, \tag{7}$$

$$\sum_{j=1}^{n} \lambda_j^2 \tilde{z}_{jp}^{(1,2)} + g\tilde{s}_{dp}^{(1,2)-} = \tilde{z}_{dp}^{(1,2)}, \tag{8}$$

$$\sum_{j=1}^{n} \lambda_j^1 \tilde{z}_{jp}^{(1,2)} - (1-g)\tilde{s}_{dp}^{(1,2)+} = \tilde{z}_{dp}^{(1,2)}, \tag{9}$$

$$\sum_{j=1}^{n} \lambda_j^1 \tilde{z}_{dp}^{(1,2)} = \sum_{j=1}^{n} \lambda_j^2 \tilde{z}_{dp}^{(1,2)}, \tag{10}$$

$$\frac{1-\frac{1}{m_1}(\sum_{i=1}^{m_1} \frac{\tilde{s}_{ip}^{1-}}{\tilde{x}_{ij}^1})}{1+\frac{1}{D}(\sum_{d=1}^{D} \frac{\tilde{s}_{dp}^{(1,2)+}}{\tilde{z}_{dp}^{(1,2)}})} \geq \tilde{\psi}_p^{*1}, \quad \forall d \in I_{(k,h)}, \forall (k,h), \tag{11}$$

$$\lambda_j^1 \geq 0, \lambda_j^2 \geq 0 \ , \tilde{s}_{ip}^{1-} \geq 0 \ \geq 0, \tilde{s}_{dp}^{(1,2)+} \geq 0, S \geq 0, (\forall j, k), \tag{12}$$

Where g is a binary variable that determines the role of intermediate measure $\tilde{z}_d^{(1,2)}$ as either an input to the follower or an output from the leader. This enables the follower to improve its objective function independently of the leader's decisions and seek its optimum efficiency by just keeping the optimum efficiency of leader. With this assumption, the leader can anticipate the follower's best move before making its own

As mentioned earlier, the presented models operate under the Constant Returns-to-Scale (CRS) assumption. However, by incorporating relations $\sum_{j=1}^{n} \lambda_j^1 = 1$ into Model (4) and $\sum_{j=1}^{n} \lambda_j^1 = \sum_{j=1}^{n} \lambda_j^2 = 1$ into Model presented in equations (5-12), calculations can be changed to the Variable Returns-to-Scale (VRS) assumption.

To ensure the feasibility and robustness of the proposed model, we allow the follower a slight deviation from the leader's optimal strategy by imposing a penalty term on the follower's objective function. This penalty ensures that the leader's score closely matches the observed value. Any slight deviation from this, denoted by $S^{penalty} where\ S^{penalty} \geq 0$, incurs a penalty for the follower, with the severity determined by the factor π. This underscores the importance of closely aligning with the leader's performance. To implement this, we modify equations (11) to (13) and the objective function (5) to (14).

$$\frac{1-\frac{1}{m_1}(\sum_{i=1}^{m_1} \frac{\tilde{s}_{ip}^{1-}}{\tilde{x}_{ij}^1})}{1+\frac{1}{D}(\sum_{d=1}^{D} \frac{\tilde{s}_{dp}^{(1,2)+}}{\tilde{z}_{dp}^{(1,2)}})} + S^{penalty} \geq \tilde{\psi}_p^{*1}, \tag{13}$$

$$\tilde{\psi}_p^{*2} = \min \left\{ \frac{1 - \frac{1}{m_{2+D}} (\sum_{i=1}^{m_1} \frac{\tilde{s}_{ip}^{2-}}{\tilde{x}_{ip}^1} + \sum_{d=1}^{D} \frac{\tilde{s}_{dp}^{(1,2)-}}{\tilde{z}_{dp}^{(1,2)}}))}{1 + \frac{1}{s} (\sum_{d=1}^{D} \frac{\tilde{s}_{rp}^{2+}}{\tilde{y}_{rp}^2})} + \pi S^{penalty} \right\}, \tag{14}$$

Essentially, π serves as a factor determining the strength of the penalty applied to the follower's performance relative to the leader's efficiency. Increasing the value of π would result in a stronger penalty for the follower if they deviate from the leader's efficiency. On the other hand, decreasing the value of π would lead to a weaker penalty for such deviations.

Since all the data presented in the models for efficiency evaluation of both the leader and follower are of the neutrosophic data type, the resulting efficiency is also expressed as a neutrosophic number. Therefore, in the next section, we aim to convert the model into an interval-based model.

3.1. Converting the proposed model to interval-based model

As previously mentioned, the efficiency derived from the proposed model is expressed as a neutrosophic number. The neutrosophic values in the model are represented by convex neutrosophic numbers $\tilde{x}_{ij}^{tk}$, $\tilde{y}_{rj}^{tk}$ and $\tilde{z}_{dj}^{t(k,h)\beta}$, each characterized by truth, indeterminacy, and falsity membership functions as presented in equations (15) to (17):

$$\tilde{X}_{ij}^{q} = \left\{ \tilde{x}_{ij}^{q}, T_{\tilde{x}_{ij}^{tq}}, I_{\tilde{x}_{ij}^{tq}}, F_{\tilde{x}_{ij}^{tq}} \middle| \tilde{x}_{ij}^{tq} \in S(\tilde{x}_{ij}^{tq}) \right\}, i \in \{1,2\}, \tag{15}$$

$$\tilde{Y}_{rj}^{q} = \left\{ \tilde{y}_{rj}^{q}, T_{\tilde{y}_{rj}^{q}}, I_{\tilde{y}_{rj}^{q}}, F_{\tilde{y}_{rj}^{q}} \middle| \tilde{y}_{rj}^{q} \in S(\tilde{y}_{rj}^{tk}) \right\}, \tag{16}$$

$$\tilde{Z}_{dj}^{t(1,2)} = \left\{ \tilde{z}_{dj}^{t(1,2)}, T_{\tilde{z}_{dj}^{t(1,2)}}, I_{\tilde{z}_{dj}^{t(1,2)}}, F_{\tilde{z}_{dj}^{t(1,2)}} \middle| \tilde{z}_{dj}^{t(1,2)} \in S(\tilde{z}_{dj}^{t(1,2)}) \right\}. \tag{17}$$

Where S(.) denotes the support set of the neutrosophic number, it is sufficient to determine the boundaries of the aggregate coefficient introduced in Definition 4. Therefore, for all the data, the upper and lower boundaries of the neutrosophic variable in a φ variation degree can be measured as presented in equations (18)-(20):

$$(X_{ij}^{q})_{\varphi}^{L} \le x_{ij}^{q} \le (X_{ij}^{q})_{\varphi}^{U}, \tag{18}$$

$$(Y_{ij}^{q})_{\varphi}^{L} \le y_{ij}^{q} \le (Y_{ij}^{q})_{\varphi}^{U}, \tag{19}$$

$$(Z_{dj}^{(1,2)})_{\varphi}^{L} \le z_{dj}^{(1,2)} \le (Z_{dj}^{(1,2)})_{\varphi}^{U}, \tag{20}$$

By utilizing the φ variation degree of neutrosophic numbers, the neutrosophic model of the leader can be simply transformed into the two-stage programming models represented in equations (21) and (22).

$$\begin{gathered}(\psi_{p}^{1*})_{\varphi}^{U} = \max \\ (X_{ij}^{1})_{\varphi}^{L} \le x_{ij}^{1} \le (X_{ij}^{1})_{\varphi}^{U} \\ (Z_{dj}^{(1,2)})_{\varphi}^{L} \le z_{dj}^{(1,2)} \le (Z_{dj}^{(1,2)})_{\varphi}^{U} \\ \forall d, \forall i\end{gathered} = \begin{cases} \min \dfrac{1 - \frac{1}{m_1}\left(\sum_{i=1}^{m_1} \frac{s_{ip}^{1-}}{x_{ip}^{1}}\right)}{1 + \frac{1}{D}\left(\sum_{d=1}^{D} \frac{s_{dp}^{(1,2)+}}{z_{dp}^{(1,2)}}\right)} \\ \sum_{j=1}^{n} \lambda_j^1 x_{ij}^1 + s_{ip}^{1-} = x_{ip}^1 \\ \sum_{j=1}^{n} \lambda_j^1 z_{dp}^{(1,2)} - s_{dp}^{(1,2)+} = z_{dp}^{(1,2)} \\ \lambda_j^1 \ge 0, s_{ip}^{1-} \ge 0 \ge 0, s_{dp}^{(1,2)+} \ge 0 \end{cases}, \tag{21}$$

$$\begin{gathered}(\psi_{p}^{1*})_{\varphi}^{U} = \min \\ (X_{ij}^{1})_{\varphi}^{L} \le x_{ij}^{1} \le (X_{ij}^{1})_{\varphi}^{U} \\ (Z_{dj}^{(1,2)})_{\varphi}^{L} \le z_{dj}^{(1,2)} \le (Z_{dj}^{(1,2)})_{\varphi}^{U} \\ \forall d, \forall i\end{gathered} = \begin{cases} \min \dfrac{1 - \frac{1}{m_1}\left(\sum_{i=1}^{m_1} \frac{s_{ip}^{1-}}{x_{ip}^{1}}\right)}{1 + \frac{1}{D}\left(\sum_{d=1}^{D} \frac{s_{dp}^{(1,2)+}}{z_{dp}^{(1,2)}}\right)} \\ \sum_{j=1}^{n} \lambda_j^1 x_{ij}^1 + s_{ip}^{1-} = x_{ip}^1 \\ \sum_{j=1}^{n} \lambda_j^1 z_{dp}^{(1,2)} - s_{dp}^{(1,2)+} = z_{dp}^{(1,2)} \\ \lambda_j^1 \ge 0, s_{ip}^{1-} \ge 0 \ge 0, s_{dp}^{(1,2)+} \ge 0 \end{cases}, \tag{22}$$

Similarly, the neutrosophic model of the follower can be transformed into the two-stage programming models represented in equations (23) and (24).

$$
\begin{cases}
(\psi_p^{2*})_\varphi^U = \text{Max} \\
(X_{ij}^1)_\varphi^L \le x_{ij}^1 \le (X_{ij}^1)_\varphi^U \\
(Z_{dj}^{(1,2)})_\varphi^L \le z_{dj}^{(1,2)} \le (Z_{dj}^{(1,2)})_\varphi^U \\
\forall d\,, \forall i \\
\min\left\{ \dfrac{1-\frac{1}{m_2+D}\left(\sum_{i=1}^{m_1}\frac{s_{ip}^{2-}}{x_{ip}^1}+\sum_{d=1}^{D}\frac{s_{dp}^{(1,2)-}}{z_{dp}^{(1,2)}}\right)}{1+\frac{1}{s}\left(\sum_{d=1}^{D}\frac{s_{rp}^{2+}}{y_{rp}^2}\right)} \right\}, \\
\sum_{j=1}^{n}\lambda_j^1 x_{ij}^1 + s_{ip}^{1-} = x_{ip}^1, \\
\sum_{j=1}^{n}\lambda_j^2 y_{rj}^2 - s_{rp}^{2+} = y_{rp}^2, \\
\sum_{j=1}^{n}\lambda_j^1 z_{dp}^{(1,2)} - (1-g)s_{dp}^{(1,2)+} = z_{dp}^{(1,2)}, \\
\sum_{j=1}^{n}\lambda_j^2 z_{dp}^{(1,2)} + g s_{dp}^{(1,2)-} = z_{dp}^{(1,2)}, \\
\sum_{j=1}^{n}\lambda_j^1 z_{dp}^{(1,2)} = \sum_{j=1}^{n}\lambda_j^2 z_{dp}^{(1,2)}, \\
\dfrac{1-\frac{1}{m_1}\left(\sum_{i=1}^{m_1}\frac{s_{ip}^{1-}}{x_{ip}^1}\right)}{1+\frac{1}{D}\left(\sum_{d=1}^{D}\frac{s_{dp}^{(1,2)+}}{z_{dp}^{(1,2)}}\right)} \ge (\psi_p^{1*})_\varphi^U, \\
\lambda_j^1, \lambda_j^1 \ge 0\,;\, s_{ip}^{1-} \ge 0\,;\, s_{rp}^{2+} \ge 0, s_{dp}^{(1,2)+} \ge 0;\; s_{dp}^{(1,2)-}; g\epsilon\{0,1\}; S \ge 0,
\end{cases}
\tag{23}
$$

$$
\begin{cases}
(\psi_p^{2*})_\varphi^L = \text{Min} \\
(X_{ij}^1)_\varphi^L \le x_{ij}^1 \le (X_{ij}^1)_\varphi^U \\
(Z_{dj}^{(1,2)})_\varphi^L \le z_{dj}^{(1,2)} \le (Z_{dj}^{(1,2)})_\varphi^U \\
\forall d\,, \forall i \\
\min\left\{ \dfrac{1-\frac{1}{m_2+D}\left(\sum_{i=1}^{m_1}\frac{s_{ip}^{2-}}{x_{ip}^1}+\sum_{d=1}^{D}\frac{s_{dp}^{(1,2)-}}{z_{dp}^{(1,2)}}\right)}{1+\frac{1}{s}\left(\sum_{d=1}^{D}\frac{s_{rp}^{2+}}{y_{rp}^2}\right)} \right\}, \\
\sum_{j=1}^{n}\lambda_j^1 x_{ij}^1 + s_{ip}^{1-} = x_{ip}^1, \\
\sum_{j=1}^{n}\lambda_j^2 y_{rj}^2 - s_{rp}^{2+} = y_{rp}^2, \\
\sum_{j=1}^{n}\lambda_j^1 z_{dp}^{(1,2)} - (1-g)s_{dp}^{(1,2)+} = z_{dp}^{(1,2)}, \\
\sum_{j=1}^{n}\lambda_j^2 z_{dp}^{(1,2)} + g s_{dp}^{(1,2)-} = z_{dp}^{(1,2)}, \\
\sum_{j=1}^{n}\lambda_j^1 z_{dp}^{(1,2)} = \sum_{j=1}^{n}\lambda_j^2 z_{dp}^{(1,2)}, \\
\dfrac{1-\frac{1}{m_1}\left(\sum_{i=1}^{m_1}\frac{s_{ip}^{1-}}{x_{ip}^1}\right)}{1+\frac{1}{D}\left(\sum_{d=1}^{D}\frac{s_{dp}^{(1,2)+}}{z_{dp}^{(1,2)}}\right)} = (\psi_p^{1*})_\varphi^L, \\
\lambda_j^1, \lambda_j^1 \ge 0\,;\, s_{ip}^{1-} \ge 0\,;\, s_{rp}^{2+} \ge 0, s_{dp}^{(1,2)+} \ge 0;\; s_{dp}^{(1,2)-}; g\epsilon\{0,1\}; S \ge 0,
\end{cases}
\tag{24}
$$

By comparing the two-stage models for both leader and follower, it becomes clear that $(\psi_p^{1*})_\varphi^L \le (\psi_p^{1*})_\varphi^U$ and $(\psi_p^{2*})_\varphi^L \le (\psi_p^{2*})_\varphi^U$. Therefore, the intervals $\left[(\psi_p^{1*})_\varphi^U, (\psi_p^{1*})_\varphi^U\right]$ and $\left[(\psi_p^{2*})_\varphi^U, (\psi_p^{2*})_\varphi^U\right]$, represent the efficiency score boundaries for the leader and follower, respectively, at the degree of variation for φ. Following the concept of Pareto efficiency, for a $DMUp$ with a specific aggregate coefficient $\varphi_{(\alpha,\beta,\gamma)} = \varphi$, the upper limit of efficiency occurs when the $DMUp$ under evaluation minimizes its input and maximizes its output, while other DMUs maximize their input and minimize their output. Conversely, the lower limit occurs when the $DMUp$ maximizes its input and minimizes

its output, while other DMUs minimize their input and maximize their output. Models (25) and (26) measure the upper and lower bounds of the leader's efficiency at a given aggregate coefficient of φ, respectively, while Models (27) and (28) determine the lower and upper scores for the follower, respectively.

$$(\psi_p^{1*})_\varphi^U = \min \frac{1-\frac{1}{m_1}\left(\sum_{i=1}^{m_1}\frac{(s_{ip}^{1-})_\varphi^L}{(X_{ip}^1)_\varphi^L}\right)}{1+\frac{1}{D}\left(\sum_{d=1}^{D}\frac{(s_{dP}^{(1,2)+})_\varphi^U}{(Z_{dP}^{(1,2)})_\varphi^U}\right)},$$

$$\sum_{\substack{j=1\\ j\neq p}}^{n}\lambda_j^1(X_{ij}^1)_\varphi^U+\lambda_p^1(X_{ip}^1)_\varphi^L+(s_{ip}^{1-})_\varphi^L=(X_{ip}^1)_\varphi^L, \tag{25}$$

$$\sum_{\substack{j=1\\ j\neq p}}^{n}\lambda_j^1(Z_{dj}^{(1,2)})_\varphi^L+\lambda_p^1(Z_{dp}^{(1,2)})_\varphi^U-(s_{dp}^{(1,2)+})_\varphi^U=(Z_{dp}^{(1,2)})_\varphi^U,$$

$$\lambda_j^1\geq 0\,,(s_{ij}^{1-})_\varphi^L\geq 0,(s_{dP}^{(1,2)+})_\varphi^U\geq 0,$$

$$(\psi_p^{1*})_\varphi^L = \min \frac{1-\frac{1}{m_1}\left(\sum_{i=1}^{m_1}\frac{(s_{ip}^{1-})_\varphi^U}{(X_{ip}^1)_\varphi^U}\right)}{1+\frac{1}{D}\left(\sum_{d=1}^{D}\frac{(s_{dP}^{(1,2)+})_\varphi^L}{(Z_{dP}^{(1,2)})_\varphi^L}\right)},$$

$$\sum_{\substack{j=1\\ j\neq p}}^{n}\lambda_j^1(X_{ij}^1)_\varphi^L+\lambda_p^1(X_{ip}^1)_\varphi^U+(s_{ip}^{1-})_\varphi^U=(X_{ip}^1)_\varphi^U, \tag{26}$$

$$\sum_{\substack{j=1\\ j\neq p}}^{n}\lambda_j^1(Z_{dj}^{(1,2)})_\varphi^U+\lambda_p^1(Z_{dp}^{(1,2)})_\varphi^L-(s_{dp}^{(1,2)+})_\varphi^L=(Z_{dp}^{(1,2)})_\varphi^L,$$

$$\lambda_j^1\geq 0\,,(s_{ij}^{1-})_\varphi^U\geq 0,(s_{dP}^{(1,2)+})_\varphi^L\geq 0,$$

The models developed to calculate the efficiency boundaries for the leader and follower are nonlinear, particularly the models for the follower, which involve mixed integer nonlinear programming that is challenging to solve. The models for the leader can be easily transformed to the linear form by applying Charnes-Cooper transformation [38], but doing the same for the follower's models is challenging. In the following, we apply Charnes-Cooper transformation along with other techniques to change variables and linearize the models for evaluating the efficiency boundaries of the follower.

$$(\psi_p^{2*})_\varphi^U = \min\left\{\frac{1-\frac{1}{m_2+D}\left(\sum_{i=1}^{m_1}\frac{(s_{ij}^{1-})_\varphi^L}{(X_{ip}^1)_\varphi^L}+\sum_{d=1}^{D}\frac{(s_{dp}^{(1,2)-})_\varphi^L}{(Z_{dj}^{(1,2)})_\varphi^L}\right)}{1+\frac{1}{s}\left(\sum_{d=1}^{D}\frac{(s_{rp}^{2+})_\varphi^U}{(Y_{ip}^2)_\varphi^U}\right)}\right\},$$

$$\sum_{\substack{j=1\\ j\neq p}}^{n}\lambda_j^1(X_{ij}^1)_\varphi^U+\lambda_p^1(X_{ip}^1)_\varphi^L+(s_{ip}^{1-})_\varphi^L=(X_{ip}^1)_\varphi^L,$$

$$\sum_{\substack{j=1\\ j\neq p}}^{n}\lambda_j^2(Y_{rj}^2)_\varphi^L+\lambda_p^2(Y_{rp}^1)_\varphi^U-(s_{rp}^{2+})_\varphi^U=(Y_{rp}^2)_\varphi^U,$$

$$\sum_{\substack{j=1\\ j\neq p}}^{n}\lambda_j^1(Z_{dj}^{(1,2)})_\varphi^U+\lambda_p^1(Z_{dj}^{(1,2)})_\varphi^L+g(s_{dp}^{(1,2)-})_\varphi^L=(Z_{dj}^{(1,2)})_\varphi^L,$$

$$\sum_{\substack{j=1\\ j\neq p}}^{n}\lambda_j^2(Z_{dj}^{(1,2)})_\varphi^L+\lambda_p^2(Z_{dj}^{(1,2)})_\varphi^U-(1-g)(s_{dp}^{(1,2)+})_\varphi^U=(Z_{dj}^{(1,2)})_\varphi^U, \tag{27}$$

$$\sum_{\substack{j=1\\ j\neq p}}^{n}\lambda_j^1(Z_{dj}^{(1,2)})_\varphi^U+\lambda_p^1(Z_{dj}^{(1,2)})_\varphi^L=\sum_{\substack{j=1\\ j\neq p}}^{n}\lambda_j^2(Z_{dj}^{(1,2)})_\varphi^L+\lambda_p^2(Z_{dj}^{(1,2)})_\varphi^U,$$

$$\frac{1-\frac{1}{m_1}\left(\sum_{i=1}^{m_1}\frac{(s_{ip}^{1-})_\varphi^L}{(X_{ip}^1)_\varphi^L}\right)}{1+\frac{1}{D}\left(\sum_{d=1}^{D}\frac{(s_{dP}^{(1,2)+})_\varphi^U}{(Z_{dP}^{(1,2)})_\varphi^U}\right)}=(\psi_p^{1*})_\varphi^U,$$

$$\lambda_j^1,\lambda_j^2\geq 0\,;(s_{ip}^{1-})_\varphi^U\geq 0\,;(s_{ip}^{2+})_\varphi^L,(s_{dp}^{(1,2)-})_\varphi^U\geq 0;$$
$$(s_{dp}^{(1,2)+})_\varphi^L;g\epsilon\{0,1\};S^U\geq 0.$$

$$
\begin{aligned}
(\psi_p^{2*})_\varphi^L = \min\left\{\frac{1-\frac{1}{m_2+D}\left(\sum_{i=1}^{m_1}\frac{(s_{ij}^{1-})_\varphi^U}{(X_{ip}^1)_\varphi^U}+\sum_{d=1}^{D}\frac{(s_{dp}^{(1,2)-})_\varphi^U}{(Z_{dj}^{(1,2)})_\varphi^U}\right)}{1+\frac{1}{s}\left(\sum_{d=1}^{D}\frac{(s_{rp}^{2+})_\varphi^L}{(Y_{ip}^2)_\varphi^L}\right)}\right\},\\
\sum_{\substack{j=1\\ j\neq p}}^{n}\lambda_j^1(X_{ij}^1)_\varphi^L+\lambda_p^1(X_{ip}^1)_\varphi^U+(s_{ip}^{1-})_\varphi^U=(X_{ip}^1)_\varphi^U,\\
\sum_{\substack{j=1\\ j\neq p}}^{n}\lambda_j^2(Y_{rj}^2)_\varphi^U+\lambda_p^2(Y_{rp}^1)_\varphi^L-(s_{rp}^{2+})_\varphi^L=(Y_{rp}^2)_\varphi^L,\\
\sum_{\substack{j=1\\ j\neq p}}^{n}\lambda_j^1(Z_{dj}^{(1,2)})_\varphi^L+\lambda_p^1(Z_{dj}^{(1,2)})_\varphi^U+g(s_{dp}^{(1,2)-})_\varphi^U=(Z_{dj}^{(1,2)})_\varphi^U,\\
\sum_{\substack{j=1\\ j\neq p}}^{n}\lambda_j^2(Z_{dj}^{(1,2)})_\varphi^U+\lambda_p^2(Z_{dj}^{(1,2)})_\varphi^L-(1-g)(s_{dp}^{(1,2)+})_\varphi^L=(Z_{dj}^{(1,2)})_\varphi^L,\\
\sum_{\substack{j=1\\ j\neq p}}^{n}\lambda_j^1(Z_{dj}^{(1,2)})_\varphi^L+\lambda_p^1(Z_{dj}^{(1,2)})_\varphi^U=\sum_{\substack{j=1\\ j\neq p}}^{n}\lambda_j^2(Z_{dj}^{(1,2)})_\varphi^U+\lambda_p^2(Z_{dj}^{(1,2)})_\varphi^L,\\
\frac{1-\frac{1}{m_1}\left(\sum_{i=1}^{m_1}\frac{(s_{ip}^{1-})_\varphi^U}{(X_{ip}^1)_\varphi^U}\right)}{1+\frac{1}{D}\left(\sum_{d=1}^{D}\frac{(s_{dP}^{(1,2)+})_\varphi^L}{(Z_{dP}^{(1,2)})_\varphi^L}\right)}=(\psi_p^{1*})_\varphi^L,\\
\lambda_j^1,\lambda_j^2\geq 0\,;\,(s_{ip}^{1-})_\varphi^U\geq 0\,;\,(s_{ip}^{2+})_\varphi^L,(s_{dp}^{(1,2)-})_\varphi^U\geq 0;\;(s_{dp}^{(1,2)+})_\varphi^L;\\
g\epsilon\{0,1\}.
\end{aligned}
\tag{28}
$$

3.2. Linearization of the proposed models

As mentioned earlier, linearizing the follower's models is challenging. In this subsection, we utilize the Charnes-Cooper transformation and some other techniques to change variables and linearize the models for evaluating the follower's efficiency boundaries. We linearize the model for measuring the lower bound score of the follower. The same approach can be applied to the leader. To do this, let's define the variable t as in equation (29):

$$
\frac{1}{1+\frac{1}{s}\left(\sum_{d=1}^{D}\frac{t(s_{rp}^{2+})_\varphi^L}{(Y_{ip}^2)_\varphi^L}\right)}=t,
\tag{29}
$$

Then the model outlined in equation (28) can be rewritten as presented in equation (30).

$$(\psi_p^{2*})_\varphi^L = \min\left\{t - \frac{1}{m_2 + D}\left(\sum_{i=1}^{m_1} \frac{t(s_{ij}^{2-})_\varphi^U}{(X_{ip}^2)_\varphi^U} + \sum_{d=1}^{D} \frac{t(s_{dp}^{(1,2)-})_\varphi^U}{(Z_{dj}^{(1,2)})_\varphi^U}\right)\right\},$$

$$t + \frac{1}{s}\left(\sum_{d=1}^{D} \frac{t(s_{rp}^{2+})_\varphi^L}{(Y_{ip}^2)_\varphi^L}\right) = 1,$$

$$\sum_{\substack{j=1 \\ j\neq p}}^{n} t\lambda_j^1 (X_{ij}^1)_\varphi^L + t\lambda_p^1 (X_{ip}^1)_\varphi^U + t(s_{ip}^{1-})_\varphi^U = t(X_{ip}^1)_\varphi^U,$$

$$\sum_{\substack{j=1 \\ j\neq p}}^{n} t\lambda_j^2 (Y_{rj}^2)_\varphi^U + t\lambda_p^2 (Y_{rp}^1)_\varphi^L - t(s_{rp}^{2+})_\varphi^L = t(Y_{rp}^2)_\varphi^L,$$

$$\sum_{\substack{j=1 \\ j\neq p}}^{n} t\lambda_j^1 (Z_{dj}^{(1,2)})_\varphi^L + t\lambda_p^1 (Z_{dj}^{(1,2)})_\varphi^U + gt(s_{dp}^{(1,2)-})_\varphi^U = t(Z_{dj}^{(1,2)})_\varphi^U, \tag{30}$$

$$\sum_{\substack{j=1 \\ j\neq p}}^{n} t\lambda_j^2 (Z_{dj}^{(1,2)})_\varphi^U + t\lambda_p^2 (Z_{dj}^{(1,2)})_\varphi^L - (1-g)t(s_{dp}^{(1,2)+})_\varphi^L = t(Z_{dj}^{(1,2)})_\varphi^L,$$

$$\sum_{\substack{j=1 \\ j\neq p}}^{n} t\lambda_j^1 (Z_{dj}^{(1,2)})_\varphi^L + t\lambda_p^1 (Z_{dj}^{(1,2)})_\varphi^U = \sum_{\substack{j=1 \\ j\neq p}}^{n} t\lambda_j^2 (Z_{dj}^{(1,2)})_\varphi^U + t\lambda_p^2 (Z_{dj}^{(1,2)})_\varphi^L,$$

$$t - \frac{1}{m_1}\left(\sum_{i=1}^{m_1} \frac{t(s_{ip}^{1-})_\varphi^U}{(X_{ip}^1)_\varphi^U}\right) - (\psi_p^{1*})_\varphi^L \left(t + \frac{1}{D}\left(\sum_{d=1}^{D} \frac{t(s_{dP}^{(1,2)+})_\varphi^L}{(Z_{dP}^{(1,2)})_\varphi^L}\right)\right) \geq 0,$$

$$\lambda_j^1, \lambda_j^2 \geq 0\ ;\ (s_{ip}^{1-})_\varphi^U \geq 0\ ;\ (s_{ip}^{2+})_\varphi^L, (s_{dp}^{(1,2)-})_\varphi^U \geq 0;\ (s_{dp}^{(1,2)+})_\varphi^L,$$

$$g \epsilon \{0,1\},$$

In the next step, we replace the nonlinear terms $g(S_{dp}^{(1,2)-})_\varphi^U$ and $(1-g)(S_{dp}^{(1,2)+})_\varphi^L$ respectively by $(h_{dp}^{(1,2)-})_\varphi^U$ and $(h_{dp}^{(1,2)+})_\varphi^U$ .Then we add the constraints (31)-(36) to the problem.

$$(h_{dp}^{(1,2)-})_\varphi^U = g(S_{dp}^{(1,2)-})_\varphi^U, \tag{31}$$

$$(h_{dp}^{(1,2)+})_\varphi^U = (1-g)(S_{dp}^{(1,2)+})_\varphi^L, \tag{32}$$

$$(S_{dp}^{(1,2)-})_\varphi^U - M(1-g) \leq (h_{dp}^{(1,2)-})_\varphi^U \leq (S_{dp}^{(1,2)-})_\varphi^U + M(1-g), \tag{33}$$

$$(h_{dp}^{(1,2)-})_\varphi^U \leq Mg, \tag{34}$$

$$(S_{dp}^{(1,2)+})_\varphi^U - Mg \leq (h_{dp}^{(1,2)+})_\varphi^U \leq (S_{dp}^{(1,2)+})_\varphi^U + Mg, \tag{35}$$

$$(h_{dp}^{(1,2)+})_\varphi^U \leq M(1-g), \tag{36}$$

Now, in the final stage of linearization, we define $t(s_{rp}^{2+})_\alpha^U = (S_{rp}^{2+})_\alpha^U$, $t(s_{dp}^{(1,2)-})_\alpha^L = (S_{dp}^{(1,2)-})_\alpha^L$, $t(s_{dp}^{(1,2)+})_\alpha^U = (S_{dp}^{(1,2)+})_\alpha^U$, $t(s_{ip}^{1-})_\alpha^L = (S_{ip}^{1-})_\alpha^L$، $t\lambda_j^1 = \Lambda_j^1$, $t\lambda_j^2 = \Lambda_j^2$ and substitute them into the model. Consequently, the equivalent linear representation of the model is obtained, as presented in Model (37).

$$
\begin{aligned}
&(\psi_p^{2*})_\varphi^L = \min\left\{t - \frac{1}{m_2 + D}\left(\sum_{i=1}^{m_1}\frac{(S_{ij}^{2-})_\varphi^U}{(X_{ip}^2)_\varphi^U} + \sum_{d=1}^{D}\frac{(S_{dp}^{(1,2)-})_\varphi^U}{(Z_{dj}^{(1,2)})_\varphi^U}\right)\right\}\\
&t + \frac{1}{s}\left(\sum_{d=1}^{D}\frac{(S_{rp}^{2+})_\varphi^L}{(Y_{ip}^2)_\varphi^L}\right) = 1,\\
&\text{s.t.}\sum_{\substack{j=1\\ j\neq p}}^{n}\Lambda_j^1(X_{ij}^1)_\varphi^L + \Lambda_p^1(X_{ip}^1)_\varphi^U + (S_{ip}^{1-})_\varphi^U = t(X_{ip}^1)_\varphi^U,\\
&\sum_{\substack{j=1\\ j\neq p}}^{n}\Lambda_j^2(Y_{rj}^2)_\varphi^U + \Lambda_p^2(Y_{rp}^1)_\varphi^L - (S_{rp}^{2+})_\varphi^L = t(Y_{rp}^2)_\varphi^L,\\
&\sum_{\substack{j=1\\ j\neq p}}^{n}\Lambda_j^1(Z_{dj}^{(1,2)})_\varphi^L + \Lambda_p^1(Z_{dj}^{(1,2)})_\varphi^U + (h_{dp}^{(1,2)-})_\varphi^U = t(Z_{dj}^{(1,2)})_\varphi^U,\\
&\sum_{\substack{j=1\\ j\neq p}}^{n}\Lambda_j^2(Z_{dj}^{(1,2)})_\varphi^U + \Lambda_p^2(Z_{dj}^{(1,2)})_\varphi^L - (h_{dp}^{(1,2)+})_\varphi^L = t(Z_{dj}^{(1,2)})_\varphi^L\\
&(S_{dp}^{(1,2)-})_\varphi^U - M(1-g) \le (h_{dp}^{(1,2)-})_\varphi^U \le (S_{dp}^{(1,2)-})_\varphi^U + M(1-g),\\
&(h_{dp}^{(1,2)-})_\varphi^U \le Mg,\\
&(S_{dp}^{(1,2)+})_\varphi^U - Mg \le (h_{dp}^{(1,2)+})_\varphi^U \le (S_{dp}^{(1,2)+})_\varphi^U + Mg,\\
&(h_{dp}^{(1,2)+})_\varphi^U \le M(1-g),\\
&\sum_{\substack{j=1\\ j\neq p}}^{n}\Lambda_j^1(Z_{dj}^{(1,2)})_\varphi^L + \Lambda_p^1(Z_{dj}^{(1,2)})_\varphi^U = \sum_{\substack{j=1\\ j\neq p}}^{n}\Lambda_j^2(Z_{dj}^{(1,2)})_\varphi^U + \Lambda_p^2(Z_{dj}^{(1,2)})_\varphi^L,\\
&t - \frac{1}{m_1}\left(\sum_{i=1}^{m_1}\frac{(S_{ip}^{1-})_\varphi^U}{(X_{ip}^1)_\varphi^U}\right) - (\psi_p^{1*})_\varphi^L\left(t + \frac{1}{D}\left(\sum_{d=1}^{D}\frac{(S_{dP}^{(1,2)+})_\varphi^L}{(Z_{dP}^{(1,2)})_\varphi^L}\right)\right) = 0,\\
&\Lambda_j^1, \Lambda_j^2 \ge 0\ ;\ (S_{ip}^{1-})_\varphi^U \ge 0\ ;\ (S_{ip}^{2+})_\varphi^L, (S_{dp}^{(1,2)-})_\varphi^U \ge 0;\ (S_{dp}^{(1,2)+})_\varphi^L,\\
&g\epsilon\{0,1\},
\end{aligned}
\tag{37}
$$

As discussed earlier, to ensure the feasibility robustness of the proposed model, we can impose a penalty on the follower for any deviation from the leader's efficiency, which is presented in Model (38).

$$
\begin{aligned}
(\psi_p^{2*})_\varphi^L = \min\left\{ t - \frac{1}{m_2 + D}\left(\sum_{i=1}^{m_1} \frac{(S_{ij}^{2-})_\varphi^U}{(X_{ip}^2)_\varphi^U} + \sum_{d=1}^{D} \frac{(S_{dp}^{(1,2)-})_\varphi^U}{(Z_{dj}^{(1,2)})_\varphi^U}\right) + \pi S^{penalty} \right\},\\
t + \frac{1}{s}\left(\sum_{d=1}^{D} \frac{(S_{rp}^{2+})_\varphi^L}{(Y_{ip}^2)_\varphi^L}\right) = 1,\\
s.t. \sum_{\substack{j=1\\ j\neq p}}^{n} \Lambda_j^1 (X_{ij}^1)_\varphi^L + \Lambda_p^1 (X_{ip}^1)_\varphi^U + (S_{ip}^{1-})_\varphi^U = t(X_{ip}^1)_\varphi^U,\\
\sum_{\substack{j=1\\ j\neq p}}^{n} \Lambda_j^2 (Y_{rj}^2)_\varphi^U + \Lambda_p^2 (Y_{rp}^1)_\varphi^L - (S_{rp}^{2+})_\varphi^L = t(Y_{rp}^2)_\varphi^L,\\
\sum_{\substack{j=1\\ j\neq p}}^{n} \Lambda_j^1 (Z_{dj}^{(1,2)})_\varphi^L + \Lambda_p^1 (Z_{dj}^{(1,2)})_\varphi^U + (h_{dp}^{(1,2)-})_\varphi^U = t(Z_{dj}^{(1,2)})_\varphi^U,\\
\sum_{\substack{j=1\\ j\neq p}}^{n} \Lambda_j^2 (Z_{dj}^{(1,2)})_\varphi^U + \Lambda_p^2 (Z_{dj}^{(1,2)})_\varphi^L - (h_{dp}^{(1,2)+})_\varphi^L = t(Z_{dj}^{(1,2)})_\varphi^L,\\
(S_{dp}^{(1,2)-})_\varphi^U - M(1-g) \le (h_{dp}^{(1,2)-})_\varphi^U \le (S_{dp}^{(1,2)-})_\varphi^U + M(1-g),\\
(h_{dp}^{(1,2)-})_\varphi^U \le Mg,\\
(S_{dp}^{(1,2)+})_\varphi^U - Mg \le (h_{dp}^{(1,2)+})_\varphi^U \le (S_{dp}^{(1,2)+})_\varphi^U + Mg,\\
(h_{dp}^{(1,2)+})_\varphi^U \le M(1-g),\\
\sum_{\substack{j=1\\ j\neq p}}^{n} \Lambda_j^1 (Z_{dj}^{(1,2)})_\varphi^L + \Lambda_p^1 (Z_{dj}^{(1,2)})_\varphi^U = \sum_{\substack{j=1\\ j\neq p}}^{n} \Lambda_j^2 (Z_{dj}^{(1,2)})_\varphi^U + \Lambda_p^2 (Z_{dj}^{(1,2)})_\varphi^L,\\
t - \frac{1}{m_1}\left(\sum_{i=1}^{m_1} \frac{(S_{ip}^{1-})_\varphi^U}{(X_{ip}^1)_\varphi^U}\right) - (\psi_p^{1*})_\varphi^L \left(t + \frac{1}{D}\left(\sum_{d=1}^{D} \frac{(S_{dP}^{(1,2)+})_\varphi^L}{(Z_{dP}^{(1,2)})_\varphi^L}\right)\right) - S^{penalty} = 0,\\
\Lambda_j^1, \Lambda_j^2 \ge 0\ ;\ (S_{ip}^{1-})_\varphi^U \ge 0\ ;\ (S_{ip}^{2+})_\varphi^L, (S_{dp}^{(1,2)-})_\varphi^U \ge 0;\ (S_{dp}^{(1,2)+})_\varphi^L,\\
g \epsilon \{0,1\}; S^{penalty} \ge 0.
\end{aligned}
\tag{37}
$$

Increasing the value of π would result in a stronger penalty for the follower if they deviate from the leader's strategy. On the other hand, decreasing the value of π would lead to a weaker penalty for such deviations. Essentially, π serves as a factor determining the strength of the penalty applied to the follower's performance relative to the leader's efficiency.

In this section, we applied the leader-follower Stackelberg game to propose a novel network SBM model in a neutrosophic environment. To address the inherent uncertainty within the model, we convert it into a two-stage programming framework. By employing the concept of Pareto efficiency, we develop the upper and lower levels of efficiency within the φ variation degree of neutrosophic numbers. Subsequently, the models are transformed into linear programming to reduce complexity. In the next section, we present a case study to validate the proposed model.

4. An Application in the Airline Industry

In this section, we utilize the proposed model to evaluate the efficiency of airlines in Iran. Initially, we explore the interconnected divisions of these airlines, structured as a network. Subsequently, we introduce and implement the model under investigation. Finally, we present and analyze the obtained results. In the following we use the example of Rasinojehdehi and Bagherzadeh Valami [10] to illustrate our methodology.

In Rasinojehdehi and Bagherzadeh Valami [10] the production and consumption efficiencies of 13 Iranian airlines are investigated. The production division (Division 1) utilizes inputs such as the

number of employees and number of aircraft to produce Tons-kilometers supplied per year and seat-kilometers supplied per year. These outputs serve as intermediate inputs for the consumption division. In the consumption division (Division 2), airlines utilize external inputs such as Airport coverage on domestic flights and Airport coverage on foreign flights, along with the intermediate inputs from the production division. From these inputs, they produce Passenger share of foreign flights, Passenger share of domestic flights, Ton-kilometers performed, and Passenger-kilometers performed. The results are summarized in Table 1, which displays the lower and upper bounds of efficiency scores for the leader and follower at various levels of truth, indeterminacy, and falsity. In this context, the production division is designated as the leader, with the consumption division serving as the follower. The results are obtained under constant return to scale assumption. It should be emphasized that, the penalty for deviation from the leader's efficiency is set to 10 ($\pi = 10$). This parameter underscores the significant consequence imposed on deviations from the optimal efficiency established by the leader.

Table 1. Scores of the leader and follower at different levels of uncertainty.

Level of Truth, indeterminacy and falsity	$\boldsymbol{\varphi_{(0,0.1,0.1)}}$				$\boldsymbol{\varphi_{(0.2,0.7,0.9)}}$				$\boldsymbol{\varphi_{(0.5,0.8,0.9)}}$			
DMU	E_L^-	E_L^+	E_F^-	E_F^+	E_L^-	E_L^+	E_F^-	E_F^+	E_L^-	E_L^+	E_F^-	E_F^+
DMU1	0.69	1.00	0.79	1.00	0.74	1.00	0.79	1.00	0.83	1.00	0.88	1.00
DMU2	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
DMU3	0.66	0.83	0.74	0.81	0.76	0.83	0.74	0.81	0.85	0.93	0.94	1.00
DMU4	0.89	1.00	1.00	1.00	0.91	1.00	1.00	1.00	0.91	1.00	1.00	1.00
DMU5	0.94	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
DMU6	0.24	0.39	0.31	0.42	0.36	0.49	0.51	0.68	0.44	0.59	0.58	0.79
DMU7	0.55	0.69	0.63	0.75	0.65	0.79	0.68	0.85	0.75	0.83	0.79	0.95
DMU8	0.52	0.90	0.68	1.00	0.56	1.00	0.79	1.00	0.65	1.00	0.81	1.00
DMU9	0.66	0.84	0.74	1.00	0.74	0.94	0.83	1.00	0.79	1.00	0.88	1.00
DMU10	0.70	0.85	0.83	0.95	0.81	0.88	0.91	0.95	1.00	1.00	1.00	1.00
DMU11	0.89	1.00	1.00	1.00	0.94	1.00	1.00	1.00	1.00	1.00	1.00	1.00
DMU12	0.76	1.00	1.00	1.00	0.83	1.00	1.00	1.00	1.00	1.00	1.00	1.00
DMU13	0.88	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
Average	0.722	0.885	0.825	0.918	0.792	0.918	0.865	0.945	0.863	0.950	0.914	0.980

In Table 1, for each DMU, efficiency is measured through both lower and upper bounds for the Leader (E_L^- and E_L^+) and Follower ($E_F^-$and E_F^+) divisions, providing a nuanced view of performance across various operational scenarios. According to the results, DMU2 consistently achieves perfect efficiency scores of 1.00 across all φ values, demonstrating exceptionally stable performance where both lower and upper bounds align perfectly. In contrast, DMU6 shows more variability in efficiency scores, especially at higher levels of indeterminacy and falsity ($\varphi_{(0.5,0.8,0.9)}$). This variability suggests opportunities for operational adjustments to enhance performance reliability. DMU6 notably has the lowest scores for both the Leader and Follower divisions. Specifically, at $\varphi_{(0,\mathbf{0.1},\mathbf{0.1})}$, DMU6 scored 0.24 for the Leader division and 0.31 for the Follower division at the lower bound. These scores

indicate significant challenges for DMU6 in achieving efficiency compared to other units in the study. These findings underscore the importance for DMU6 to address operational issues, improve resource management, and enhance overall performance. By doing so, DMU6 can better align with efficiency targets and improve its competitiveness in the studied environment. Fig. 2 to Fig. 5 compare the scores of the leader and follower at different levels of uncertainty.

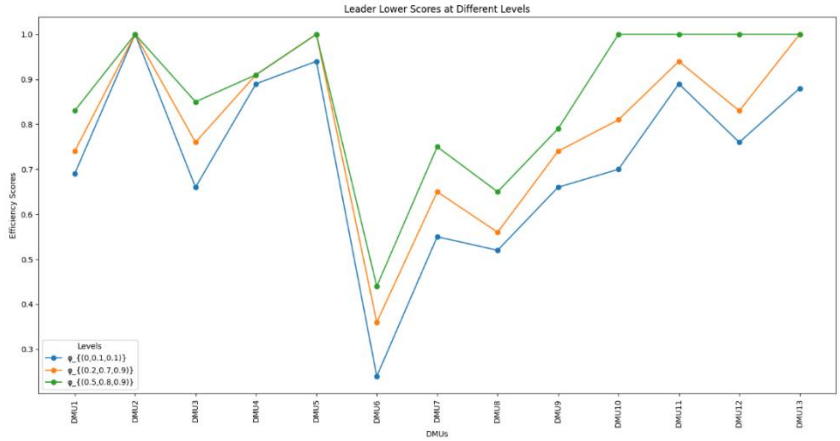

Figure 2. Leader's lower score at different levels of uncertainty.

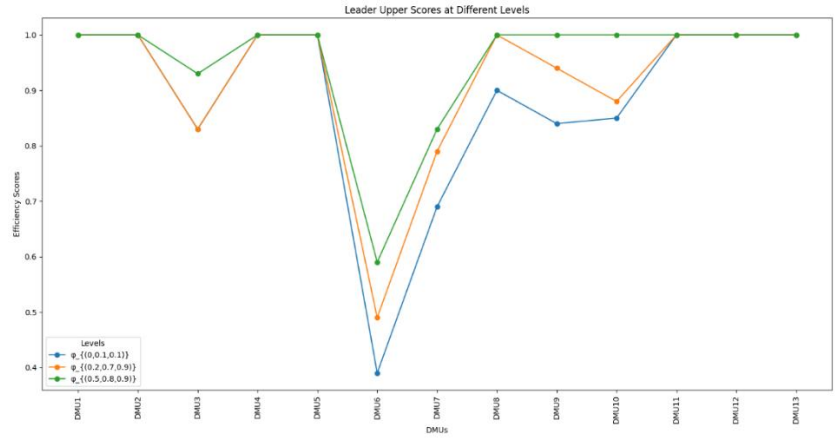

Figure 3. Leader's upper score at different levels of uncertainty.

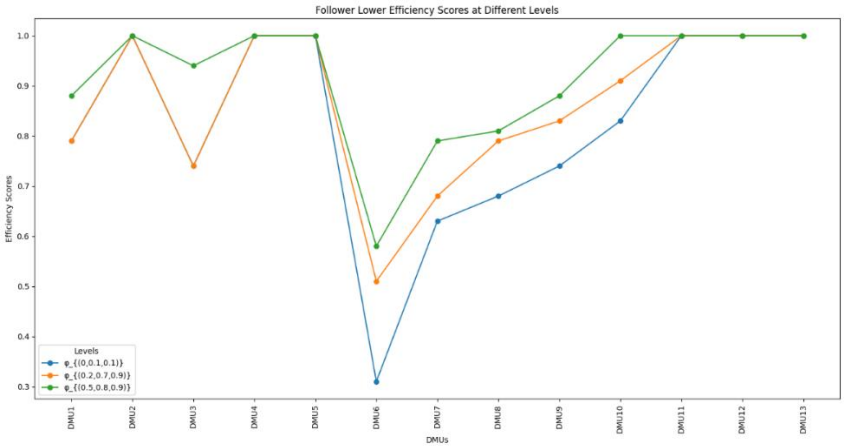

Figure 4. Follower's lower score at different levels of uncertainty.

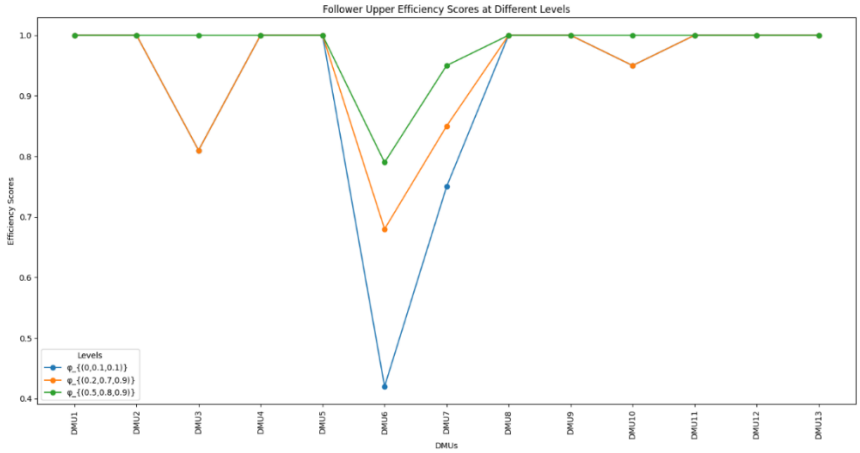

Figure 5. Follower's upper score at different levels of uncertainty.

4.1. Policy recommendations

Based on the findings of our model, several policy recommendations can be drawn to enhance operational efficiency in the Iranian airline industry:

1. Improve Resource Allocation for Underperforming Units:

Airlines such as DMU6, which consistently show lower efficiency scores across various levels of uncertainty, should focus on improving resource management in both their production and consumption divisions. This could involve optimizing workforce deployment and better utilization of aircraft to align closer with the performance of more efficient airlines like DMU2.

2. Targeted Interventions for Efficiency Improvement:

The variability in efficiency scores at different levels of uncertainty, particularly for DMU6, suggests that management should invest in operational adjustments that specifically address areas prone to inefficiencies. This could include investing in better route planning, aircraft maintenance, or optimizing flight schedules to reduce inefficiency in both divisions.

3. Implement a Penalty-Based Performance Framework:

The penalty for deviation from the leader's efficiency score, as shown in our model, could be used by airlines to incentivize higher performance. Policymakers should consider adopting a framework where airlines are rewarded for staying within optimal efficiency ranges and penalized for significant deviations, especially in the follower division, to promote consistency across all operational stages.

4.2.Comparison of results with an existing approach

In this subsection, we compare the results of the proposed model with those of Rasinojehdehi et al. [10]. Fig. 6 and Fig. 7 illustrate comparison of the average efficiency scores for the generated by our model and the corresponding results from Rasinojehdehi and Bagherzadeh Valami [10]. This comparison highlights the improvements and differences in performance evaluation, showcasing the strengths of the proposed dynamic network DEA model in handling the dual role of intermediate measures.

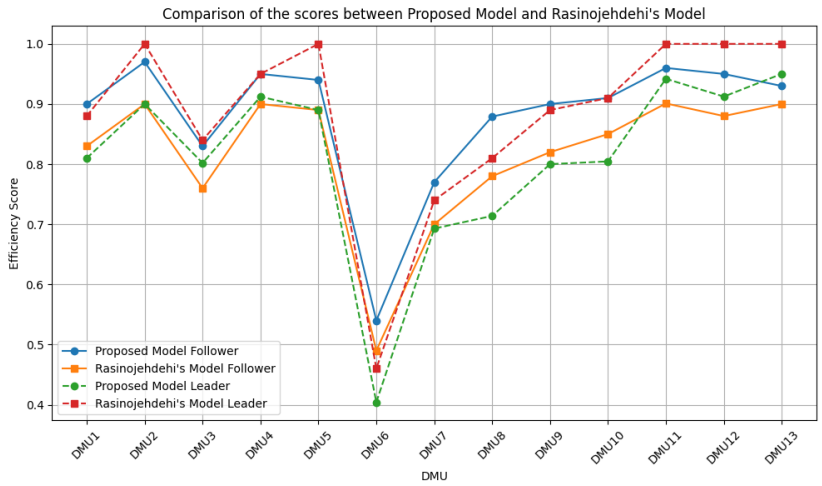

Figure 6. Comparison of average stage scores between our model and the model by Rasinojehdehi and Bagherzadeh Valami [10].

The comparison reveals that the average scores for the leader's stage in the proposed model are lower than those of Rasinojehdehi et al. [10], reflecting a more stringent evaluation of the leader's performance. This stricter assessment highlights the proposed model's ability to differentiate between varying levels of efficiency, especially in the leader's stage. Furthermore, the proposed model exhibits superior discriminating power, enabling a clearer distinction between efficient and inefficient DMUs. This enhancement stems from the model's focus on optimizing the leader's performance, which drives the system by compelling followers to adopt the strategy set by the leader, thereby improving overall system cohesion and performance stability.

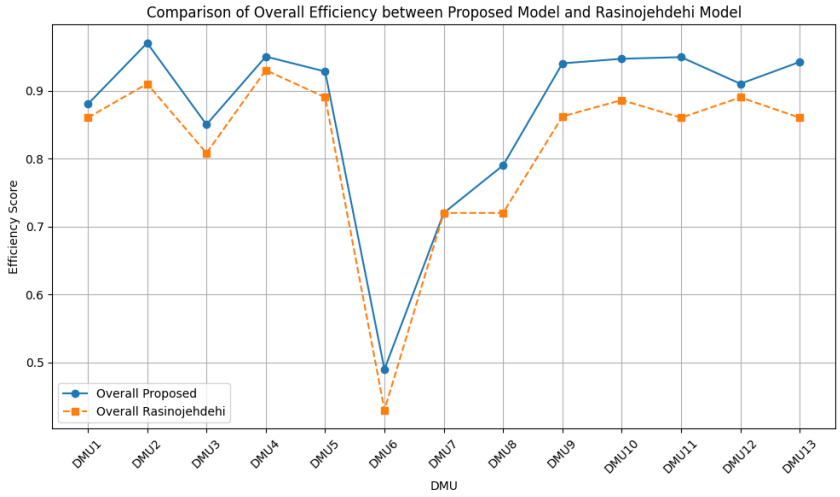

Figure 7. Comparison of the average overall score between our model and the model by Rasinojehdehi and Bagherzadeh Valami [10].

In the proposed model, the leader is tasked with determining the allocation of intermediate measures, ensuring that resources are allocated in a manner that optimizes the leader's efficiency. In contrast, Rasinojehdehi et al.'s [10] model focuses on distributing intermediate measures to minimize the overall system score, emphasizing a more collective approach. This fundamental difference explains why the average leader's score in the proposed model is lower, as it places greater emphasis on the leader's strategic role. However, despite the stricter assessment of the leader's efficiency, the overall system score remains comparable between the two models. This is because the followers in

the proposed model are guided by the leader's strategy, leading to a more balanced, aligned, and cohesive system performance.

5. Conclusion

In this paper, a novel approach for the performance evaluation of two-stage network systems, addressing the dual role of intermediate measures, is presented. To handle uncertainty within real-world data, SVNSs, a special case of neutrosophic sets, are employed. The Stackelberg game theory is utilized to explain the relationship between the stages and address the issue of the dual role of intermediate measures. The proposed model is developed within the SBM framework, which not only maintains the continuity of linking activities among the stages but also incorporates their inefficiency into the efficiency evaluation. Additionally, the model incorporates a penalty on the follower's objective for deviations from the leader's efficiency. This penalty enhances the model's feasibility robustness by underscoring the significant consequences imposed on deviations from the optimal efficiency established by the leader. To address uncertainty, the model is refined into a two-stage framework using Pareto efficiency concepts to establish lower and upper bounds for DMU efficiencies. Data from the Iranian Airline Industry, was used to validate our approach. This empirical study evaluates the production and consumption efficiencies of 13 Iranian airlines, demonstrating how our methodology effectively captures and quantifies operational efficiencies amidst uncertainty.

5.1. Limitations and future directions

While the proposed model has proven effective in capturing operational efficiencies under uncertainty, there are several limitations that should be considered. First, the model assumes a constant return to scale, which may not reflect real-world variability in airline operations. Future studies could explore the use of variable returns to scale to account for this. Additionally, the current model focuses on a two-stage process; however, many real-world systems are dynamic and involve more complex multi-stage interactions. Extending the model to dynamic NDEA or multi-stage systems could provide deeper insights into efficiency changes over time.

Furthermore, while Stackelberg game theory is used to explain the relationship between stages, other game theory models, such as cooperative games, could be integrated to explore different interaction dynamics between stages. Comparative studies could enhance the robustness of efficiency measurements by analyzing how various game theory models affect the outcomes.

Lastly, the model's applicability has been demonstrated in the airline industry, but future research should test its adaptability in other industries or sectors where uncertainty and multi-stage processes are prevalent.

Statements & Declarations

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All authors contributed to the study conception and design.

Compliance with Ethical Standards

Ethical approval: This article does not contain any studies with human participants or animals performed by any of the authors.

Conflict of Interest

The authors declare that they have no conflict of interest.

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Combine knowledge and similarity measure to solve Single valued Neutrosophic set MCDM problems

Nguyen Xuan Thao[1,2], Truong Thi Thuy Duong [3,*]

1 Banking Academy of Vietnam; thaonx@hvnh.edu.vn
2 International School, Vietnam National University, Hanoi, Vietnam, 24078014@vnu.edu.vn
3 Banking Academy of Vietnam; duongtt@hvnh.edu.vn

* Correspondence: duongtt@hvnh.edu.vn

Abstract: The Neutrosophics set have been paid attention of researchers using in many fields in the practice. It is a useful tool to deal with those complex problems. In this paper, we first investigat the knowledge measures on the Single neutrosophic set. Then, we introduce a new similarity measure of SNS that overcomes the restriction of some already existing similarity measure cases. A proposed measure-based multi-criteria decision-making (MCDM) model was constructed. In this model the knowledge measure to determine the weight of each criterion and similarity measure to rank the alternatives. This model applied to solve the agricultural land selection. Finally, the comparison results are presented to illustrate the effectiveness of the proposed method.

Keywords: SNS; similarity; MCDM; Knowledge measure

1. Introduction

In the real world, a lot of uncertainty exists to serve in decision-making problems. For instance, information gathered about a subject or the decision maker's knowledge is incomplete. To deal with such uncertain situations, fuzzy sets and intuitionistic fuzzy sets have proven as useful tools [27]. By the complexity and the continuous improvements of many fields in the practices, the fuzzy sets, intuitionistics sets cannot represent enough information. The single Neutrosophic set proposed by Smarandache [23] to overcome this limitation. In the environment of SNS, three membership $T_A(x), I_A(x), F_A(x)$ are valued in $[0,1]$ for all x in X. The SNS is the extension of fuzzy sets, intuitionistic fuzzy sets ([4], [5], [12], [20]) to deal with the uncertain information ([1], [3], [6], [7], [8], [9]). The neutrosophic set have been recognized effective tool and applied in many fields ([13], [22-24]). In addition, much attention has generalized functions and measure using Neutrosophics numbers to deal with decision making problems. Broumi and Smarandache [6] gave some similarity measures to solve MCDM problem using Hausdorff distance. Huang [10] built the distance SNS for clustering analysis problem. Kharal [14] proposed the novel score functions of SNS to deal with MCDM problems. Liu et al. [15] proposed some new aggregation operators to MCDM. Thao & Smarandache [31] pioneered introduced the divergence measure of SNS and applied to the medical diagnosis. The entropy and similarity measures are effectively function in MCDM model.

They are generalized on the SNS set and widely applied ([16], [34]). We can mention some outstanding results in [16], [21], [35-38]).

The motivation and novelties of this study: The knowledge measure of fuzzy set is used to evaluate the degree of fuzziness of the fuzzy set. If a data set having high degree of fuzziness, then the knowledge about it is low, while conversely, if the degree of fuzziness is low, then the knowledge about the data is high. Some similarity measures were proposed based on knowledge measure on intuitionistic fuzzy sets ([11], [17]). It is effective to determine the weight of criteria or rank the alternatives in MCDM model [11]. Tan and Chen [27] pointed out that there are always counter examples that demonstrate the ineffectiveness of similarity measures in some cases based on the analysis of most published studies. But, the knowledge measure of SNS is not found in any the existing works. It is a gap in the study about neutrosophic set. In the MCDM model based Single neutrosophic set the knowledge measures of SNS is using to determine the weight of each criterion.

On the other hand, neutrosophic is an extension of fuzzy sets, intuitionistic sets, so similarity measures on SNS are effective tools in MCDM problems. This motivates us to construct a new similarity measure between NSs, and providing reliable results. At the same time, to check the effectiveness, we make some comparisons with previous measures.

Population growth increases pressure on both natural and agricultural resources. This can lead to degradation of available land resources [2]. Therefore, there is a need for planning and evaluating land areas suitable for each subject and goal towards sustainable growth. If self-sufficiency in agricultural production is to be achieved in developing and transitional countries, land evaluation techniques will be necessary to develop models to predict the suitability of land for different types of agriculture. In which, multi-criteria decision-making model is one of the effective methods. This motivates us to construct an MCDM model to select agricultural land in order to illustrate our measure.

The contributions of this paper in terms of some aspects (Figure 1).

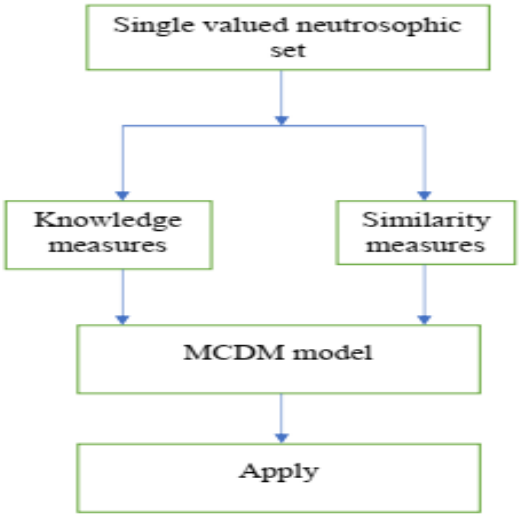

Figure 1. The flow chart of works in this study

Firstly, we construct the knowledge measure of SNS. It is used to determine the weight of the criteria. Secondly, the paper introduces a new similarity measure of the SNS set. Thirdly, we compare the proposed similarity measure to other similarity measures. Finally, we propose the TOPSIS model based on new measures to choose the best option for agricultural land selection and compare the proposed TOPSIS to some other models.

The rest of the paper is organized as follows: some concepts in SNS are introduced in section 2. The SNS's new measure (knowledge measures of SNS) is introduced in section 3. The similarity measures of SNS is proposed in section 4, in this section, we also analysis the new similarity with the existing similarity measures of SNS. The knowledge measure and new similarity measure-based TOPSIS model was applied to the agricultural land selection is shown in section 5. Next, we discuss about the ranking results of new model with other MCDM models in section 6. Finally, we give out the conclusion of this work.

2. Preliminary

In this section, the concept of SNS and their similarity measures are mentioned as follows

Definition 1. ([33]) In the universal set, a SNS A structure by three functions $T_A, I_A, F_A : X \to [0,1]$ named truth-membership, indeterminacy-membership, and falsity-membership function, respectively, such that $T_A(x) + I_A(x) + F_A(x) \le 3$ for all $x \in X$.

We denoted $A = \{(x, T_A(x), I_A(x), F_A(x)) \mid x \in X\}$ is a SNS A in X. The collection of SNS on X is denoted by $SNS(X)$.

For $A, B \in SNS(X)$ and for all $x \in X$, some operator is mentioned as follows:

- Complement of a SNS A: $\overline{A} = \{(x, F_A(x), 1 - I_A(x), T_A(x)) \mid x \in X\}$
- $A \subseteq B$ iff $T_A(x) \le T_B(x)$, $I_A(x) \ge I_B(x)$ and $F_A(x) \ge F_B(x)$ for all $x \in X$.
- $A = B$ iff $A \subseteq B$ and $B \subseteq A$.

IF $A \in SNS(X)$, then triple $(T_A(x), I_A(x), F_A(x))$ named a single valued neutrosophic number (SNN) and denoted by $a = (t_A, i_A, f_A)$.

- For $\lambda \in R^+$, we have:

$$\lambda a = \left(1 - (1 - t_A)^{\lambda}, i_A^{\lambda}, f_A^{\lambda}\right)$$

$$a^{\lambda} = \left(t_A^{\lambda}, 1 - (1 - i_A)^{\lambda}, 1 - (1 - f_A)^{\lambda}\right).$$

Now, we consider the universal set $X = \{x_1, x_2, ..., x_n\}$.

Now, we define the similarity measure on $SNS(X)$. It is often used to rank the alternatives in the MCDM problem.

Definition 2. ([6]) A function $S: SNS(X)^2 \to [0,1]$ is a similarity measure on $SNS(X)$, if it fulfills the following conditions:

(s1) $S(A,B) = S(B,A)$

(s2) $S(A,B) = 1$ iff $A = B$

(s3) If $A \subset B \subset C$ then $S(A,C) \le \min\{S(A,B), S(B,C)\}$, for all $A, B, C \in SNS(X)$.

The similarity measures are useful to handle the degree of similarity between alternatives, calculate the close alternatives and ranking to make decisions. The typical measurements are shown below.

+ Ye's similarity measures of SNSs ([35- 39]). Given $A, B \in SVNS(X)$:

- Cosine similarity measure

$$S_C(A,B) = \frac{1}{n}\sum_{i=1}^{n} \frac{T_A(x_i)T_B(x_i) + I_A(x_i)I_B(x_i) + F_A(x_i)F_B(x_i)}{\sqrt{T_A^2(x_i) + I_A^2(x_i) + F_A^2(x_i)}\sqrt{T_B^2(x_i) + I_B^2(x_i) + F_B^2(x_i)}} \quad (1)$$

- Dice similarity measure

$$S_D(A,B) = \frac{1}{n}\sum_{i=1}^{n} \frac{2(T_A(x_i)T_B(x_i) + I_A(x_i)I_B(x_i) + F_A(x_i)F_B(x_i))}{T_A^2(x_i) + I_A^2(x_i) + F_A^2(x_i) + T_B^2(x_i) + I_B^2(x_i) + F_B^2(x_i)} \quad (2)$$

- Jicard similarity measure

$$S_J(A,B) = \frac{1}{n}\sum_{i=1}^{n} \frac{T_A(x_i)T_B(x_i) + I_A(x_i)I_B(x_i) + F_A(x_i)F_B(x_i)}{\begin{bmatrix} T_A^2(x_i) + I_A^2(x_i) + F_A^2(x_i) + T_B^2(x_i) + I_B^2(x_i) + F_B^2(x_i) \\ -(T_A(x_i)T_B(x_i) + I_A(x_i)I_B(x_i) + F_A(x_i)F_B(x_i)) \end{bmatrix}} \quad (3)$$

- Cosine functions-based similarity measure

$$SCOS_1(A,B) = \frac{1}{n}\sum_{i=1}^{n} \cos\left\{\frac{\pi}{2} max\left\{|T_A(x_i) - T_B(x_i)|, |F_A(x_i) - F_B(x_i)|, |I_A(x_i) - I_B(x_i)|\right\}\right\} \quad (4)$$

$$SCOS_2(A,B) = \frac{1}{n}\sum_{i=1}^{n} \cos\left\{\frac{\pi}{6} sum\left\{|T_A(x_i) - T_B(x_i)|, |F_A(x_i) - F_B(x_i)|, |I_A(x_i) - I_B(x_i)|\right\}\right\} \quad (5)$$

- Tangent functions -based similarity measure

$$ST_1(A,B) = 1 - \frac{1}{n}\sum_{i=1}^{n} \tan\left\{\frac{\pi}{4} max\left\{|T_A(x_i) - T_B(x_i)|, |F_A(x_i) - F_B(x_i)|, |I_A(x_i) - I_B(x_i)|\right\}\right\} \quad (6)$$

$$ST_2(A,B) = 1 - \frac{1}{n}\sum_{i=1}^{n} \tan\left\{\frac{\pi}{12} sum\left\{|T_A(x_i) - T_B(x_i)|, |F_A(x_i) - F_B(x_i)|, |I_A(x_i) - I_B(x_i)|\right\}\right\} \quad (7)$$

- Cotangent functions -based similarity measure

$$SCOT_1(A,B) = \frac{1}{n}\sum_{i=1}^{n} \cot\left\{\frac{\pi}{4} + \frac{\pi}{4} max\left\{|T_A(x_i) - T_B(x_i)|, |F_A(x_i) - F_B(x_i)|, |I_A(x_i) - I_B(x_i)|\right\}\right\} \quad (8)$$

$$SCOT_2(A,B)=\frac{1}{n}\sum_{i=1}^{n}\cot\left\{\frac{\pi}{4}+\frac{\pi}{12}\,sum\left\{\left|T_A(x_i)-T_B(x_i)\right|,\left|F_A(x_i)-F_B(x_i)\right|,\left|I_A(x_i)-I_B(x_i)\right|\right\}\right\} \quad (9)$$

+ Ye's measure ([31], [32]).

$$SY_1(A,B)=1-\frac{1}{3n}\sum_{i=1}^{n}\left\{\left|T_A(x_i)-T_B(x_i)\right|+\left|F_A(x_i)-F_B(x_i)\right|+\left|I_A(x_i)-I_B(x_i)\right|\right\} \quad (10)$$

$$SY_2(A,B)=1-\frac{1}{3n}\sum_{i=1}^{n}\left\{\left|T_A(x_i)-T_B(x_i)\right|^2+\left|F_A(x_i)-F_B(x_i)\right|^2+\left|I_A(x_i)-I_B(x_i)\right|^2\right\}^{\frac{1}{2}} \quad (11)$$

$$SY_3(A,B)=\frac{1-\frac{1}{3}\sum_{i=1}^{n}\left\{\left|T_A(x_i)-T_B(x_i)\right|+\left|F_A(x_i)-F_B(x_i)\right|+\left|I_A(x_i)-I_B(x_i)\right|\right\}}{1+\frac{1}{3}\sum_{i=1}^{n}\left\{\left|T_A(x_i)-T_B(x_i)\right|+\left|F_A(x_i)-F_B(x_i)\right|+\left|I_A(x_i)-I_B(x_i)\right|\right\}} \quad (12)$$

+ Ye and Du's measure in ([38]).

$$SYD(A,B)=1-\frac{1}{n}\sum_{i=1}^{n}\max\left\{\left|T_A(x_i)-T_B(x_i)\right|,\left|F_A(x_i)-F_B(x_i)\right|,\left|I_A(x_i)-I_B(x_i)\right|\right\} \quad (13)$$

+ Thao and Smarandache ([31]).

$$S_T(A,B)=1-\frac{1}{4n}\sum_{i=1}^{n}\left[\left|T_A(x_i)-T_B(x_i)\right|+2\left|I_A(x_i)-I_B(x_i)\right|+\left|F_A(x_i)-F_B(x_i)\right|\right] \quad (14)$$

$$S_{MT}(A,B)=\frac{1}{n}\sum_{i=1}^{n}\frac{\min\left[1+\left|T_A(x_i)-T_B(x_i)\right|,1+\left|I_A(x_i)-I_B(x_i)\right|,1-\left|I_A(x_i)-I_B(x_i)\right|,1+\left|F_A(x_i)-F_B(x_i)\right|\right]}{\max\left[1+\left|T_A(x_i)-T_B(x_i)\right|,1+\left|I_A(x_i)-I_B(x_i)\right|,1-\left|I_A(x_i)-I_B(x_i)\right|,1+\left|F_A(x_i)-F_B(x_i)\right|\right]} \quad (15)$$

+ Qin and Wang's measure in ([21])

$$S_{QW}(A,B)=\frac{1}{n}\sum_{i=1}^{n}sqw(i) \quad (16)$$

where $sqw(i)=\begin{cases}1-\dfrac{\left|I_A(x_i)-I_B(x_i)\right|}{2} & \text{if } T_A(x_i)=T_B(x_i), F_A(x_i)=F_B(x_i)\\ \dfrac{2-\left|T_A(x_i)-T_B(x_i)\right|-\left|F_A(x_i)-F_B(x_i)\right|}{4} & otherwise\end{cases}$

for all $x_i\in X$.

3. Knowledge measure of SNS

Let $X=\{x_1,x_2,...,x_n\}$ be a universal set, given $A\in SNS(X)$. The knowledge measure of SNS A is determined by the Eq.(17).

$$K_{TH}(A)=\frac{2}{3n}\sum_{i=1}^{n}\left[\left|T_A(x_i)-F_A(x_i)\right|+\frac{\left|2I_A(x_i)-1\right|}{1+\left|2I_A(x_i)-1\right|}\right] \quad (17)$$

Example 1. For a SNS $A=\{(x_1,0.4,0.5,0.7),(x_2,1,0.8,0.7)\}$,

The knowledge measure of SNS A is $K_{TH}(A)=\frac{2}{3\times 2}\left[\begin{array}{l}\left(|0.4-0.7|+\frac{2|2\times 0.5-1|}{1+|2\times 0.5-1|}\right)\\+\left(|1-0.7|+\frac{2|2\times 0.8-1|}{1+|2\times 0.8-1|}\right)\end{array}\right]=0.45$.

It is easy to verify the following properties of knowledge measure of SNS:

Property 1. $K_{TH}(A)=1$ if A is crisp set.

Property 2. $K_{TH}(A)=0$ if $A=\{(x_i,T_A(x_i),0.5,T_A(x_i))\,|\,x_i\in X\}$

Property 3. $K_{TH}(A)=K_{TH}(A^C)$ for all $A\in SNS(X)$.

Theses properties are axioms to determine the knowledge measure of SNS. This is a kind of new measure on SNS. In assessing the quality of information. From the nature of knowledge measures, we see that if the information is clear, the value of knowledge measures is high, on the contrary, if the information is ambiguous, the value of knowledge measures is low. Therefore, we can use this measure to assess the weight of criteria in the MCDM model.

4. New similarity measures of SNS

In this section, we introduce a new similarity measure of SNS. Let $X=\{x_1,x_2,...,x_n\}$ be a universal set, given $A,B\in SNS(X)$. We consider a new function $S:\mathrm{SN}S(\mathrm{X})^2\to[0,1]$ by

$$S_{TH}(A,B)=1-\frac{1}{3n}\sum_{i=1}^{n}\left[|T_A(x_i)-T_B(x_i)|+\frac{2|I_A(x_i)-I_B(x_i)|}{1+|I_A(x_i)-I_B(x_i)|}+|F_A(x_i)-F_B(x_i)|\right] \quad (18)$$

for all $x_i\in X$.

Theorem 1. The Eq.(18) determine a similarity measure of two SNS sets.

Proof

For all $A,B\in SNS(X)$, we have $0\le|T_A(x_i)-T_B(x_i)|\le 1$,

$0\le\frac{2|I_A(x_i)-I_B(x_i)|}{1+|I_A(x_i)-I_B(x_i)|}\le 1, 0\le|F_A(x_i)-F_B(x_i)|\le 1$ for all $x_i\in X$ then $0\le S_{TH}(A,B)\le 1$.

(s1) It is easy to see that $S_{TH}(A,B)=S_{TH}(A,B)$ for all $A,B\in SNS(X)$.

(s2) If $A=B$ we have $|T_A(x_i)-T_B(x_i)|=0$, $|I_A(x_i)-I_B(x_i)|=0$ and $|F_A(x_i)-F_B(x_i)|=0.$ So that $S_{TH}(A,B)=1$ if $A=B$.

(s3) For all $A,B,C\in SNS(X)$ and $A\subseteq B\subseteq C,$ we have $T_A(x_i)\le T_B(x_i)\le T_C(x_i),$ $I_A(x_i)\ge I_B(x_i)\ge I_C(x_i)$ and $F_A(x_i)\ge F_B(x_i)\ge F_C(x_i)$ for all $x_i\in X.$ Because of the increased monotony of the functions $f(a)=a$ and $g(a)=\dfrac{2a}{1+a}$ (for all $a\in[0,1]$). These imply that

$$\max\left\{|T_A(x_i)-T_B(x_i)|,|T_B(x_i)-T_C(x_i)|\right\}\le|T_A(x_i)-T_C(x_i)|,$$

$$\max\left\{\frac{2|I_A(x_i)-I_B(x_i)|}{1+|I_A(x_i)-I_B(x_i)|},\frac{2|I_B(x_i)-I_C(x_i)|}{1+|I_B(x_i)-I_C(x_i)|}\right\}\le\frac{2|I_A(x_i)-I_C(x_i)|}{1+|I_A(x_i)-I_C(x_i)|},$$

$$\max\left\{|F_A(x_i)-F_B(x_i)|,|F_B(x_i)-F_C(x_i)|\right\}\le|F_A(x_i)-F_C(x_i)|$$

for all $x_i\in X.$ So that

$$S_{TH}(A,C)=1-\frac{1}{3}\left[|T_A(x_i)-T_C(x_i)|+\frac{2|I_A(x_i)-I_C(x_i)|}{1+|I_A(x_i)-I_C(x_i)|}+|F_A(x_i)-F_C(x_i)|\right]$$

$$\le S_{TH}(A,B)=1-\frac{1}{3}\left[|T_A(x_i)-T_B(x_i)|+\frac{2|I_A(x_i)-I_B(x_i)|}{1+|I_A(x_i)-I_B(x_i)|}+|F_A(x_i)-F_B(x_i)|\right]$$

and

$$S_{TH}(A,C)=1-\frac{1}{3}\left[|T_A(x_i)-T_C(x_i)|+\frac{2|I_A(x_i)-I_C(x_i)|}{1+|I_A(x_i)-I_C(x_i)|}+|F_A(x_i)-F_C(x_i)|\right]$$

$$\le S_{TH}(B,C)=1-\frac{1}{3}\left[|T_B(x_i)-T_C(x_i)|+\frac{2|I_B(x_i)-I_C(x_i)|}{1+|I_B(x_i)-I_C(x_i)|}+|F_B(x_i)-F_C(x_i)|\right]$$

This means that $S(A,C)\le\min\{S(A,B),S(B,C)\}$ for all $A\subseteq B\subseteq C$. □

In general that, with $\omega=(\omega_1,\omega_2,...,\omega_n)$ is the vector weight $(\omega_i\ge 0,\sum_{i=1}^{n}\omega_i=1)$ on X we obtain a general similarity measure is defined by

$$S_{TH}^{\omega}(A,B)=\sum_{i=1}^{n}\omega_i\times\left\{1-\frac{1}{3}\left[|T_A(x_i)-T_B(x_i)|+\frac{2|I_A(x_i)-I_B(x_i)|}{1+2|I_A(x_i)-I_B(x_i)|}+|F_A(x_i)-F_B(x_i)|\right]\right\} \quad (19)$$

Theorem 2. The Eq.(19) determines a similarity measure of two SNS sets.

Proof. The proof is similar Theorem 1. □

The effectiveness of the proposed measure

To evaluate the effectiveness of the proposed measure, we compare the proposed measure with some existing measures in classification problems. Given m patterns $\{A_1, A_2, ..., A_m\}$ in the form of SNS on $X = \{x_1, x_2, ..., x_n\}$. Having C is a new SNS sample on $X = \{x_1, x_2, ..., x_n\}$.

Question: What pattern does C belong to?

To solve this problem, we first compute $S(A_i, C)$ $(i = 1, 2, ..., m)$. Then, C belongs to pattern A_{i*} having $S(A_{i*}, C) = max\{S(A_i, C) \mid i = 1, 2, ..., m\}$. We compare new similarity with some existing measures.

Example 2. First of all, we can easy that the similarity measures $S_C(A,B)$, $S_D(A,B)$ and $S_J(A,B)$ is non-determine when $A = \{(x,0,0,0)\}$ and $B = \{(x,0,0,0)\}$. Meanwhile, in this case, we have

$$SCOS_1(A,B) = SCOS_2(A,B) = ST_1(A,B) = ST_1(A,B) = SCOT_1(A,B)$$
$$= SCOT_2(A,B) = SY_1(A,B) = SY_2(A,B) = SY_3(A,B) = SYD(A,B) = S_{MT}(A,B) = S_T(A,B)$$
$$= S_{QW}(A,B) = S_{TH}(A,B) = 1.$$

It means that, in this case, the similarity measures $S_C(A,B)$, $S_D(A,B)$ and $S_J(A,B)$ are not shown the similar measure on $A = \{(x,0,0,0)\}$ and $B = \{(x,0,0,0)\}$, but other similarity measures (including the proposed measure) are shown that A, B are similar.

Example 3. Given two patterns SNSs on $X = \{x_1, x_2\}$ and two neutrosophic sets $A_1 = \{(x_1, 0.7, 0.3, 0.3), (x_2, 0.5, 0.2, 0.3)\}$, $A_2 = \{(x_1, 0.3, 0.6, 0.7), (x_2, 0.3, 0.4, 0.5)\}$. Let a new sample $C = \{(x_1, 0.5, 0.4, 0.5), (x_2, 0.4, 0.3, 0.4)\}$ on *X*. We will check which set C belong to using similarity measures.

Then, Table 1 gives the classification results based on similarity measures. In this case, we see that only the cosine similarity measure S_C having decision C belongs to pattern A_2; five similarity measures $SCOS_1$, ST_1, $SCOT_1$, SYD and S_{QW} shown that we non-determine which pattern (A_1 or A_2) that C belongs to; but our proposed measure and ten other similarity measures shown that C belongs to pattern A_1. This obtains that we can put C belongs to pattern A_1.

Table 1. The classification results

Measures	The results from C to	Classification results

	A_1	A_2	
$S_C(A_i,C)$ (in the Eq.(1))	0.9475	**0.9556**	A_2
$S_D(A_i,C)$ (in the Eq.(2))	**0.9472**	0.9460	A_1
$S_J(A_i,C)$ (in the Eq.(3))	**0.9000**	0.8983	A_1
$SCOS_1(A_i,C)$ (in the Eq.(4))	0.9694	0.9694	Non-determine
$SCOS_2(A_i,C)$ (in the Eq.(5))	**0.9768**	0.9694	A_1
$ST_1(A_i,C)$ (in the Eq.(6))	0.8815	0.8815	Non-determine
$ST_2(A_i,C)$ (in the Eq.(7))	**0.8948**	0.8815	A_1
$SCOT_1(A_i,C)$ (in the Eq.(8))	0.7903	0.7903	Non-determine
$SCOT_2(A_i,C)$ (in the Eq.(9))	**0.8107**	0.7903	A_1
$SY_1(A_i,C)$ (in the Eq.(10))	**0.8667**	0.8500	A_1
$SY_2(A_i,C)$ (in the Eq.(11))	**0.9800**	0.9750	A_1
$SY_3(A_i,C)$ (in the Eq.(12))	**0.5790**	0.5385	A_1
$SYD(A_i,C)$ (in the Eq.(13))	0.8500	0.8500	Non-determine
$S_T(A_i,C)$ (in the Eq.(14))	**0.8750**	0.8500	A_1
$S_{MT}(A_i,C)$ (in the Eq.(15))	**0.7841**	0.7424	A_1
$S_{QW}(A_i,C)$ (in the Eq.(16))	0.4250	0.4250	Non-determine
$S_{TH}(A_i,C)$ (proposed in the Eq.(18)	**0.8394**	0.8141	A_1

Example 4. Given two patterns SNSs

$A_1 = \{(x_1, 0.8, 0.8, 0.2), (x_2, 0.6, 0.6, 0.2)\}$ and $A_2 = \{(x_1, 1, 0.4, 0.4), (x_2, 0.8, 0.4, 0.4)\}$ on $X = \{x_1, x_2\}$ and the sample $C = \{(x_1, 0.6, 0.6, 0.6), (x_2, 0.8, 0.4, 0.6)\}$.

The classification results are shown in Table 2 based on measures. In this case, we see that only the similarity measure S_{MT} having decision C belongs to pattern A_1; five similarity measures $SCOS_1$, ST_1, $SCOT_1$, SYD and S_T shown that we non-determine which pattern (A_1 or A_2) that C belongs to; but our proposed measure and ten other similarity measures shown that C belongs to pattern A_2. This mean that we can put C belongs to pattern A_2.

Table 2. The classification results

Measures	The results from C to		Classification results
	A_1	A_2	
$S_C(A_i,C)$ (in the eq.(1))	0.8996	**0.9165**	A_2
$S_D(A_i,C)$ (in the eq.(2))	0.8875	**0.9115**	A_2
$S_J(A_i,C)$ (in the eq.(3))	0.7980	**0.8377**	A_2
$SCOS_1(A_i,C)$ (in the eq.(4))	0.8090	0.8090	non-determine
$SCOS_2(A_i,C)$ (in the eq.(5))	0.9135	**0.9323**	A_2
$ST_1(A_i,C)$ (in the eq.(6))	0.6751	0.6751	non-determine
$ST_2(A_i,C)$ (in the eq.(7))	0.7874	**0.8145**	A_2
$SCOT_1(A_i,C)$ (in the eq.(8))	0.5095	0.5095	non-determine
$SCOT_2(A_i,C)$ (in the eq.(9))	0.6494	**0.6880**	A_2
$SY_1(A_i,C)$ (in the eq.(10))	0.7333	**0.7667**	A_2
$SY_2(A_i,C)$ (in the eq.(11))	0.9200	**0.9267**	A_2
$SY_3(A_i,C)$ (in the eq.(12))	0.3044	**0.3636**	A_2
$SYD(A_i,C)$ (in the eq.(13))	0.6000	0.6000	non-determine
$S_T(A_i,C)$ (in the eq.(14))	0.7500	0.7500	non-determine’

$S_{MT}(A_i, C)$ (in the eq.(2))	**0.5714**	0.5000	A_1
$S_{QW}(A_i, C)$ (in the eq.(3))	0.3500	**0.4000**	A_2
$S_{TH}(A_i, C)$ (proposed in the eq.(18)	0.6889	**0.7159**	A_2

Example 5 ([27]). Given two patterns SNSs on $X = \{x_1, x_2\}$,

$A_1 = \{(x_1, 0.8, 0.8, 0.2), (x_2, 0.6, 0.8, 0.2)\}$, $A_2 = \{(x_1, 1, 0.4, 0.4), (x_2, 1, 0.2, 0)\}$ and a new sample C on X, $C = \{(x_1, 0.6, 0.6, 0.6), (x_2, 0.6, 0.2, 0.6)\}$.

Table 3 gives the classification results based on similarity measures. In this case, we see that having thirteen similarity measures S_C, S_D, S_J, $SCOS_1$, $SCOS_2$, ST_1, ST_2, $SCOT_1$, $SCOT_2$, SY_1, SY_2, SY_3, SYD shown that C would not belong to pattern A_1 or A_2; only S_{QW} having decision C belongs to pattern A_1; but our proposed measure and two other similarity measures shown that C belongs to pattern A_2. This mean that we can put C belongs to pattern A_2. It also indicates the effectiveness of the proposed similarity measure.

Table 3. The classification results

Measures	The results from C to		Classification results
	A_1	A_2	
$S_C(A_i, C)$ (in the Eq.(1))	0.8122	0.8122	non-determine
$S_D(A_i, C)$ (in the Eq.(2))	0.8056	0.8056	non-determine
$S_J(A_i, C)$ (in the Eq.(3))	0.6850	06850	non-determine
$SCOS_1(A_i, C)$ (in the Eq.(4))	0.6984	0.6984	non-determine
$SCOS_2(A_i, C)$ (in the eq.(5))	0.8898	0.8898	non-determine
$ST_1(A_i, C)$ (in the Eq.(6))	0.5828	0.5828	non-determine
$ST_2(A_i, C)$ (in the Eq.(7))	0.7598	0.7598	non-determine

$SCOT_1(A_i,C)$ (in the Eq.(8))	0.4172	0.4172	non-determine
$SCOT_2(A_i,C)$ (in the Eq.(9))	06134	0.6134	non-determine
$SY_1(A_i,C)$ (in the Eq.(10))	0.7000	0.7000	non-determine
$SY_2(A_i,C)$ (in the Eq.(11))	0.8733	0.8733	non-determine
$SY_3(A_i,C)$ (in the Eq.(12))	0.2500	0.2500	non-determine
$SYD(A_i,C)$ (in the Eq.(13))	0.5000	0.5000	non-determine
$S_T(A_i,C)$ (in the Eq.(14))	0.6750	**0.7500**	A_2
$S_{MT}(A_i,C)$ (in the Eq.(2))	0.4107	**0.5982**	A_2
$S_{QW}(A_i,C)$ (in the Eq.(3))	**0.3750**	0.3000	A_1
$S_{TH}(A_i,C)$ (proposed in the Eq.(18)	0.6528	**0.6778**	A_2

The above four examples have demonstrated the feasibility of the proposed new measure. Specifically, as in Example 2, the new measure is better than the ones (such as those in Eq.(4), Eq.(6), Eq.(8), Eq.(13), and Eq.(16)). In Example 3, the new measure is better than the ones (such as those in Eq.(4), Eq.(6), Eq.(8), Eq.(13), Eq.(14), and Eq.(15)). In Example 4, the new measure has superiority over the previous ones (such as those in Eq.(1) - (13)).

In the next section, we apply this new measure to build a TOPSIS model to solve the MCDM problem, then we also compare the proposed method with some existing methods.

5. The proposed knowledge and similarity measures-based TOPSIS model for agricultural land selection

In MCDM problem, we have to determine an optimal option from set of m alternatives $A=\{A_1,A_2,...,A_m\}$ based on n criteria $C=\{C_1,C_2,...,C_n\}$. In this problem, we consider each A_i is a SNS on C, i.e. $A_i=\{(C_j,D_{ij})\,|\,C_j\in C\}$ where $D_{ij}=(T_{ij},I_{ij},F_{ij})$ in which $T_{ij}=T_{ij}(C_j),I_{ij}=I_{ij}(C_j),F_{ij}=F_{ij}(C_j)$ for all $i=1,2,...,m,\ j=1,2,...,n$. So that, we a SNS decision matrix $D=\left[D_{ij}\right]_{m\times n}$.

The proposed TOPSIS model based on similarity measure of SNSs is presented as in Algorithm 1. Figure 2 displays the flow chart of new similarity-based SNS TOPSIS model.

Algorithm 1. New similarity-based SNS TOPSIS model

***Input**: a SNS decision matrix.*

Output: *ranking of* $A=\{A_1,A_2,...,A_m\}$.

Step 1. Determine the weight ω_j of each criteria C_j (the decision maker can choose the weight ω_j of each criteria C_j), for all $j=1,2,...,n$.

$$\omega_j=\frac{K_{C_j}(A_i)}{\sum_{j=1}^{n}K_{C_j}(A_i)} \qquad (20)$$

where $K_{C_j}(A_i)$ is the knowledge measure of criteria C_j for all $j=1,2,...,n$

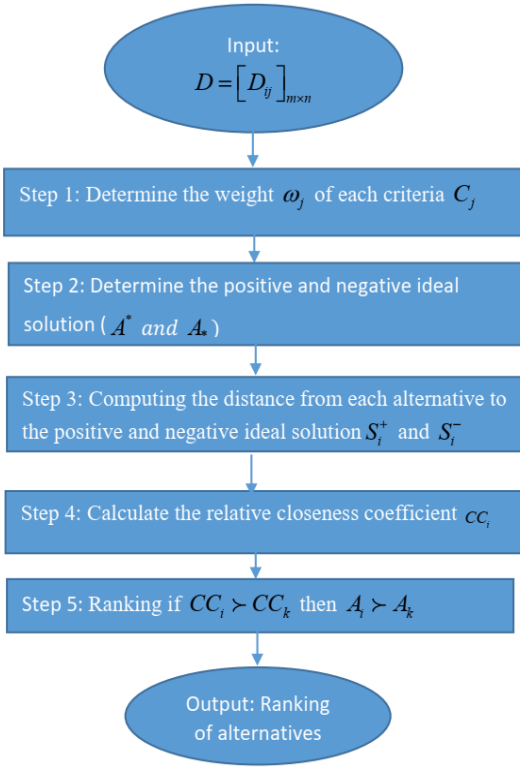
Input:
$D=[D_{ij}]_{m\times n}$
Step 1: Determine the weight ω_j of each criteria C_j
Step 2: Determine the positive and negative ideal solution (A^* and A_*)
Step 3: Computing the distance from each alternative to the positive and negative ideal solution S_i^+ and S_i^-
Step 4: Calculate the relative closeness coefficient CC_i
Step 5: Ranking if $CC_i \succ CC_k$ then $A_i \succ A_k$
Output: Ranking of alternatives

Fig 2. *A*lgorithm diagram of ***new similarity-based SNS TOPSIS model***

Step 2. Compute the SNS best solution A^* and the SNS worst solution A_* as follow

+ $A^*=\{(C_j,T_j^*,I_j^*,F_j^*)\,|\,C_j\in C\}$

where

$$T_j^*=\begin{cases}\min\limits_{i=1,...,m}T_{ij} \text{ if } C_j \text{ is non-benifite}\\ \max\limits_{i=1,...,m}T_{ij} \text{ if } C_j \text{ is benifite}\end{cases}$$

$$I_j^* = \begin{cases} \max\limits_{i=1,...,m} \mathrm{I}_{\mathrm{ij}} \text{ if } C_j \text{ is non-benifit} \\ \min\limits_{i=1,...,m} I_{\mathrm{ij}} \text{ if } C_j \text{ is benifite} \end{cases}$$

$$F_j^* = \begin{cases} \max\limits_{i=1,...,m} F_{\mathrm{ij}} \text{ if } C_j \text{ is non-benifit} \\ \min\limits_{i=1,...,m} F_{\mathrm{ij}} \text{ if } C_j \text{ is benifite} \end{cases} \quad (21)$$

+ $A_* = \left\{ \left(C_j, T_j^{'}, I_j^{'}, F_j^{'} \right) | C_j \in C \right\}$

where

$$T_j^{'} = \begin{cases} \max T_{\mathrm{ij}} \text{ if } C_j \text{ is non-benifite} \\ \min\limits_{i=1,...,m} T_{\mathrm{ij}} \text{ if } C_j \text{ is benifite} \end{cases}$$

$$I_j^{'} = \begin{cases} \min\limits_{i=1,...,m} \mathrm{I}_{\mathrm{ij}} \text{ if } C_j \text{ is non-benifite} \\ \max\limits_{i=1,...,m} I_{\mathrm{ij}} \text{ if } C_j \text{ is benifite} \end{cases}$$

$$F_j^{'} = \begin{cases} \min\limits_{i=1,...,m} F_{\mathrm{ij}} \text{ if } C_j \text{ is non-benifite} \\ \max\limits_{i=1,...,m} F_{\mathrm{ij}} \text{ if } C_j \text{ is benifite} \end{cases} \quad (22)$$

Step 3. Using the eq.(3) to determine S_i^+ and S_i^- are the similarity measures from A_i to A^* and A_*, respectively, in detail as follows:

$$S_{TH}^{\omega}(A_i, A^*) = \sum_{j=1}^{n} \omega_j \times \left\{ 1 - \frac{1}{3} \left[\left| T_{\mathrm{ij}}(C_j) - T_j^*(C_j) \right| + \frac{2\left| I_{\mathrm{ij}}(C_j) - I_j^*(C_j) \right|}{1 + 2\left| I_{\mathrm{ij}}(C_j) - I_j^*(C_j) \right|} + \left| F_{\mathrm{ij}}(C_j) - F_j^*(C_j) \right| \right] \right\} \quad (23)$$

$$S_{TH}^{\omega}(A_i, A_*) = \sum_{j=1}^{n} \omega_j \times \left\{ 1 - \frac{1}{3} \left[\left| T_{\mathrm{ij}}(C_j) - T_j^{'}(C_j) \right| + \frac{2\left| I_{\mathrm{ij}}(C_j) - I_j^{'}(C_j) \right|}{1 + 2\left| I_{\mathrm{ij}}(C_j) - I_j^{'}(C_j) \right|} + \left| F_{\mathrm{ij}}(C_j) - F_j^{'}(C_j) \right| \right] \right\} \quad (24)$$

Step 4. Determine the relative closeness coefficient of $A_i, (i = 1, 2, ..., m)$ as

$$CC_i = \frac{S_i^+}{S_i^+ + S_i^-} \quad (25)$$

for all $i = 1, 2, ..., m.$

Step 5. Ranking: if $CC_i \succ CC_k$ then $A_i \succ A_k$ for all $i, k = 1, 2, ..., m.$

Now, we use Algorithm 1 to solve a support decision problem to select the agricultural land as follows:

Example 6 ([3]). A trader wants to invest in the agricultural sector and find suitable land. He/she has to choose the optimal alternative in a set of five alternatives $A = \{A_1, A_2, A_3, A_4, A_5\}$ depend on the set of five criteria $C = \{C_1, C_2, C_3, C_4, C_5\}$ (where C_1: location, C_2: climate, C_3: fertility, C_4: price,

C_5:water availability). In which, the criteria C_2(climate) and C_4 (price) are the cost criteria, and other are benefit criteria. The relationship between the alternatives and criteria shown in the Table 4.

Table 4. SNS decision matrix (Ashraf and Abdullah, 2020)

	C_1	C_2	C_3	C_4	C_5
A_1	(0.5,0.3,0.4)	(0.3,0.2,0.5)	(0.2,0.2,0.6)	(0.4,0.2,0.3)	(0.3,0.3,0.4)
A_2	(0.7,0.1,0.3)	(0.3,0.2,0.7)	(0.6,0.3,0.2)	(0.2,0.4,0.6)	(0.7,0.1,0.2)
A_3	(0.5,0.3,0.4)	(0.4,0.2,0.6)	(0.6,0.1,0.2)	(0.3,0.1,0.5)	(0.6,0.4,0.3)
A_4	(0.7,0.3,0.2)	(0.2,0.2,0.7)	(0.4,0.5,0.2)	(0.2,0.2,0.5)	(0.4,0.5,0.4)
A_5	(0.4,0.1,0.3)	(0.2,0.1,0.5)	(0.4,0.1,0.5)	(0.6,0.3,0.4)	(0.3,0.2,0.4)

We use Algithm1 to solve this* agricultural land selection problem as follows*:

Step 1. By considering that, each criterion is a SNS on the set of alternative sets $A=\{A_1,A_2,A_3,A_4,A_5\}$. Using the Eq.(21), we compute the weight vector of $C=\{C_1,C_2,C_3,C_4,C_5\}$ is $\omega=(0.203,0.2437,0.2081,0.1929,0.1523)$.

Step 2. Determine SNS best solution A^* and the SNS worst solution A_* by using the Eq.(21) and Eq.(22) (see Table 5).

Table 5. The values of A^* and A_*

	C_1	C_2	C_3	C_4	C_5
A^*	(0.7,0.1,0.2)	(0.2,0.2,0.7)	(0.6,0.1,0.2)	(0.2,0.4,0.6)	(0.7,0.1,0.2)
A_*	(0.4,0.3,0.4)	(0.4,0.1,0.5)	(0.2,0.5,0.6)	(0.6,0.1,0.3)	(0.3,0.5,0.4)

Step 3. Using the Eq.(23) and the Eq.(24), we get S_i^+ and S_i^- from A_i to A^* and A_*, respectively (see Table 6).

Table 6. The results of S_i^+ and S_i^-

	A_1	A_2	A_3	A_4	A_5
$S(A_i,A^*)$	0.7569	0.9620	0.8499	0.8417	0.8172
$S(A_i,A_*)$	0.8967	0.6985	0.8135	0.8219	0.8427

Step 4. Determine CC_i of $A_i,(i=1,2,...,5)$ by using the Eq.(25) (see Table 7)

Table 7. The values of $CC_i,(i=1,2,...,5)$ and ranking

	A_1	A_2	A_3	A_4	A_5
CC_i	0.4577	0.5794	0.5109	0.5059	0.4923

Ranking	5	**1**	2	3	4

Step 5. Ranking (in Table 7) $A_2 \succ A_3 \succ A_4 \succ A_5 \succ A_1$. Thus A_2 is the best alternative.

6. Discussion

In this section, we compare result when using our proposed method with some methods as: SVNFWA, SVNWG, SVNOWG, SVNOWA and SVNWA [18], [19], L-SVNHWA and L-SVNHWG [3], TS-MCDM [31]. Raking results shown that $A_2 \succ A_3 \succ A_4 \succ A_5 \succ A_1$. It means that A_2 is the best alternative and A_1 is the worst alternative (see Fig. 3).

Moreover, the above methods have more computational steps and are more complex than the proposed new method. The methods SVNFWA, VNWG, SVNOWG, SVNOWA, and SVNWA, L-SVNHWA and L-SVNHWG used either the aggregation operator and the score function or the outranking relation matrix to rank the alternatives. Even the L-SVNHWA and L-SVNHWG methods also use trigonometric aggregation operators under the NSN environment. Therefore, the implementation steps of the above methods are not less than 7 steps, while the new method proposed here only needs 5 calculation steps. This proves that the proposed new method has simpler calculation steps, but the efficiency of the method is still achieved as the above methods.

It is evident that this presentation confirms the reasonable and applicability of the proposed similarity-based SNS TOPSIS model under SNS environments.

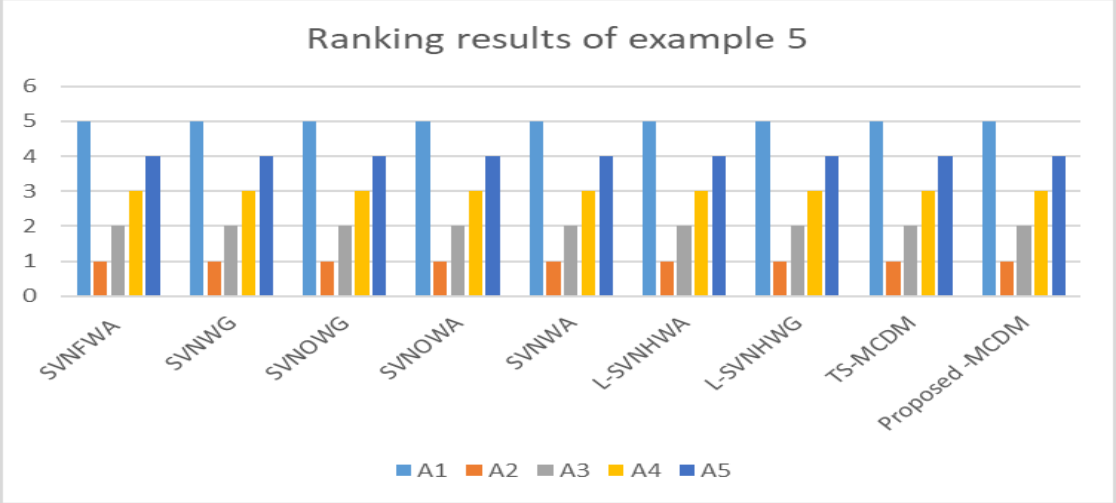

Fig. 3: Raking results of Example 5 when using some different ranking methods

7. Conclusion

In this paper, we construct the knowledge measure of SNS. It is a new measure of SNSs. This is a new main contribution of this paper. The knowledge measures allowed us to determine the weights of criteria in the MCDM model. In the same time, we also proposed a new similarity measure for SNSs. Comparing results shows the benefit of the new measure can overcome a lot of the limitations encountered in some cases of already existing similarity measures. Next, the knowledge measures and new similarity measure based-MCDM is introduced to apply to the problem of agricultural land selection. The new model is simple easy to use and efficiency. To

demonstrate the feasibility of the proposed method, we also compared to some other decision-making methods in the agricultural land selection: the rating using the suggested method is also in line with the rating results using some of the existing ranking methods.

The disadvantage of this method is that it has not yet given the relationship between the knowledge measure of SNS with other measures of SNS. This is also something that needs to be expanded in the direction of future research. In the future, the connection of the knowledge and similarity measure need to study. Moreover, we will continue to work to find other metrics on SNS as dissimilarity measures shown in [9], divergence measures in [27] and the extended Neutrosophic sets, rough fuzzy set, or rough standard neutrosophic set. At the same time, they apply them to solving real-world problems, and other problem classes such as pattern recognition problem, cluster analysis problem, and other MCDM models as in [9], [38].

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Data availability: Adopted datasets using in Example 5 come from the paper: *Ashraf, S., & Abdullah, S. (2020). Decision Support Modeling For Agriculture Land Selection Based On Sine Trigonometric Single Valued Neutrosophic Information. International Journal of Neutrosophic Science (IJNS) Vol.9, 60-73.*

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University Mathematics Classroom Teaching Quality Assessment Based on Core Competencies Using the DEMATEL Approach and Single-Valued Neutrosophic Hypersoft Sets

Jingjing Qiao*
University of Perpetual Help System DALTA, Manila, Philippines
*Corresponding author E-mail: qjjedu@163.com

Abstract: This study proposes a multi-criteria decision-making (MCDM) approach to evaluate the quality of University Mathematics Classroom Teaching based on core competencies. The MCDM approach is applied within the framework of the neutrosophic set to address vague and uncertain data. Unlike the hypersoft set, which handles multiple disjoint attribute-valued sets corresponding to various characteristics, the soft set operates with a single set of attributes. This study introduces the concept of Single-Valued Neutrosophic Hypersoft Expert Sets (SVNS), which integrate single-valued neutrosophic sets and hypersoft expert sets. Eight criteria were employed to construct the pairwise comparison matrix. The Decision-Making Trial and Evaluation Laboratory (DEMATEL) method was utilized to analyze the interrelationships among the criteria within the SVNS and hypersoft set frameworks. The results indicate that Conceptual Understanding utilizes the highest impact among the criteria.

Keywords: University Mathematics Classroom; Uncertainty; MCDM; DEMATEL Approach.

1. Introduction

The quality of teaching in university mathematics classrooms is critical for developing students' analytical thinking, logical reasoning, and problem-solving skills. Mathematics forms the backbone of many technical and scientific disciplines, and its teaching demands a high level of expertise. Effective teaching in this field does not merely involve the accurate delivery of content but also necessitates fostering deep conceptual understanding and encouraging critical thinking among students. Educators must strike a balance between subject-matter expertise and pedagogical strategies to accommodate diverse learning needs and backgrounds [1][2].

Traditional evaluation methods, such as student feedback surveys, standardized tests, and peer reviews, are frequently employed to assess teaching quality. While these methods provide valuable insights, they often fall short of capturing the multifaceted nature of teaching. For instance, student feedback can be influenced by factors unrelated to teaching, such as the difficulty of the course or the instructor's demeanor. Standardized assessments, on the other hand, tend to focus narrowly on measurable outcomes, overlooking crucial qualitative aspects like classroom engagement and the ability to foster critical thinking [3][4].
MCDM approaches have emerged as effective tools for addressing these challenges. By allowing the analysis of multiple, often interrelated criteria, MCDM approaches provide a structured way to evaluate teaching effectiveness comprehensively. Among these, the DEMATEL method stands out due to its ability to identify and analyze causal relationships among criteria. Unlike other MCDM methods, such as the Analytic Hierarchy Process (AHP) and Technique for Order Preference by Similarity to Ideal Solution

(TOPSIS), DEMATEL goes beyond ranking and prioritizing to uncover interdependencies and influence levels among factors, making it particularly well-suited for evaluating complex educational settings [5][6].

In addition to MCDM methods, advancements in mathematical modeling, such as neutrosophic sets, have significantly enhanced decision-making processes. Neutrosophic sets, introduced by Smarandache, extend traditional fuzzy logic by incorporating three independent membership degrees: truth, indeterminacy, and falsity. This method provides a nuanced way to represent uncertainty, ambiguity, and inconsistency, which are often inherent in educational data [7]. Building on this foundation, single-valued neutrosophic sets (SVNS) and hypersoft sets were developed to address decision-making problems characterized by multi-attribute and uncertain data. These tools are particularly relevant in evaluating teaching quality, where subjective factors like student perceptions and classroom dynamics play a significant role [8][9].
The integration of DEMATEL with Single-Valued Neutrosophic Hypersoft Sets offers a novel and comprehensive framework for evaluating teaching quality. This hybrid approach combines the strengths of both methodologies: the causal analysis capabilities of DEMATEL and the uncertainty-handling flexibility of neutrosophic sets. By focusing on core competencies such as conceptual understanding, problem-solving skills, classroom engagement, and feedback quality, this framework addresses the limitations of traditional evaluation methods and provides actionable insights for educators and policymakers [10][11].

This study aims to contribute to the field of educational evaluation by introducing a robust framework that integrates DEMATEL and SVNS for assessing teaching quality in university mathematics classrooms. The proposed approach is designed to account for the multidimensional and interdependent nature of teaching quality criteria while addressing the inherent uncertainties and subjectivities in educational data. By leveraging these advanced methodologies, this research seeks to provide a more comprehensive and reliable tool for evaluating and improving teaching practices in mathematics education [12][13].
Moreover, the study emphasizes the importance of adopting evidence-based evaluation frameworks in higher education, especially in disciplines like mathematics, where the complexity of the subject matter and the diversity of student needs present unique challenges. By integrating quantitative and qualitative analysis, the proposed framework not only evaluates the effectiveness of teaching practices but also identifies areas for targeted intervention, ultimately enhancing the overall quality of education [14][15].

2. Related Work and Literature Review

The evaluation of teaching quality has been a topic of ongoing debate and research due to the multifaceted nature of teaching and learning. Traditional evaluation systems often rely on student feedback, standardized tests, and peer reviews, which are inherently limited in capturing the complexities of effective teaching. According to Broumi et al. [16], these approaches often fail to account for interdependencies between different factors affecting teaching quality, such as conceptual understanding, student engagement, and the integration of technology in teaching practices. This gap has led to the exploration of more advanced frameworks that can address the intricate nature of educational evaluations.
One of the major criticisms of conventional evaluation methods is their inability to represent uncertainty and subjective biases effectively. Smarandache et al. [20] emphasized that subjective data, such as student feedback, often involves ambiguity and inconsistencies that traditional tools cannot process adequately. These limitations have prompted researchers to seek methods capable of incorporating uncertainty while simultaneously analyzing interrelated criteria.

Neutrosophic sets have gained significant traction in decision-making processes due to their ability to handle uncertainty and inconsistency effectively. Unlike traditional fuzzy sets, neutrosophic sets incorporate three independent membership degrees: truth, indeterminacy, and falsity. This allows them to

model complex scenarios with greater flexibility. Liu et al. [17] applied neutrosophic sets in the selection of transportation service providers, demonstrating their capacity to manage multi-criteria problems involving subjective and uncertain data.

In the context of education, Rahman et al. [7] introduced interval-valued neutrosophic hypersoft sets to represent multi-attribute decision-making scenarios. These sets are particularly useful for modeling subjective factors, such as student perceptions of teaching quality or the impact of classroom engagement. Their findings highlighted the potential of neutrosophic sets to enhance decision-making frameworks by addressing ambiguities inherent in educational data.

The integration of DEMATEL with neutrosophic sets represents a significant advancement in multi-criteria decision-making methodologies. DEMATEL is widely recognized for its ability to map causal relationships between criteria, providing a visual representation of how different factors influence one another. When combined with neutrosophic sets, this method gains the ability to incorporate uncertainty into causal analysis, making it particularly suitable for complex systems like education. Abdullah et al. [18] demonstrated this integration in analyzing subcontractor selection processes, highlighting its utility in evaluating systems characterized by interrelated and uncertain criteria.

Moreover, Rodzi et al. [19] applied a hybrid framework combining neutrosophic sets with DEMATEL to analyze barriers to Halal certification adoption in Malaysia. Their findings underscored the framework's ability to handle complex relationships and subjective biases, which are also prevalent in teaching quality evaluations. This approach offers significant promise for educational contexts, where factors such as teaching effectiveness, student engagement, and technology use are deeply interdependent.

The evolution of hypersoft sets has further expanded the applicability of the neutrosophic theory in educational evaluations. Hypersoft sets allow for the representation of multiple attributes associated with each decision-making entity, providing a richer analysis of complex scenarios. Zhao et al. [5] applied single-valued neutrosophic hypersoft sets to investment decisions, showcasing their ability to model intricate relationships between criteria. Similarly, Rahman et al. [11] extended this concept to site selection for solid waste management, highlighting its versatility across various domains.

In the educational domain, hypersoft sets can be adapted to evaluate teaching quality by analyzing multiple competencies simultaneously. Saqlain et al. [6] explored the use of neutrosophic linguistic-valued hypersoft sets in medical diagnosis, emphasizing their effectiveness in scenarios involving subjective evaluations. These findings suggest that hypersoft sets can provide a robust framework for addressing the complexities of teaching quality assessments in mathematics classrooms. While traditional MCDM methods such as AHP and TOPSIS have been extensively used in educational research, their limitations have become increasingly apparent. AHP is well-suited for ranking and prioritizing criteria but assumes independence among them, which is rarely the case in educational settings. TOPSIS, on the other hand, focuses on proximity to an ideal solution but cannot analyze causal relationships.

The hybrid integration of DEMATEL with neutrosophic sets offers a more comprehensive approach. DEMATEL excels in identifying and mapping interdependencies, while neutrosophic sets enhance the framework's ability to handle uncertainty. This makes the combined methodology particularly advantageous for evaluating teaching quality, where criteria such as conceptual understanding, classroom engagement, and technology use are deeply interconnected.

3. Theoretical Background

Definition 3.1

Neutrosophic sets, introduced by Smarandache, extend fuzzy logic and intuitionistic fuzzy sets by introducing three independent membership functions for an element x in a set A:

Truth-Membership ($T_A(x)$): Degree of truth in the statement.

Indeterminacy-Membership ($I_A(x)$): Degree of indeterminacy or uncertainty in the statement.

Falsity-Membership ($F_A(x)$): Degree of falsity in the statement.

These values are defined on the interval [0,1], with the condition: $0 \leq T_A(x), I_A(x), F_A(x) \leq 1$

Unlike fuzzy sets, neutrosophic sets allow $T_A(x)+I_A(x)+F_A(x) \leq 3$, enabling the representation of incomplete, inconsistent, or contradictory information. For example, in evaluating teaching quality, a student's feedback about a teacher's ability to engage may be represented as: $T_A(x)=0.7$, $I_A(x)=0.2$, $F_A(x)=0.1$

This implies that the students' feedback is 70% true, 20% uncertain, and 10% false.

Definition 3.2 [13], [14]

The soft set can be defined as:

Let u be a universe discourse $p(u)$ is the power set of u and A set of criteria. Then the pair $(F, u), F: \rightarrow (u)$ is called a soft set over u.

Definition 3.3

Hypersoft sets, generalize soft sets by associating parameters with multi-attribute valued functions. A hypersoft set over a universal set U is defined as: H= $\{(p,f(p))|p \in P, f(p) \subseteq P(U)\}$

where:

P is a set of parameters.

$f(p)$ is a multi-attribute valued function for each p.

$P(U)$ denotes the power set of U.

Hypersoft sets enable the representation of scenarios where a single parameter (e.g., a teacher's competency) may depend on multiple interrelated attributes (e.g., conceptual understanding, classroom engagement, technology use). For example, H= $(p1, \{(x1,v1),(x2,v2)\}),(p2,\{(x3,v3),(x4,v4)\})\}$

Here, $p1$ and $p2$ represent parameters such as teaching skills and engagement, and xi,vi are attributes and their corresponding values.

3.1 DEMATEL Approach

The DEMATEL method analyzes and visualizes the causal relationships among criteria. The approach consists of the following steps:

i. Constructing the Direct-Relation Matrix

Experts evaluate the pairwise influence of criteria Ci and Cj on a scale of 0 (*no influence*) to 4 (*very high influence*). The direct-relation matrix D is defined as: D=$[d_{ij}]$ (1)

where d_{ij} represents the influence of Ci on Cj

ii. Normalizing the Direct-Relation Matrix

The matrix D is normalized to ensure all values fall within the range [0,1]: (2)

$$D^{norm} = \frac{D}{\max \sum_j d_{ij}}$$

iii. Calculating the Total-Relation Matrix

The total-relation matrix T is computed using: (3)

$$T = D^{norm}(I - D^{norm})^{-1}$$

where I is the identity matrix.

iv. Determining the Cause-and-Effect Relationships

The sums of rows and columns in T are calculated as: (4)

$$R_i = \sum_j t_{ij}, \quad C_j = \sum_i t_{ij}$$

where R_i is the total influence of Ci on others, and Cj is the total influence received by Cj the difference $Ri-Cj$ indicates whether a criterion is a net cause ($Ri>Cj$) or a net effect ($Ri<Cj$)

v. Visualizing the Cause-and-Effect Diagram

A diagram is plotted with $Ri+Cj$ on the horizontal axis (importance) and $Ri-Cj$ on the vertical axis (cause or effect nature).

3.2 DEMATEL Educational Evaluation

To demonstrate the application of the DEMATEL method in evaluating teaching quality in university mathematics classrooms, consider four key criteria for assessment: Conceptual Understanding (C1), referring to the teacher's ability to facilitate deep comprehension of mathematical concepts; Classroom Engagement (C2), which evaluates the teacher's effectiveness in actively engaging students during lessons; Use of Technology (C3), assessing the integration of technological tools to enhance teaching; and Problem-Solving Skills (C4), representing the teacher's capability to develop students' ability to solve complex mathematical problems.

Step 1, Expert judgments are collected to evaluate the influence of one criterion on another using a scale ranging from 0 (no influence) to 4 (very high influence). Based on these judgments, the Direct-Relation Matrix (D) is constructed as follows:

$$D = \begin{bmatrix} 0 & 3 & 2 & 1 \\ 0 & 0 & 4 & 2 \\ 1 & 0 & 0 & 3 \\ 2 & 1 & 0 & 0 \end{bmatrix}$$

Each element d_{ij} represents the degree of influence criterion Ci exerts on Cj.

d_{12}=3: Conceptual Understanding (C1) has a high influence on Classroom Engagement (C2).

d_{34}=3: Use of Technology (C3) strongly influences Problem-Solving Skills (C4).

Step 2, To normalize the matrix, calculate the maximum row sum:

Row sums of D= [6,6,4,3], Max row sum=6

Divide each element of D by the maximum row sum:

$$D^{norm} = \frac{D}{6} = \begin{bmatrix} 0 & 0.5 & 0.333 & 0.167 \\ 0 & 0 & 0.667 & 0.333 \\ 0.167 & 0 & 0 & 0.5 \\ 0.333 & 0.167 & 0 & 0 \end{bmatrix}$$

Step 3, The Total-Relation Matrix (T) is calculated using eq. (3) as:

$$T = \begin{bmatrix} 0.665 & 0.797 & 0.915 & 0.720 \\ 0.295 & 0.444 & 1.004 & 0.796 \\ 0.459 & 0.260 & 0.241 & 0.682 \\ 0.608 & 0.484 & 0.528 & 0.362 \end{bmatrix}$$

Step 4, The row sums (Ri) and column sums (Cj) are calculated using eq. (4) as:

R= [3.097,2.539,1.642,1.982], C= [2.028,1.985,2.688,2.558]

Step 5, To determine causal relationships, compute $Ri+Cj$ (overall importance) and $Ri-Cj$ (causal nature):

R+C= [5.125,4.524,4.330,4.540], R−C= [1.069,0.554, −1.046, −0.576]

C1(R1−C1=1.069): Net cause; the most influential criterion.

C3(R3−C3 = −1.046): Net effect; heavily influenced by other criteria.

The Cause-Effect Diagram is plotted with:

R+C on the horizontal axis, representing the overall importance of each criterion.

R−C on the vertical axis, indicating whether the criterion is a cause ($R-C>0$) or an effect ($R-C<0$).
Causes: C1(Conceptual Understanding) and C2(Classroom Engagement).
Effects: C3 (Use of Technology) and C4 (Problem-Solving Skills).

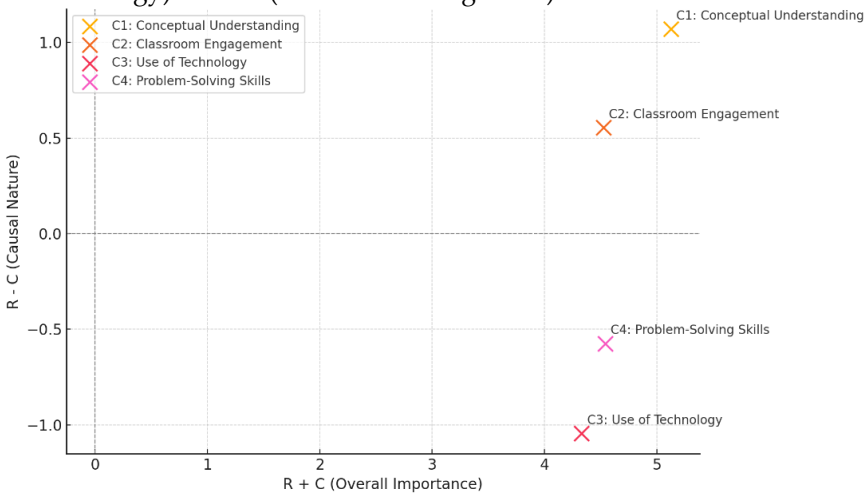

Figure 1. The Cause-Effect Diagram

This analysis provides actionable insights into the teaching quality evaluation process using the DEMATEL method. Conceptual Understanding (C1) emerges as the most influential criterion. Improving conceptual teaching methods can have positive effects on other criteria, such as engagement (C2) and problem-solving skills (C4). Teachers should prioritize strategies that enhance students' understanding of mathematical concepts, ensuring a solid foundation for further learning.

3.3 Integration of DEMATEL and SVNSs Framework

In the proposed methodology, DEMATEL identifies and quantifies the relationships between criteria, classifying them as causes (influential factors) or effects (dependent factors). SVNSs handle uncertainty in expert evaluations by capturing truth, indeterminacy, and falsity values, ensuring accurate and reliable data. Together, they provide a structured and precise framework for analyzing interdependent criteria under uncertainty, supporting better decision-making. This integration is especially relevant in educational evaluations, where subjective data (e.g., student feedback) often involves ambiguity and vagueness. Figure 2 shows the steps of the proposed approach.

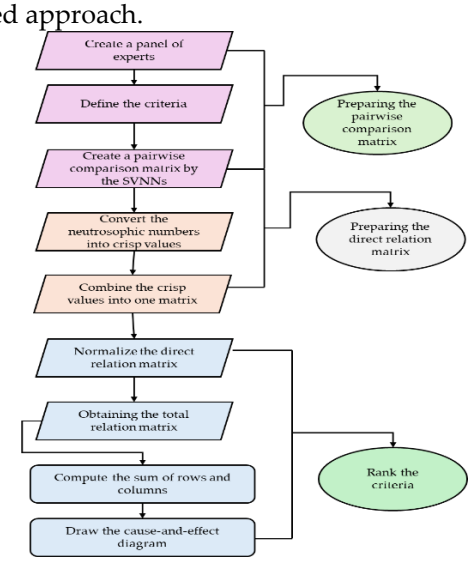
Create a panel of experts
Define the criteria
Create a pairwise comparison matrix by the SVNNs
Convert the neutrosophic numbers into crisp values
Combine the crisp values into one matrix
Normalize the direct relation matrix
Obtaining the total relation matrix
Compute the sum of rows and columns
Draw the cause-and-effect diagram
Preparing the pairwise comparison matrix
Preparing the direct relation matrix
Rank the criteria

Figure 2. Steps of SVN-DEMATEL

4. The Proposed Approach in the Case Study: Application of HS-SVN-DEMATEL

This section elaborates on the application of the proposed HS-SVN-DEMATEL methodology to evaluate university mathematics classroom teaching quality. This hybrid approach, which integrates HS Sets and SVNSs, is designed to effectively address uncertainty and complex decision-making. By combining these two mathematical tools, the methodology provides a systematic evaluation of interrelations among criteria, offering insights into their individual and collective impacts.

4.1 The Role of HyperSoft Sets in the Proposed Methodology

The HS Sets generalize traditional soft sets by incorporating multiple attribute-valued functions, allowing them to model complex systems more comprehensively. In this study, HS Sets were employed to accommodate MCDM, enabling the representation of criteria that involve multiple, often overlapping attributes. This capability ensured a robust framework for analyzing the complex interdependencies among the evaluation criteria in an uncertain environment.

4.2 Evaluation Framework and Criteria

The study focuses on evaluating eight critical criteria related to classroom teaching quality, as defined in Table 1**:**

1. Conceptual Understanding
2. Problem-Solving Skills
3. Mathematical Communication
4. Classroom Engagement
5. Use of Technology
6. Critical Thinking Development
7. Assessment and Feedback Quality
8. Application of Mathematics in Real-Life Contexts

Three domain experts were selected to evaluate these criteria. They used qualitative descriptors such as "High," "Excellent," and "Highly Relevant," which were mapped to SVNS terms characterized by truth-membership (T), indeterminacy-membership (I), and falsity-membership (F).

Table 1. HyperSoft Criteria and Values.

Criteria	Values
Conceptual Understanding	High, Moderate, Low
Problem-Solving Skills	Excellent, Adequate, Inadequate
Mathematical Communication	Highly Effective, Moderately Effective, Ineffective
Classroom Engagement	Highly Engaged, Partially Engaged, Not Engaged
Use of Technology	Optimal, Satisfactory, Suboptimal
Critical Thinking Development	Strong, Moderate, Weak
Assessment and Feedback Quality	Comprehensive, Sufficient, Insufficient
Application of Mathematics in Real-Life Contexts	Highly Relevant, Moderately Relevant, Irrelevant

4.2.1 Construction of Pairwise Comparison Matrices

Each expert independently assessed the relative importance of the criteria, resulting in three pairwise comparison matrices, presented in Tables 2, 3, and 4. These matrices contained SVNS values (T, I, F), which quantify the influence of one criterion over another.

Table 2. First Pairwise Comparison Matrix.

	C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8
C_1	(0.9,0.1,0.2)	(0.8,0.2,0.3)	(0.7,0.3,0.4)	(0.6,0.4,0.5)	(0.5,0.5,0.5)	(0.4,0.6,0.7)	(0.3,0.7,0.8)	(0.2,0.8,0.9)
C_2	(0.5,0.5,0.5)	(0.4,0.6,0.7)	(0.3,0.7,0.8)	(0.2,0.8,0.9)	(0.9,0.1,0.2)	(0.8,0.2,0.3)	(0.7,0.3,0.4)	(0.9,0.1,0.2)
C_3	(0.6,0.4,0.5)	(0.3,0.7,0.8)	(0.2,0.8,0.9)	(0.1,0.9,0.9)	(0.1,0.9,0.9)	(0.9,0.1,0.2)	(0.6,0.4,0.5)	(0.8,0.2,0.3)
C_4	(0.7,0.3,0.4)	(0.4,0.6,0.7)	(0.1,0.9,0.9)	(0.9,0.1,0.2)	(0.1,0.9,0.9)	(0.8,0.2,0.3)	(0.5,0.5,0.5)	(0.7,0.3,0.4)
C_5	(0.8,0.2,0.3)	(0.5,0.5,0.5)	(0.2,0.8,0.9)	(0.8,0.2,0.3)	(0.9,0.1,0.2)	(0.7,0.3,0.4)	(0.4,0.6,0.7)	(0.6,0.4,0.5)
C_6	(0.9,0.1,0.2)	(0.6,0.4,0.5)	(0.3,0.7,0.8)	(0.4,0.6,0.7)	(0.5,0.5,0.5)	(0.6,0.4,0.5)	(0.3,0.7,0.8)	(0.5,0.5,0.5)
C_7	(0.2,0.8,0.9)	(0.7,0.3,0.4)	(0.8,0.2,0.3)	(0.9,0.1,0.2)	(0.1,0.9,0.9)	(0.1,0.9,0.9)	(0.2,0.8,0.9)	(0.4,0.6,0.7)
C_8	(0.3,0.7,0.8)	(0.4,0.6,0.7)	(0.5,0.5,0.5)	(0.6,0.4,0.5)	(0.7,0.3,0.4)	(0.8,0.2,0.3)	(0.9,0.1,0.2)	(0.3,0.7,0.8)

Table 3. The Second Pairwise Comparison Matrix

	C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8
C_1	(0.1,0.9,0.9)	(0.8,0.2,0.3)	(0.7,0.3,0.4)	(0.6,0.4,0.5)	(0.5,0.5,0.5)	(0.4,0.6,0.7)	(0.3,0.7,0.8)	(0.1,0.9,0.9)
C_2	(0.9,0.1,0.2)	(0.4,0.6,0.7)	(0.3,0.7,0.8)	(0.2,0.8,0.9)	(0.9,0.1,0.2)	(0.8,0.2,0.3)	(0.7,0.3,0.4)	(0.9,0.1,0.2)
C_3	(0.5,0.5,0.5)	(0.1,0.9,0.9)	(0.2,0.8,0.9)	(0.1,0.9,0.9)	(0.1,0.9,0.9)	(0.1,0.9,0.9)	(0.6,0.4,0.5)	(0.5,0.5,0.5)
C_4	(0.1,0.9,0.9)	(0.9,0.1,0.2)	(0.1,0.9,0.9)	(0.1,0.9,0.9)	(0.1,0.9,0.9)	(0.9,0.1,0.2)	(0.1,0.9,0.9)	(0.1,0.9,0.9)
C_5	(0.8,0.2,0.3)	(0.5,0.5,0.5)	(0.9,0.1,0.2)	(0.9,0.1,0.2)	(0.9,0.1,0.2)	(0.5,0.5,0.5)	(0.9,0.1,0.2)	(0.1,0.9,0.9)
C_6	(0.9,0.1,0.2)	(0.1,0.9,0.9)	(0.5,0.5,0.5)	(0.5,0.5,0.5)	(0.5,0.5,0.5)	(0.1,0.9,0.9)	(0.5,0.5,0.5)	(0.9,0.1,0.2)
C_7	(0.2,0.8,0.9)	(0.7,0.3,0.4)	(0.1,0.9,0.9)	(0.1,0.9,0.9)	(0.1,0.9,0.9)	(0.1,0.9,0.9)	(0.1,0.9,0.9)	(0.5,0.5,0.5)
C_8	(0.3,0.7,0.8)	(0.4,0.6,0.7)	(0.5,0.5,0.5)	(0.6,0.4,0.5)	(0.7,0.3,0.4)	(0.8,0.2,0.3)	(0.9,0.1,0.2)	(0.1,0.9,0.9)

Table 4. Third Pairwise Comparison Matrix

	C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8
C_1	(0.5,0.5,0.5)	(0.8,0.2,0.3)	(0.7,0.3,0.4)	(0.6,0.4,0.5)	(0.5,0.5,0.5)	(0.5,0.5,0.5)	(0.3,0.7,0.8)	(0.2,0.8,0.9)
C_2	(0.9,0.1,0.2)	(0.5,0.5,0.5)	(0.3,0.7,0.8)	(0.2,0.8,0.9)	(0.9,0.1,0.2)	(0.9,0.1,0.2)	(0.5,0.5,0.5)	(0.9,0.1,0.2)
C_3	(0.1,0.9,0.9)	(0.9,0.1,0.2)	(0.5,0.5,0.5)	(0.5,0.5,0.5)	(0.1,0.9,0.9)	(0.1,0.9,0.9)	(0.9,0.1,0.2)	(0.8,0.2,0.3)
C_4	(0.1,0.9,0.9)	(0.1,0.9,0.9)	(0.9,0.1,0.2)	(0.9,0.1,0.2)	(0.1,0.9,0.9)	(0.1,0.9,0.9)	(0.1,0.9,0.9)	(0.5,0.5,0.5)
C_5	(0.9,0.1,0.2)	(0.1,0.9,0.9)	(0.1,0.9,0.9)	(0.1,0.9,0.9)	(0.9,0.1,0.2)	(0.9,0.1,0.2)	(0.1,0.9,0.9)	(0.9,0.1,0.2)
C_6	(0.9,0.1,0.2)	(0.9,0.1,0.2)	(0.1,0.9,0.9)	(0.1,0.9,0.9)	(0.5,0.5,0.5)	(0.6,0.4,0.5)	(0.9,0.1,0.2)	(0.1,0.9,0.9)
C_7	(0.2,0.8,0.9)	(0.7,0.3,0.4)	(0.9,0.1,0.2)	(0.9,0.1,0.2)	(0.1,0.9,0.9)	(0.1,0.9,0.9)	(0.2,0.8,0.9)	(0.1,0.9,0.9)
C_8	(0.3,0.7,0.8)	(0.4,0.6,0.7)	(0.5,0.5,0.5)	(0.6,0.4,0.5)	(0.7,0.3,0.4)	(0.8,0.2,0.3)	(0.9,0.1,0.2)	(0.9,0.1,0.2)

4.2.2 Building the Direct Relation Matrix

The crisp values derived from the pairwise comparison matrices were used to construct the direct relation matrix (D), where each entry a_{ij} represents the direct influence of criterion i on criterion j. The direct relation matrix was normalized using Equation (2), ensuring that all values fell within the range [0,1] The normalized direct relation matrix is presented in Table 5.

Table 5. Normalize Direct Relation Matrix.

	C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8
C_1	0.151448	0.082405	0.089087	0.095768	0.100223	0.100223	0.115813	0.126949
C_2	0.051225	0.100223	0.115813	0.122494	0.075724	0.075724	0.100223	0.075724
C_3	0.131403	0.08686	0.100223	0.100223	0.126949	0.178174	0.075724	0.100223
C_4	0.164811	0.093541	0.075724	0.126949	0.126949	0.120267	0.153675	0.138085
C_5	0.075724	0.126949	0.080178	0.120267	0.075724	0.08686	0.093541	0.106904
C_6	0.075724	0.106904	0.111359	0.11804	0.100223	0.126949	0.060134	0.10245
C_7	0.122494	0.089087	0.120267	0.126949	0.126949	0.126949	0.126949	0.11804
C_8	0.115813	0.109131	0.100223	0.095768	0.089087	0.082405	0.075724	0.08686

4.2.3 Total Relation Matrix and Its Calculation

The total relation matrix (T) was computed using Equation (3). The total relation matrix, shown in Table 6, accounts for both direct and indirect relationships among the criteria.

Table 6. Total Relation Matrix.

	C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8
C_1	0.78052	0.63804	0.641779	0.728486	0.675818	0.725878	0.678997	0.729379
C_2	0.578897	0.567612	0.581145	0.656973	0.562045	0.605992	0.575211	0.581755
C_3	0.777086	0.663658	0.671981	0.755927	0.721335	0.827244	0.654296	0.722731
C_4	0.896803	0.740791	0.719018	0.864778	0.797122	0.847405	0.811261	0.840283
C_5	0.630744	0.620556	0.571392	0.68379	0.586471	0.641714	0.594503	0.639846
C_6	0.652552	0.621959	0.622008	0.704268	0.631759	0.707185	0.578163	0.657304
C_7	0.822579	0.709648	0.736538	0.834103	0.769943	0.827511	0.754703	0.789989
C_8	0.662991	0.594323	0.582726	0.648888	0.591295	0.629447	0.567446	0.61186

4.2.3 Cause-and-Effect Analysis

To identify the causal and effectual roles of the criteria, the following metrics were computed:
Sum of Rows (Di): Represents the total influence exerted by a criterion on others.
Sum of Columns (Ri): Represents the total influence received by a criterion from others.
The net influence of each criterion was calculated as: Di−Ri
A positive value indicates a criterion is primarily a cause, while a negative value denotes it as an effect. The total importance of each criterion was determined as: Di+Ri
The cause-and-effect diagram, depicted in Figure 3, visually represents interdependence among the criteria. Key insights from the analysis include:

1. Criterion 4 emerged as the most influential cause, highlighting its central role in enhancing teaching quality and its strong impact on other criteria.
2. Criterion 2 exhibited the lowest influence, suggesting its relatively peripheral role within the evaluated context and limited impact on other criteria.

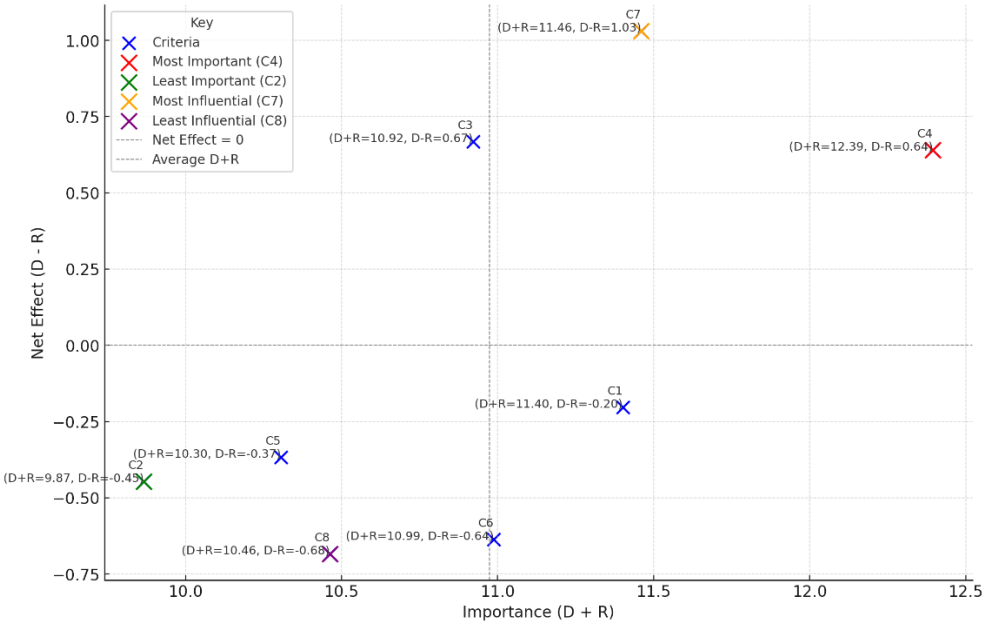

Figure 3. The cause-and-effect diagram.

5. Discussion

The proposed methodology's integration of HyperSoft Sets and SVNSs within the DEMATEL framework introduces several pivotal contributions to the field of multi-criteria decision-making under uncertainty. First, the use of HyperSoft Sets enhances the representation of complex systems by enabling multi-attribute evaluations. Unlike traditional soft sets, HyperSoft Sets allow for the simultaneous consideration of overlapping and interrelated attributes, which is crucial in scenarios involving multiple interconnected criteria. This capability ensures a more nuanced and accurate analysis of the relationships among the criteria, particularly in environments where attributes are not mutually exclusive.

Second, the incorporation of SVNSs provides a robust mechanism for managing uncertainty and vagueness inherent in expert evaluations. By explicitly accounting for truth-membership, indeterminacy-membership, and falsity-membership, SVNSs allow decision-makers to model and analyze criteria with incomplete or ambiguous information effectively. This characteristic ensures the reliability of the decision-making process even in scenarios characterized by imprecise or conflicting data.

Finally, the methodology offers a holistic analytical approach by not only quantifying the direct relationships between criteria but also capturing the indirect influences that emerge within the system. This comprehensive analysis provides a deeper understanding of interdependence and dynamics among the criteria. As a result, decision-makers can identify key drivers (causal factors) and outcomes (effects) with greater precision, facilitating strategic interventions and informed decision-making. These contributions collectively highlight the practical and theoretical significance of the proposed methodology in addressing complex evaluation problems across various domains.

5.1 Sensitivity Analysis

This section evaluates the robustness of the proposed methodology by conducting a sensitivity analysis. The analysis investigates how variations in input values, specifically D (sum of rows) and R (sum of columns) from the total relation matrix, affect the calculated metrics D+R (Total Importance) and D−R (Net Effect).

By applying adjustments of −10%, −5%, 0%(baseline), +5%, and + 10 % to these values, we aim to understand the stability of the results and identify criteria that are more sensitive to changes.

Figure 4 shows the sensitivity of D+R, which represents the total importance of each criterion. This graph highlights how the influence of criteria such as C4 remains dominant across all scenarios, with its values increasing or decreasing proportionally with adjustments. Conversely, criteria like C2, which have lower baseline values, exhibit relatively smaller variations in D+R.

Figure 5 focuses on the sensitivity of D−R, representing the net effect of each criterion. This graph illustrates how C4 consistently emerges as the most influential cause (positive D−R) regardless of changes in input values. On the other hand, C2 remains the least impactful criterion (negative D−R), indicating its reliance on improvements in other areas.

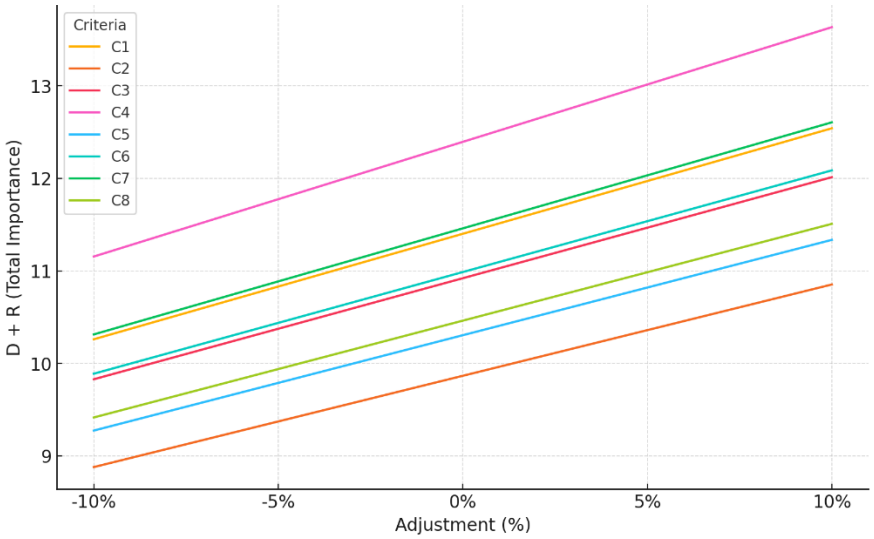

Figure 4: Sensitivity of D+R (Total Importance)

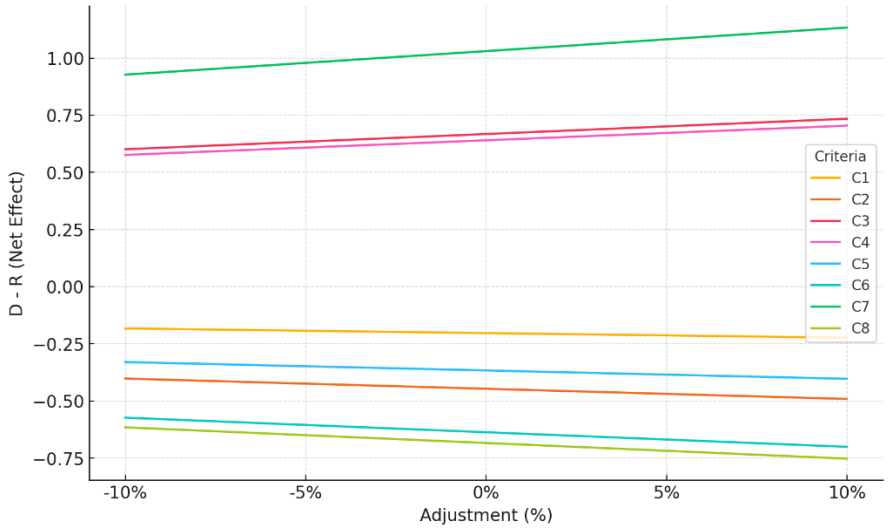

Figure 5: Sensitivity of D-R (Net Effect)

C4 (Problem-Solving Skills) consistently holds the highest total importance across all adjustment scenarios, while C7 (Assessment and Feedback Quality) emerges as the most influential criterion with the highest positive net effect. On the other hand, C2 (Classroom Engagement) demonstrates the lowest total importance, and C8 (Use of Technology) consistently shows the lowest net effect, indicating their peripheral roles. The stability of these rankings and classifications across all scenarios confirms the robustness and reliability of the DEMATEL-based framework integrated with the hypersoft set approach. These findings emphasize the critical roles of C4 and C7 in enhancing teaching quality and offer valuable guidance for educators and policymakers in optimizing resource allocation and strategic planning.

4. Conclusions

This paper proposed a multi-criteria decision-making approach to evaluate teaching quality in university mathematics classrooms based on core competencies. By employing the DEMATEL method, the study analyzed the relationships among various evaluation criteria, uncovering their interdependence and influence levels. A panel of three experts provided assessments to construct the pairwise comparison matrix, which served as the foundation for subsequent computations.

SVNNs were used to address uncertainty and ambiguity in the evaluation process. A scoring function was applied to transform the neutrosophic data into crisp values, which were then aggregated into a unified matrix. The hypersoft set framework was employed to model the criteria and their relationships, enabling a more comprehensive and robust analysis. The normalized direct relation matrix and the total relation matrix provided detailed insights into the causal structure of the criteria.

Among the eight criteria evaluated, C4 (Problem-Solving Skills) emerged as the most impactful, highlighting its critical role in enhancing teaching quality. In contrast, C2 (Classroom Engagement) was identified as having the lowest impact, indicating its reliance on improvements in other areas. These findings offer actionable insights for educators and policymakers to focus on key drivers, allocate resources strategically, and optimize teaching practices to achieve better outcomes.

4.1 Limitations

While this study presents a robust evaluation framework, it is not without its limitations. Firstly, the analysis relies on input from only three experts, which may limit the generalizability of the findings. Expanding the pool of experts could provide a broader perspective and enhance the reliability of the results. Secondly, the evaluation is context-specific, focusing on university mathematics classrooms. The criteria and their relationships may differ in other educational settings or disciplines, requiring adaptations to the proposed framework.

Another limitation is the reliance on the DEMATEL method alone. While effective for analyzing causal relationships, integrating other MCDM methods, such as AHP or TOPSIS, could provide complementary insights and further validate the findings. Additionally, the use of neutrosophic sets, while powerful in handling uncertainty, requires expertise to interpret the results, potentially limiting its application for non-specialists.

4.2 Future Work

Future research could address these limitations by involving a larger and more diverse group of experts from various educational institutions. This would improve the robustness and applicability of the framework across different contexts. Expanding the study to include other academic disciplines and educational levels could also provide a more comprehensive understanding of teaching quality evaluation.

The integration of complementary MCDM methods, such as fuzzy AHP or hybrid approaches combining DEMATEL with TOPSIS, could enhance the depth of analysis and provide additional perspectives on the interdependencies among criteria. Furthermore, incorporating dynamic data collection methods, such as real-time feedback from students or advanced learning analytics, could make the evaluation process more responsive and adaptive to changing classroom dynamics.

Developing user-friendly software tools to automate the framework's calculations and visualizations would also be a valuable step. Such tools could make the proposed methodology accessible to a wider audience, including educators and policymakers with limited technical expertise.

4.3 Practical Implications

The findings of this study have significant practical implications for improving teaching quality in university mathematics classrooms. Prioritizing C4 (Problem-Solving Skills) can have a cascading positive effect on other dependent criteria, such as conceptual understanding and classroom engagement. Policymakers and educators should design interventions that promote active participation, including interactive teaching methods, collaborative activities, and the integration of digital tools.

Although C2 (Classroom Engagement) was identified as having the lowest direct impact, it remains a critical outcome of effective teaching. Indirect improvements to classroom engagement can be achieved by focusing on foundational and influential criteria, such as problem-solving skills and assessment quality.

These insights can guide the allocation of resources, and the development of professional training programs aimed at equipping educators with effective strategies to address these areas.

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Intuitionistic Quadri-Partitioned Neutrosophic Sets with Extended Operators

Sai Anu P[1], Arulselvam A[2] and Bobin A[3,*]

[1]Department of Mathematics, Bharath Institute of Higher Education and Research, Chennai, Tamilnadu; saianupriya@gmail.com

[2]Department of Mathematics, Bharath Institute of Higher Education and Research, Chennai, Tamilnadu; arulselvam.a91@gmail.com.

[3]Department of Mathematics, IFET College of Engineering (An Autonomous Institution), Villupuram, Tamilnadu.

*Correspondence: bobinalbert@gmail.com

Abstract. We present the concept of intuitionistic quadri-partitioned neutrosophic sets (IQ-PNSs) and extend it to intuitionistic quadri-partitioned neutrosophic soft sets (IQ-PNSSs). These sets introduce a constraint where the combined membership values for truth and falsity must not exceed one, while the indeterminacy and contradiction membership values are restricted to the interval $[0, 1]$, ensuring the total sum of all membership values remains below three. In addition, we define necessity and possibility operators, along with two novel operators, ($\oplus$ and $\ominus$), on IQ-PNSSs and explore their properties. Furthermore, we introduce and analyze the operators Δ_μ, $\Delta_{\mu,\nu}$, and $\delta_{\mu,\nu}$, as well as the concentration ($\mathcal{CO}$) and dilation ($\mathcal{DO}$) operators on IQ-PNSSs. This framework effectively addresses complex uncertainties, particularly in situations where conventional approaches, such as those used for modeling ambivalent psychological behaviors, are insufficient.

Keywords:soft set; intuitionistic set; neutrosophic set; quadri-partitioned neutrosophic set.

1. Introduction

The concept of fuzzy sets (FS) was first introduced by Zadeh [31], where the membership of each element is represented by a value within the closed interval $[0, 1]$. Later, Atanassov [3] expanded this concept by proposing intuitionistic fuzzy sets (IFS), incorporating both membership and non-membership values, with the condition that their combined total does not exceed one. Building on this foundation, Atanassov [4] explored various properties of IFS. Subsequent advancements were made by Smarandache [26], who introduced the neutrosophic set (NS), characterized by three membership functions: truth, indeterminacy, and

falsity. Neutrosophic theory has been applied extensively across numerous fields, particularly in decision-making. To make neutrosophic theory more practical, Wang et al. [30] introduced the single-valued neutrosophic set (SVNS), which provides a more adaptable framework for real-world applications. Researchers like Smarandache [27] have applied neutrosophic theory to psychological concepts such as the soul, and Christianto and Smarandache [17] reviewed the use of neutrosophic logic in cultural psychology. Similarly, Chicaiza et al. [12] investigated emotional intelligence among students using neutrosophic psychology, while Smarandache [28] introduced the concepts of dependence and independence in FS and NS. Smarandache [29] also discussed hypersoft sets as an extension of soft set theory. The growing use of NS and SVNS in psychology demonstrates their significance.

In the field of medical diagnosis, Shahzadi et al. [25] employed SVNSs, while Broumi et al. [9] developed single-valued neutrosophic graphs as a generalization of fuzzy and intuitionistic fuzzy graphs. Other significant advancements include Hamidi et al.'s [18] application of neutrosophic graphs in sensor network analysis, and Beg and Tabasam's [5] exploration of comparative linguistic expressions using hesitant IFSs. Researchers such as Anita et al. [2] utilized interval-valued IFS for multi-criteria decision-making (MCDM), while Jianming et al. [20] proposed weighted aggregation operators for neutrosophic cubic sets (NCSs) in similar applications. The development of algebraic and Einstein operators on NCSs by Majid Khan et al. [22] has further advanced MCDM methodologies. Other contributions, such as Chinnadurai et al.'s [13] creation of a ranking system for neutrosophic environments, and Chinnadurai and Bobin's [14] application of prospect theory to MCDM, emphasize the practical applications of these models. Additionally, Chatterjee et al. [10] and Abdel Basset et al. [1] have contributed to the development of similarity measures (SMs) within the neutrosophic framework, including bipolar SVNSs for diagnosing bipolar disorder behaviors. Programming tools for comparing NSs, as developed by Saranya et al. [24], and similarity measures based on Hausdorff and Euclidean distances, introduced by Broumi and Smarandache [7] and Lix et al. [21], have further strengthened these approaches. In a specialized context, Hashim et al. [19] applied similarity measures in neutrosophic bipolar fuzzy sets to assist the HOPE foundation in establishing a children's hospital.

In set theory, Bhowmik and Pal [6] introduced the intuitionistic neutrosophic set (INS) and studied its properties, while Broumi and Smarandache [8] extended this work by defining the intuitionistic neutrosophic soft set (INSS) and examining its characteristics. Chinnadurai and Bobin [15] made a significant contribution to decision-making methodologies by introducing the Simplified-Intuitionistic Neutrosophic Soft Set (INSS) framework. Additionally, they [16] proposed a novel approach to addressing decision-making challenges in uncertain

environments through the development of the Interval-Valued INSS. This framework is particularly focused on its application in diagnosing psychiatric disorders, a domain where ambiguity and incomplete information often hinder effective decision-making. However, no existing set addresses the scenario where truth and falsity membership values are interdependent, while indeterminacy and contradiction are independent. To fill this gap, we propose intuitionistic quadri-partitioned neutrosophic soft sets (IQ-PNSS), which offer more flexible decision-making conditions. This study highlights the relevance of IQ-PNSS, particularly in situations where experts assign membership values for truth, indeterminacy, contradiction, and falsity under specific constraints.

2. **Preliminaries**

Let $\mathcal{U}$ be the universal set, with $\triangle \in \mathcal{U}$, and let $\mathcal{M}$ represent a set of parameters, where Λ is a subset of $\mathcal{M}$. Moreover, let $\mathcal{I}^{Q}$ stand for the set containing all IQ-PNS over $\mathcal{U}$. The following notations will be used consistently throughout the manuscript, unless explicitly stated otherwise.

Definition 2.1. [31] A FS is represented as $\mathcal{F} = \{(\triangle, \mathcal{P}_{\mathcal{F}}(\triangle)) : \triangle \in \mathcal{U}\}$, where the function $\mathcal{P}_{\mathcal{F}} : \mathcal{U} \to [0,1]$ assigns a membership degree to each element $\triangle \in \mathcal{U}$.

Definition 2.2. [3] An IFS is given as $\mathcal{I} = \{(\triangle, \mathcal{P}_{\mathcal{I}}(\triangle), \mathcal{S}_{\mathcal{I}}(\triangle)) : \triangle \in \mathcal{U}\}$, where $\mathcal{P}_{\mathcal{I}}(\triangle) : \mathcal{U} \to [0,1]$ and $\mathcal{S}_{\mathcal{I}}(\triangle) : \mathcal{U} \to [0,1]$ denote the membership and non-membership degrees of each element $\triangle \in \mathcal{U}$, respectively. For all $\triangle \in \mathcal{U}$, $0 \leq \mathcal{P}_{\mathcal{I}}(\triangle) + \mathcal{S}_{\mathcal{I}}(\triangle) \leq 1$, and the degree of hesitation is given by $\pi_{\mathcal{I}}(\triangle) = 1 - \mathcal{P}_{\mathcal{I}}(\triangle) - \mathcal{S}_{\mathcal{I}}(\triangle)$.

Definition 2.3. [30] A SVNS is defined as $\mathcal{N} = \{\langle \triangle, \mathcal{P}_{\mathcal{N}(\triangle)}, \mathcal{Q}_{\mathcal{N}}(\triangle), \mathcal{S}_{\mathcal{N}}(\triangle)\rangle : \triangle \in \mathcal{U}\}$, where $\mathcal{P}_{\mathcal{N}}(\triangle) : \mathcal{U} \to [0,1]$, $\mathcal{Q}_{\mathcal{N}}(\triangle) : \mathcal{U} \to [0,1]$ and $\mathcal{S}_{\mathcal{N}}(\triangle) : \mathcal{U} \to [0,1]$ represent the degrees of truth, indeterminacy, and falsity membership, respectively. For each $\triangle \in \mathcal{U}$, $0 \leq \mathcal{P}_{\mathcal{N}}(\triangle) + \mathcal{Q}_{\mathcal{N}}(\triangle) + \mathcal{S}_{\mathcal{N}}(\triangle) \leq 3$.

Definition 2.4. [23] A soft set (SS) over a universe $\mathcal{U}$ is a pair (δ, Λ), where $\delta : \Lambda \to \mathcal{P}(\mathcal{U})$. For each $e \in \Lambda$, $\delta(e) = 1$ implies $\triangle \in \delta(e)$, and $\delta(e) = 0$ implies $\triangle \notin \delta(e)$. Therefore, a soft set is not an ordinary set but rather a parameterized collection of subsets of $\mathcal{U}$.

Definition 2.5. [11] A quadri-partitioned neutrosophic set (Q-PNS) is given as $\mathfrak{N} = \{\langle \triangle, \mathcal{P}_{\mathfrak{N}}(\triangle), \mathcal{Q}_{\mathfrak{N}}(\triangle), \mathcal{R}_{\mathfrak{N}}(\triangle), \mathcal{S}_{\mathfrak{N}}(\triangle)\rangle : \triangle \in \mathcal{U}\}$, where $\mathcal{P}_{\mathfrak{N}}(\triangle) : \mathcal{U} \to [0,1]$, $\mathcal{Q}_{\mathfrak{N}}(\triangle) : \mathcal{U} \to [0,1]$, $\mathcal{R}_{\mathfrak{N}}(\triangle) : \mathcal{U} \to [0,1]$ and $\mathcal{S}_{\mathfrak{N}}(\triangle) : \mathcal{U} \to [0,1]$ represent the membership degrees for truth, indeterminacy, contradiction, and falsity, respectively. For every $\triangle \in \mathcal{U}$, it holds that $0 \leq \mathcal{P}_{\mathfrak{N}}(\triangle) + \mathcal{Q}_{\mathfrak{N}}(\triangle) + \mathcal{R}_{\mathfrak{N}}(\triangle) + \mathcal{S}_{\mathfrak{N}}(\triangle) \leq 4$.

3. Intuitionistic quadri-partitioned neutrosophic set

We introduce the notions of IQ-PNS and IQ-PNSS, along with a detailed examination of their properties.

Definition 3.1. An IQ-PNS within the universe $\mathcal{U}$ can be expressed as $\mathcal{A} = \{\langle \triangle, \mathcal{P}_{\mathcal{A}}(\triangle), \mathcal{Q}_{\mathcal{A}}(\triangle), \mathcal{R}_{\mathcal{A}}(\triangle), \mathcal{S}_{\mathcal{A}(\triangle)}, \rangle\}$, $\mathcal{P}_{\mathcal{A}}(\triangle) : \mathcal{U} \to [0,1]$, $\mathcal{Q}_{\mathcal{A}}(\triangle) : \mathcal{U} \to [0,1]$, $\mathcal{R}_{\mathcal{A}}(\triangle) : \mathcal{U} \to [0,1]$ and $\mathcal{S}_{\mathcal{A}}(\triangle) : \mathcal{U} \to [0,1]$ denote the degrees of truth, indeterminacy, contradiction, and falsity, respectively, for an element $\triangle \in \mathcal{U}$. These functions must satisfy the following conditions: $0 \leq \mathcal{P}_{\mathcal{A}}(\triangle) + \mathcal{S}_{\mathcal{A}(\triangle)} \leq 1$ and $0 \leq \mathcal{P}_{\mathcal{A}}(\triangle) + \mathcal{Q}_{\mathcal{A}(\triangle)} + \mathcal{R}_{\mathcal{A}(\triangle)} + \mathcal{S}_{\mathcal{A}(\triangle)} \leq 3$.

Example 3.2. Let $\mathcal{U} = \{\triangle_1, \triangle_2, \triangle_3\}$ be a set. Then an IQ-PNS is given as, $\mathcal{A} = \{\langle \triangle_1, .2, .9, .7, .8\rangle, \langle \triangle_2, .4, .7, .8, .2\rangle, \langle \triangle_3, 0, .9, .6, .7\rangle\}$.

Definition 3.3. A pair (δ, Λ) is called IQ-PNSS over $\mathcal{U}$, where $\delta : \Lambda \to \mathcal{I}^{Q}$. Thus, for a parameter $e \in \Lambda$, $\delta(e)$ is an IQ-PNSS.

Example 3.4. Let $\mathcal{U} = \{\triangle_1, \triangle_2, \triangle_3, \triangle_4\}$ represent a group of patients with psychological conditions, and let $\Lambda = \{e_1, e_2, e_3, e_4\}$ denote a set of symptoms corresponding to anxiety (ANX), depression (DEP), mood swings (MS), and difficulty concentrating (DC), respectively. An IQ-PNSS (δ, Λ) be a subsets of $\mathcal{U}$, as diagnosed by a psychologist, based on the information provided in Table 1.

Table 1. Subjects with psychological disorders in IQ-PNSS (δ, Λ) form.

$\mathcal{U}$	ANX(e_1)	DEP(e_2)	MS(e_3)	DC(e_4)
$\triangle_1$	$\langle .4, .2, .3, .5\rangle$	$\langle .6, .4, .3, .2\rangle$	$\langle .4, .5, .7, .4\rangle$	$\langle .1, .2, .3, .5\rangle$
$\triangle_2$	$\langle 0, .3, .4, .8\rangle$	$\langle 1, .3, .9, .0\rangle$	$\langle .4, .8, .9, .3\rangle$	$\langle .4, .6, .7, .3\rangle$
$\triangle_3$	$\langle .5, .4, .2, .5\rangle$	$\langle .3, .5, .7, .6\rangle$	$\langle .1, .3, .7, .8\rangle$	$\langle .3, .5, .9, .5\rangle$
$\triangle_4$	$\langle .6, .9, .4, .4\rangle$	$\langle .2, .4, .6, .7\rangle$	$\langle .6, .7, .8, .4\rangle$	$\langle .2, .5, .8, .7\rangle$

Definition 3.5. Let (δ_1, Θ_1) and (δ_2, Θ_2) be two IQ-PNSS over $\mathcal{U}$. Then,
(i) (δ_1, Θ_1) AND (δ_2, Θ_2) is an IQ-PNSS represented as $(\delta_1, \Theta_1) \wedge (\delta_2, \Theta_2) = (\delta_{\wedge}, \Theta_1 \times \Theta_2)$, where $\delta_{\wedge}(\varrho_1, \varrho_2) = \delta_1(\varrho_1) \cap \delta_2(\varrho_2)$, $\forall\ (\varrho_1, \varrho_2) \in \Theta_1 \times \Theta_2$.

$$\delta_{\wedge}(\varrho_1, \varrho_2) = \big\langle \min(\mathcal{P}_{\delta_1(\varrho_1)}(\triangle), \mathcal{P}_{\delta_2(\varrho_2)}(\triangle)), \min(\mathcal{Q}_{\delta_1(\varrho_1)}(\triangle), \mathcal{Q}_{\delta_2(\varrho_2)}(\triangle)),$$
$$\min(\mathcal{R}_{\delta_1(\varrho_1)}(\triangle), \mathcal{R}_{\delta_2(\varrho_2)}(\triangle)), \max(\mathcal{S}_{\delta_1(\varrho_1)}(\triangle), \mathcal{S}_{\delta_2(\varrho_2)}(\triangle))\big\rangle, \forall\ (\varrho_1, \varrho_2) \in \Theta_1 \times \Theta_2;$$

(ii) (δ_1, Θ_1) OR (δ_2, Θ_2) is an IQ-PNSS represented as $(\delta_1, \Theta_1) \vee (\delta_2, \Theta_2) = (\delta_{\vee}, \Theta_1 \times \Theta_2)$, where $\delta_{\vee}(\varrho_1, \varrho_2) = \delta_1(\varrho_1) \cup \delta_2(\varrho_2)$, $\forall\ (\varrho_1, \varrho_2) \in \Theta_1 \times \Theta_2$.

$$\delta_{\vee}(\varrho_1, \varrho_2) = \big\langle \max(\mathcal{P}_{\delta_1(\varrho_1)}(\triangle), \mathcal{P}_{\delta_2(\varrho_2)}(\triangle)), \max(\mathcal{Q}_{\delta_1(\varrho_1)}(\triangle), \mathcal{Q}_{\delta_2(\varrho_2)}(\triangle)),$$
$$\max(\mathcal{R}_{\delta_1(\varrho_1)}(\triangle), \mathcal{R}_{\delta_2(\varrho_2)}(\triangle)), \min(\mathcal{S}_{\delta_1(\varrho_1)}(\triangle), \mathcal{S}_{\delta_2(\varrho_2)}(\triangle))\big\rangle, \forall\ (\varrho_1, \varrho_2) \in \Theta_1 \times \Theta_2.$$

Example 3.6. Let $\delta_1 = \langle .1, .4, .3, .9\rangle$ and $\delta_1 = \langle 1, .1, .5, .0\rangle$ be two IQ-PNSS. Then
(i) $\delta_1 \wedge \delta_2 = \langle .1, .1, .3, .9\rangle$;
(ii) $\delta_1 \vee \delta_2 = \langle 1, .4, .5, 0\rangle$.

Definition 3.7. Let (δ_1, Θ_1) and (δ_2, Θ_2) be two IQ-PNSS over $\mathcal{U}$. Then,
(i) (δ_1, Θ_1) union (δ_2, Θ_2) is an IQ-PNSS represented as $(\delta_1, \Theta_1) \Cup (\delta_2, \Theta_2) = (\delta_{\Cup}, \Theta_{\Cup})$, where $\Theta_{\Cup} = \Theta_1 \cup \Theta_2$ and $\forall\ \varrho \in \Theta_{\Cup}$,

$$\delta_{\Cup}(\varrho) = \begin{cases} \left\{\left\langle \triangle, (\mathcal{P}_{\delta_1(\varrho)}(\triangle), \mathcal{Q}_{\delta_1(\varrho)}(\triangle), \mathcal{R}_{\delta_1(\varrho)}(\triangle), \mathcal{S}_{\delta_1(\varrho)}(\triangle))\right\rangle ; \right. & \text{if } \varrho \in \Theta_1 - \Theta_2\Big\}, \\ \left\{\left\langle \triangle, (\mathcal{P}_{\delta_2(\varrho)}(\triangle), \mathcal{Q}_{\delta_2(\varrho)}(\triangle), \mathcal{R}_{\delta_1(\varrho)}(\triangle), \mathcal{S}_{\delta_2(\varrho)}(\triangle))\right\rangle ; \right. & \text{if } \varrho \in \Theta_2 - \Theta_1\Big\}, \\ \Big\{\Big\langle\ \triangle, \max(\mathcal{P}_{\delta_1(\varrho)}(\triangle), \mathcal{P}_{\delta_2(\varrho)}(\triangle)), \max(\mathcal{Q}_{\delta_1(\varrho)}(\triangle), \mathcal{Q}_{\delta_2(\varrho)}(\triangle)), & \\ \qquad \max(\mathcal{R}_{\delta_1(\varrho)}(\triangle), \mathcal{P}_{\delta_2(\varrho)}(\triangle)), \min(\mathcal{S}_{\delta_1(\varrho)}(\triangle), \mathcal{S}_{\delta_2(\varrho)}(\triangle))\Big\rangle ; & \text{if } \varrho \in \Theta_1 \cap \Theta_2\Big\}. \end{cases}$$

(ii) (δ_1, Θ_1) intersection (δ_2, Θ_2) is a IQ-PNSS represented as $(\delta_1, \Theta_1) \Cap (\delta_2, \Theta_2) = (\delta_{\Cap}, \Theta_{\Cap})$, where $\Theta_{\Cap} = \Theta_1 \cup \Theta_2$ and $\forall\ \varrho \in \Theta_{\Cap}$,

$$\delta_{\Cap}(\varrho) = \begin{cases} \left\{\left\langle \triangle, (\mathcal{P}_{\delta_1(\varrho)}(\triangle), \mathcal{Q}_{\delta_1(\varrho)}(\triangle), \mathcal{S}_{\delta_1(\varrho)}(\triangle))\right\rangle ; \right. & \text{if } \varrho \in \Theta_1 - \Theta_2\Big\}, \\ \left\{\left\langle \triangle, (\mathcal{P}_{\delta_2(\varrho)}(\triangle), \mathcal{Q}_{\delta_2(\varrho)}(\triangle), \mathcal{S}_{\delta_2(\varrho)}(\triangle))\right\rangle ; \right. & \text{if } \varrho \in \Theta_2 - \Theta_1\Big\}, \\ \Big\{\Big\langle\ \triangle, \min(\mathcal{P}_{\delta_1(\varrho)}(\triangle), \mathcal{P}_{\delta_2(\varrho)}(\triangle)), \min(\mathcal{Q}_{\delta_1(\varrho)}(\triangle), \mathcal{Q}_{\delta_2(\varrho)}(\triangle)), & \\ \qquad \min(\mathcal{R}_{\delta_1(\varrho)}(\triangle), \mathcal{Q}_{\delta_2(\varrho)}(\triangle)), \max(\mathcal{S}_{\delta_1(\varrho)}(\triangle), \mathcal{S}_{\delta_2(\varrho)}(\triangle))\Big\rangle ; & \text{if } \varrho \in \Theta_1 \cap \Theta_2\Big\}. \end{cases}$$

Definition 3.8. The complement of an IQ-PNSS (δ, Λ) can be described as follows:

$$(\delta, \Lambda)^c = \left\{\left\langle \triangle, \mathcal{S}_{\delta(\varrho)}(\triangle), (1 - \mathcal{Q}_{\delta(\varrho)})(\triangle), (1 - \mathcal{R}_{\delta(\varrho)})(\triangle), \mathcal{P}_{\delta(\varrho)}(\triangle)\right\rangle ; \text{and } \varrho \in \Lambda\right\}.$$

Example 3.9. Let $\delta = \langle .5, .2, .3, .2\rangle$ be an IQ-PNSS. Then $\delta^c = \langle .2, .8, .7, .5\rangle$.

Theorem 3.10. *Let (δ_1, Θ_1) and (δ_2, Θ_2) be two IQ-PNSS over $\mathcal{U}$. Then,*
(i) $\langle(\delta_1, \Theta_1) \wedge (\delta_2, \Theta_2)\rangle^c = (\delta_1, \Theta_1)^c \vee (\delta_2, \Theta_2)^c$;
(ii) $\langle(\delta_1, \Theta_1) \vee (\delta_2, \Theta_2)\rangle^c = (\delta_1, \Theta_1)^c \wedge (\delta_2, \Theta_2)^c$.

Proof. We provide the proof for (i); the proof for (ii) follows similarly.
(i) $(\delta_1, \Theta_1) \wedge (\delta_2, \Theta_2) = (\delta_{\wedge}, \Theta_1 \times \Theta_2)$.
$\langle(\delta_1, \Theta_1) \wedge (\delta_2, \Theta_2)\rangle^c = (\delta_{\wedge}, \Theta_1 \times \Theta_2)^c$.

$$\begin{aligned} \delta^c_{\wedge}(\varrho_1, \varrho_2) = \big\langle &\max(\mathcal{S}_{\delta_1(\varrho_1)}(\triangle), \mathcal{S}_{\delta_2(\varrho_2)}(\triangle)), \max((1 - \mathcal{Q}_{\delta_1(\varrho_1)}(\triangle)), (1 - \mathcal{Q}_{\delta_2(\varrho_2)}(\triangle))), \\ &\max((1 - \mathcal{R}_{\delta_1(\varrho_1)}(\triangle)), (1 - \mathcal{R}_{\delta_2(\varrho_2)}(\triangle))), \\ &\min(\mathcal{P}_{\delta_1(\varrho_1)}(\triangle), \mathcal{P}_{\delta_2(\varrho_2)}(\triangle))\big\rangle, \forall(\varrho_1, \varrho_2) \in \Theta_1 \times \Theta_2. \end{aligned}$$

$(\delta_1, \Theta_1)^c \vee (\delta_2, \Theta_2)^c = (\delta_\vee, \Theta_1 \times \Theta_2)$.

$$\delta_\vee(\varrho_1, \varrho_2) = \big\langle \max(\mathcal{S}_{\delta_1(\varrho_1)}(\triangle), \mathcal{S}_{\delta_2(\varrho_2)}(\triangle)), \max((1 - \mathcal{Q}_{\delta_1(\varrho_1)}(\triangle)), (1 - \mathcal{Q}_{\delta_2(\varrho_2)}(\triangle))),$$
$$\max((1 - \mathcal{R}_{\delta_1(\varrho_1)}(\triangle)), (1 - \mathcal{R}_{\delta_2(\varrho_2)}(\triangle))),$$
$$\min(\mathcal{P}_{\delta_1(\varrho_1)}(\triangle), \mathcal{P}_{\delta_2(\varrho_2)}(\triangle))\big\rangle, \forall(\varrho_1, \varrho_2) \in \Theta_1 \times \Theta_2.$$

Thus, $\langle(\delta_1, \Theta_1) \wedge (\delta_2, \Theta_2)\rangle^c = (\delta_1, \Theta_1)^c \vee (\delta_2, \Theta_2)^c$. □

Definition 3.11. Let Θ_1 and Θ_2 be subsets of Λ. The pair (δ_1, Θ_1) is called an intuitionistic quadri-partitioned neutrosophic soft subset (IQ-PNSSS) of (δ_2, Θ_2), denoted by $(\delta_1, \Theta_1) \Subset (\delta_2, \Theta_2)$, iff the below given conditions are satisfied:
(i) $\Theta_1 \subseteq \Theta_2$;
(ii) For each $\varrho \in \Theta_1$, $\delta_1(\varrho)$ is an IQ-PNSSS of $\delta_2(\varrho)$, that is, $\mathcal{P}_{\delta_1(\varrho)}(\triangle) \leq \mathcal{P}_{\delta_2(\varrho)}(\triangle)$, $\mathcal{Q}_{\delta_1(\varrho)}(\triangle) \leq \mathcal{Q}_{\delta_2(\varrho)}(\triangle)$, $\mathcal{R}_{\delta_1(\varrho)}(\triangle) \leq \mathcal{R}_{\delta_2(\varrho)}(\triangle)$, and $\mathcal{S}_{\delta_1(\varrho)}(\triangle) \geq \mathcal{S}\delta_2(\varrho)(\triangle)$.
Additionally, (δ_2, Θ_2) is referred to as an intuitionistic quadri-partitioned neutrosophic soft superset of (δ_1, Θ_1) and is represented as $(\delta_2, \Theta_2) \Supseteq (\delta_1, \Theta_1)$.

Definition 3.12. Let (δ_1, Θ_1) and (δ_2, Θ_2) be IQ-PNSSs, then $(\delta_1, \Theta_1) = (\delta_2, \Theta_2)$ iff $(\delta_1, \Theta_1) \Subset (\delta_2, \Theta_2)$ and $(\delta_2, \Theta_2) \Subset (\delta_1, \Theta_1)$.

4. Necessity ($\circledcirc$) and possibility ($\circledast$) operators on IQ-PNSS

We present $\circledcirc$ and $\circledast$ operators on IQ-PNSS and discuss its properties.

Definition 4.1. If (δ, Λ) is an IQ-PNSS over $\mathcal{U}$ and $\delta : \Lambda \to \mathcal{I}^Q$, then,
(i) the necessity operator ($\circledcirc$) is represented as,

$$\circledcirc(\delta, \Lambda) = \left\{\left\langle \triangle, \mathcal{P}_{\circledcirc\delta(\varrho)}(\triangle), \mathcal{Q}_{\circledcirc\delta(\varrho)}(\triangle), \mathcal{R}_{\circledcirc\delta(\varrho)}(\triangle), \mathcal{S}_{\circledcirc\delta(\varrho)}(\triangle)\right\rangle ;\ \varrho \in \Lambda\right\}.$$

Here, $\mathcal{P}_{\circledcirc\delta(\varrho)}(\triangle) = \mathcal{P}_{\delta(\varrho)}(\triangle)$, $\mathcal{Q}_{\circledcirc\delta(\varrho)}(\triangle) = \mathcal{Q}_{\delta(\varrho)}(\triangle)$, $\mathcal{R}_{\circledcirc\delta(\varrho)}(\triangle) = \mathcal{R}_{\delta(\varrho)}(\triangle)$ and $\mathcal{S}_{\circledcirc\delta(\varrho)}(\triangle) = (1 - \mathcal{P}_{\delta(\varrho)}(\triangle))$ denote the degrees of truth, indeterminacy, contradiction and falsity for the object $\triangle$ on the parameter ϱ.

(ii) the possibility operator ($\circledast$) is represented as,

$$\circledast(\delta, \Lambda) = \left\{\left\langle \triangle, \mathcal{P}_{\circledast\delta(\varrho)}(\triangle), \mathcal{Q}_{\circledast\delta(\varrho)}(\triangle), \mathcal{R}_{\circledast\delta(\varrho)}(\triangle), \mathcal{S}_{\circledast\delta(\varrho)}(\triangle)\right\rangle ;\ \varrho \in \Lambda\right\}.$$

Here, $\mathcal{P}_{\circledast\delta(\varrho)}(\triangle) = (1 - \mathcal{S}_{\delta(\varrho)}(\triangle))$, $\mathcal{Q}_{\circledast\delta(\varrho)}(\triangle) = \mathcal{Q}_{\delta(\varrho)}(\triangle)$, $\mathcal{R}_{\circledast\delta(\varrho)}(\triangle) = \mathcal{R}_{\delta(\varrho)}(\triangle)$ and $\mathcal{S}_{\circledast\delta(\varrho)}(\triangle) = \mathcal{S}_{\delta(\varrho)}(\triangle)$ denote the degrees of truth, indeterminacy, contradiction and falsity for the object $\triangle$ on the parameter ϱ.

Example 4.2. (i) The IQ-PNSS $\circledcirc(\delta, \Lambda)$ for Example 3.2 is presented in Table 2.
(ii) The IQ-PNSS $\circledast(\delta, \Lambda)$ for Example 3.2 is presented in Table 3.

TABLE 2. Subjects with psychological disorders by utilizing the $\circledcirc$ operator.

$\mathcal{U}$	ANX(e_1)	DEP(e_2)	MS(e_3)	DC(e_4)
$\triangle_1$	$\langle .4, .2, .3, .6\rangle$	$\langle .6, .4, .3, .4\rangle$	$\langle .4, .5, .7, .6\rangle$	$\langle .1, .2, .3, .9\rangle$
$\triangle_2$	$\langle 0, .3, .4, 1\rangle$	$\langle 1, .3, .9, .0\rangle$	$\langle .4, .8, .9, ..6\rangle$	$\langle .4, .6, .7, .6\rangle$
$\triangle_3$	$\langle .5, .4, .2, .5\rangle$	$\langle .3, .5, .7, .7\rangle$	$\langle .1, .3, .7, .9\rangle$	$\langle .3, .5, .9, .7\rangle$
$\triangle_4$	$\langle .6, .9, .4, .4\rangle$	$\langle .2, .4, .6, .8\rangle$	$\langle .6, .7, .8, .4\rangle$	$\langle .2, .5, .8, .8\rangle$

TABLE 3. Subjects with cognitive disorders by utilizing the $\circledast$ operator.

$\mathcal{U}$	ANX(e_1)	DEP(e_2)	MS(e_3)	DC(e_4)
$\triangle_1$	$\langle .5, .2, .3, .5\rangle$	$\langle .8, .4, .3, .2\rangle$	$\langle .6, .5, .7, .4\rangle$	$\langle .5, .2, .3, .5\rangle$
$\triangle_2$	$\langle .2, .3, .4, .8\rangle$	$\langle 1, .3, .9, .0\rangle$	$\langle .7, .8, .9, .3\rangle$	$\langle .7, .6, .7, .3\rangle$
$\triangle_3$	$\langle .5, .4, .2, .5\rangle$	$\langle .4, .5, .7, .6\rangle$	$\langle .2, .3, .7, .8\rangle$	$\langle .5, .5, .9, .5\rangle$
$\triangle_4$	$\langle .6, .9, .4, .4\rangle$	$\langle .3, .4, .6, .7\rangle$	$\langle .6, .7, .8, .4\rangle$	$\langle .3, .5, .8, .7\rangle$

Theorem 4.3. *Let (δ_1, Θ_1) and (δ_2, Θ_2) be IQ-PNSSs over $\mathcal{U}$. Then,*
(i) $\circledcirc\langle(\delta_1, \Theta_1) \Cup (\delta_2, \Theta_2)\rangle = \circledcirc(\delta_1, \Theta_1) \Cup \circledcirc(\delta_2, \Theta_2)$;
(ii) $\circledcirc\langle(\delta_1, \Theta_1) \Cap (\delta_2, \Theta_2)\rangle = \circledcirc(\delta_1, \Theta_1) \Cap \circledcirc(\delta_2, \Theta_2)$;
(iii) $\circledcirc\circledcirc(\delta_1, \Theta_1) = \circledcirc(\delta_1, \Theta_1)$.

Proof. We provide the proof for (i); the proof for (ii) follows similarly.

(i) Let $(\delta_1, \Theta_1) \Cup (\delta_2, \Theta_2) = (\delta_{\Cup}, \Theta_{\Cup})$, where $\Theta_{\Cup} = \Theta_1 \cup \Theta_2$, $\forall\ \varrho \in \Theta_{\Cup}$.

Consider,

$$\circledcirc\delta_{\Cup}(\varrho) = \begin{cases} \{\langle \triangle, \mathcal{P}_{\delta_1(\varrho)}(\triangle), \mathcal{Q}_{\delta_1(\varrho)}(\triangle), \mathcal{R}_{\delta_1(\varrho)}(\triangle), (1 - \mathcal{P}_{\delta_1(\varrho)}(\triangle))\rangle; & \text{if } \varrho \in \Theta_1 - \Theta_2\}, \\ \left\{\left\langle \triangle, \mathcal{P}_{\delta_2(\varrho)}(\triangle), \mathcal{Q}_{\delta_2(\varrho)}(\triangle), \mathcal{R}_{\delta_1(\varrho)}(\triangle), (1 - \mathcal{P}_{\delta_2(\varrho)}(\triangle))\right\rangle; \right. & \left. \text{if } \varrho_{\in}\Theta_2 - \Theta_1\right\}, \\ \{\langle\ \triangle, \max(\mathcal{P}_{\delta_1(\varrho)}(\triangle), \mathcal{P}_{\delta_2(\varrho)}(\triangle)), \max(\mathcal{Q}_{\delta_1(\varrho)}(\triangle), \mathcal{Q}_{\delta_2(\varrho)}(\triangle)), & \\ \quad \max(\mathcal{R}_{\delta_1(\varrho)}(\triangle), \mathcal{Q}_{\delta_2(\varrho)}(\triangle)), (1 - \max(\mathcal{P}_{\delta_1(\varrho)}(\triangle), \mathcal{P}_{\delta_2(\varrho)}(\triangle)))\rangle; & \text{if } \varrho_{\in}\Theta_1 \cap \Theta_2\}. \end{cases}$$

$$= \begin{cases} \left\{\left\langle \triangle, \mathcal{P}_{\delta_1(\varrho)}(\triangle), \mathcal{Q}_{\delta_1(\varrho)}(\triangle), \mathcal{R}_{\delta_1(\varrho)}(\triangle), (1 - \mathcal{P}_{\delta_1(\varrho)}(\triangle))\right\rangle; \right. & \left. \text{if } \varrho_{\in}\Theta_1 - \Theta_2\right\}, \\ \left\{\left\langle \triangle, \mathcal{P}_{\delta_2(\varrho)}(\triangle), \mathcal{Q}_{\delta_2(\varrho)}(\triangle), \mathcal{R}_{\delta_1(\varrho)}(\triangle), (1 - \mathcal{P}_{\delta_2(\varrho)}(\triangle))\right\rangle; \right. & \left. \text{if } \varrho_{\in}\Theta_2 - \Theta_1\right\}, \\ \{\langle\ \triangle, \max(\mathcal{P}_{\delta_1(\varrho)}(\triangle), \mathcal{P}_{\delta_2(\varrho)}(\triangle)), \max(\mathcal{Q}_{\delta_1(\varrho)}(\triangle), \mathcal{Q}_{\delta_2(\varrho)}(\triangle)), & \\ \quad \max(\mathcal{R}_{\delta_1(\varrho)}(\triangle), \mathcal{R}_{\delta_2(\varrho)}(\triangle)), \min((1 - \mathcal{P}_{\delta_1(\varrho)}(\triangle)), (1 - \mathcal{P}_{\delta_2(\varrho)}(\triangle)))\rangle; & \text{if } \varrho_{\in}\Theta_1 \cap \Theta_2\}. \end{cases}$$

$$\circledcirc(\delta_1, \Theta_1) = \left\{\left\langle \triangle, \mathcal{P}_{\delta_1(\varrho)}(\triangle), \mathcal{Q}_{\delta_1(\varrho)}(\triangle), \mathcal{R}_{\delta_1(\varrho)}(\triangle), (1 - \mathcal{P}_{\delta_1(\varrho)}(\triangle))\right\rangle; \varrho_{\in}\Theta_1\right\},$$

$$\circledcirc(\delta_2, \Theta_2) = \left\{\left\langle \triangle, \mathcal{P}_{\delta_2(\varrho)}(\triangle), \mathcal{Q}_{\delta_2(\varrho)}(\triangle), \mathcal{R}_{\delta_1(\varrho)}(\triangle), (1 - \mathcal{P}_{\delta_2(\varrho)}(\triangle))\right\rangle; \varrho_{\in}\Theta_2\right\}.$$

Let $\circledcirc(\delta_1, \Theta_1) \Cup \circledcirc(\delta_2, \Theta_2) = (\delta_{\circledcirc\Cup}, \Theta_{\circledcirc\Cup})$, where $\Theta_{\circledcirc\Cup} = \Theta_1 \cup \Theta_2$.

For $\varrho_{\in}\Theta_{\circledcirc\Cup}$,

$$\delta_{\circledcirc\Cup}(\varrho_{)} = \begin{cases} \left\{\left\langle \triangle, \mathcal{P}_{\delta_1(\varrho_{)}}(\triangle), \mathcal{Q}_{\delta_1(\varrho_{)}}(\triangle), \mathcal{R}_{\delta_1(\varrho_{)}}(\triangle), (1-\mathcal{P}_{\delta_1(\varrho_{)}}(\triangle))\right\rangle; \quad \text{if } \varrho_{\in}\Theta_1-\Theta_2\right\}, \\ \left\{\left\langle \triangle, \mathcal{P}_{\delta_2(\varrho_{)}}(\triangle), \mathcal{Q}_{\delta_2(\varrho_{)}}(\triangle), \mathcal{R}_{\delta_1(\varrho_{)}}(\triangle), (1-\mathcal{P}_{\delta_2(\varrho_{)}}(\triangle))\right\rangle; \quad \text{if } \varrho_{\in}\Theta_2-\Theta_1\right\}, \\ \{\langle\ \triangle, \max(\mathcal{P}_{\delta_1(\varrho_{)}}(\triangle), \mathcal{P}_{\delta_2(\varrho_{)}}(\triangle)), \max(\mathcal{Q}_{\delta_1(\varrho_{)}}(\triangle), \mathcal{Q}_{\delta_2(\varrho_{)}}(\triangle)), \\ \quad \max(\mathcal{R}_{\delta_1(\varrho_{)}}(\triangle), \mathcal{R}_{\delta_2(\varrho_{)}}(\triangle)), \min((1-\mathcal{P}_{\delta_1(\varrho_{)}}(\triangle)), (1-\mathcal{P}_{\delta_2(\varrho_{)}}(\triangle)))\rangle; \text{ if } \varrho_{\in}\Theta_1\cap\Theta_2\}. \end{cases}$$

Thus, $\circledcirc\langle(\delta_1,\Theta_1)\Cup(\delta_2,\Theta_2)\rangle = \circledcirc(\delta_1,\Theta_1)\Cup\circledcirc(\delta_2,\Theta_2)$.

(iii) $\circledcirc\circledcirc(\delta_1,\Theta_1) = \circledcirc\left\{\left\langle \triangle, \mathcal{P}_{\delta_1(\varrho_{)}}(\triangle), \mathcal{Q}_{\delta_1(\varrho_{)}}(\triangle), \mathcal{R}_{\delta_1(\varrho_{)}}(\triangle), (1-\mathcal{P}_{\delta_1(\varrho_{)}}(\triangle))\right\rangle; \varrho_{\in}\Theta_1\right\}$

$= \left\{\left\langle \triangle, \mathcal{P}_{\delta_1(\varrho_{)}}(\triangle), \mathcal{Q}_{\delta_1(\varrho_{)}}(\triangle), \mathcal{R}_{\delta_1(\varrho_{)}}(\triangle), (1-\mathcal{P}_{\delta_1(\varrho_{)}}(\triangle))\right\rangle; \varrho_{\in}\Theta_1\right\}$

$= \circledcirc(\delta_1,\Theta_1)$. □

Theorem 4.4. *Let (δ_1,Θ_1) and (δ_2,Θ_2) be IQ-PNSSs over $\mathcal{U}$. Then,*

(i) $\circledast\langle(\delta_1,\Theta_1)\Cup(\delta_2,\Theta_2)\rangle = \circledast(\delta_1,\Theta_1)\Cup\circledast(\delta_2,\Theta_2)$;

(ii) $\circledast\langle(\delta_1,\Theta_1)\Cap(\delta_2,\Theta_2)\rangle = \circledast(\delta_1,\Theta_1)\Cap\circledast(\delta_2,\Theta_2)$;

(iii) $\circledast\circledast(\delta_1,\Theta_1) = \circledast(\delta_1,\Theta_1)$.

Proof. We provide the proof for (i); the proof for (ii) follows similarly.

(i) Let $(\delta_1,\Theta_1)\Cup(\delta_2,\Theta_2) = (\delta_{\Cup},\Theta_{\Cup})$, where $\Theta_{\Cup} = \Theta_1\cup\Theta_2$, $\forall\ \varrho_{\in}\Theta_{\Cup}$.

Consider,

$$\circledast\delta_{\Cup}(\varrho_{)} = \begin{cases} \left\{\left\langle \triangle, (1-\mathcal{S}_{\delta_1(\varrho_{)}}(\triangle)), \mathcal{Q}_{\delta_1(\varrho_{)}}(\triangle), \mathcal{R}_{\delta_1(\varrho_{)}}(\triangle), \mathcal{S}_{\delta_1(\varrho_{)}}(\triangle)\right\rangle; \quad \text{if } \varrho_{\in}\Theta_1-\Theta_2\right\}, \\ \left\{\left\langle \triangle, (1-\mathcal{S}_{\delta_2(\varrho_{)}}(\triangle)), \mathcal{Q}_{\delta_2(\varrho_{)}}(\triangle), \mathcal{R}_{\delta_1(\varrho_{)}}(\triangle), \mathcal{S}_{\delta_2(\varrho_{)}}(\triangle)\right\rangle; \quad \text{if } \varrho_{\in}\Theta_2-\Theta_1\right\}, \\ \{\langle\ \triangle, (1-\min(\mathcal{S}_{\delta_1(\varrho_{)}}(\triangle), \mathcal{S}_{\delta_2(\varrho_{)}}(\triangle))), \max(\mathcal{Q}_{\delta_1(\varrho_{)}}(\triangle), \mathcal{Q}_{\delta_2(\varrho_{)}}(\triangle)), \\ \quad \max(\mathcal{R}_{\delta_1(\varrho_{)}}(\triangle), \mathcal{R}_{\delta_2(\varrho_{)}}(\triangle)), \min(\mathcal{S}_{\delta_1(\varrho_{)}}(\triangle), \mathcal{S}_{\delta_2(\varrho_{)}}(\triangle))\rangle; \text{ if } \varrho_{\in}\Theta_1\cap\Theta_2\}. \end{cases}$$

$$= \begin{cases} \left\{\left\langle \triangle, (1-\mathcal{S}_{\delta_1(\varrho_{)}}(\triangle)), \mathcal{Q}_{\delta_1(\varrho_{)}}(\triangle), \mathcal{R}_{\delta_1(\varrho_{)}}(\triangle), \mathcal{S}_{\delta_1(\varrho_{)}}(\triangle)\right\rangle; \quad \text{if } \varrho_{\in}\Theta_1-\Theta_2\right\}, \\ \left\{\left\langle \triangle, (1-\mathcal{S}_{\delta_2(\varrho_{)}}(\triangle)), \mathcal{Q}_{\delta_2(\varrho_{)}}(\triangle), \mathcal{R}_{\delta_1(\varrho_{)}}(\triangle), \mathcal{S}_{\delta_2(\varrho_{)}}(\triangle)\right\rangle; \quad \text{if } \varrho_{\in}\Theta_2-\Theta_1\right\}, \\ \{\langle\ \triangle, \max((1-\mathcal{S}_{\delta_1(\varrho_{)}}(\triangle)), (1-\mathcal{S}_{\delta_2(\varrho_{)}}(\triangle))), \max(\mathcal{Q}_{\delta_1(\varrho_{)}}(\triangle), \mathcal{Q}_{\delta_2(\varrho_{)}}(\triangle)), \\ \quad \max(\mathcal{R}_{\delta_1(\varrho_{)}}(\triangle), \mathcal{R}_{\delta_2(\varrho_{)}}(\triangle)), \min(\mathcal{S}_{\delta_1(\varrho_{)}}(\triangle), \mathcal{S}_{\delta_2(\varrho_{)}}(\triangle))\rangle; \text{ if } \varrho_{\in}\Theta_1\cap\Theta_2\}. \end{cases}$$

$$\circledast(\delta_1,\Theta_1) = \left\{\left\langle \triangle, (1-\mathcal{S}_{\delta_1(\varrho_{)}}(\triangle)), \mathcal{Q}_{\delta_1(\varrho_{)}}(\triangle), \mathcal{R}_{\delta_1(\varrho_{)}}(\triangle), \mathcal{S}_{\delta_1(\varrho_{)}}(\triangle)\right\rangle; \varrho_{\in}\Theta_1\right\},$$

$$\circledast(\delta_2,\Theta_2) = \left\{\left\langle \triangle, (1-\mathcal{S}_{\delta_2(\varrho_{)}}(\triangle)), \mathcal{Q}_{\delta_2(\varrho_{)}}(\triangle), \mathcal{R}_{\delta_1(\varrho_{)}}(\triangle), \mathcal{S}_{\delta_2(\varrho_{)}}(\triangle)\right\rangle; \varrho_{\in}\Theta_2\right\}.$$

Let $\circledast(\delta_1,\Theta_1)\Cup\circledast(\delta_2,\Theta_2) = (\delta_{\circledast\Cup},\Theta_{\circledast\Cup})$, where $\Theta_{\circledast\Cup} = \Theta_1\cup\Theta_2$.

For $\varrho_{\in}\Theta_{\circledast\Cup}$,

$$\delta_{\circledast\Cup}(\varrho_) = \begin{cases} \left\{\left\langle \triangle, (1-\mathcal{S}_{\delta_1(\varrho_)}(\triangle)), \mathcal{Q}_{\delta_1(\varrho_)}(\triangle), \mathcal{R}_{\delta_1(\varrho_)}(\triangle), \mathcal{S}_{\delta_1(\varrho_)}(\triangle)\right\rangle; \quad \text{if } \varrho_{\in}\Theta_1 - \Theta_2\right\}, \\ \left\{\left\langle \triangle, (1-\mathcal{S}_{\delta_2(\varrho_)}(\triangle)), \mathcal{Q}_{\delta_2(\varrho_)}(\triangle), \mathcal{R}_{\delta_1(\varrho_)}(\triangle), \mathcal{S}_{\delta_2(\varrho_)}(\triangle)\right\rangle; \quad \text{if } \varrho_{\in}\Theta_2 - \Theta_1\right\}, \\ \big\{\big\langle \triangle, \max((1-\mathcal{S}_{\delta_1(\varrho_)}(\triangle)), (1-\mathcal{S}_{\delta_2(\varrho_)}(\triangle))), \max(\mathcal{Q}_{\delta_1(\varrho_)}(\triangle), \mathcal{Q}_{\delta_2(\varrho_)}(\triangle)), \\ \quad \max(\mathcal{R}_{\delta_1(\varrho_)}(\triangle), \mathcal{R}_{\delta_2(\varrho_)}(\triangle)), \min(\mathcal{S}_{\delta_1(\varrho_)}(\triangle), \mathcal{S}_{\delta_2(\varrho_)}(\triangle))\big\rangle; \quad \text{if } \varrho_{\in}\Theta_1 \cap \Theta_2\big\}. \end{cases}$$

Thus, $\circledast\langle(\delta_1,\Theta_1) \Cup (\delta_2,\Theta_2)\rangle = \circledast(\delta_1,\Theta_1) \Cup \circledast(\delta_2,\Theta_2)$.

(iii) $\circledast\circledast(\delta_1,\Theta_1) = \circledast\left\{\left\langle \triangle, (1-\mathcal{S}_{\delta_1(\varrho_)}(\triangle)), \mathcal{Q}_{\delta_1(\varrho_)}(\triangle), \mathcal{R}_{\delta_1(\varrho_)}(\triangle), \mathcal{S}_{\delta_1(\varrho_)}(\triangle)\right\rangle; \varrho_{\in}\Theta_1\right\}$

$= \left\{\left\langle \triangle, (1-\mathcal{S}_{\delta_1(\varrho_)}(\triangle)), \mathcal{Q}_{\delta_1(\varrho_)}(\triangle), \mathcal{R}_{\delta_1(\varrho_)}(\triangle), \mathcal{S}_{\delta_1(\varrho_)}(\triangle)\right\rangle; \varrho_{\in}\Theta_1\right\}$

$= \circledast(\delta_1,\Theta_1)$. □

Theorem 4.5. *Let (δ,Λ) be an IQ-PNSS over $\mathcal{U}$. Then,*

(i) $\circledast\circledcirc(\delta,\Lambda) = \circledcirc(\delta,\Lambda)$;

(ii) $\circledcirc\circledast(\delta,\Lambda) = \circledast(\delta,\Lambda)$.

Proof. (i) $\circledast\circledcirc(\delta,\Lambda) = \left\{\left\langle \triangle, (1-(1-\mathcal{P}_{\delta(\varrho_)}(\triangle))), \mathcal{Q}_{\delta(\varrho_)}(\triangle), \mathcal{R}_{\delta(\varrho_)}(\triangle), (1-\mathcal{P}_{\delta(\varrho_)}(\triangle))\right\rangle; \varrho_{\in}\Lambda\right\}$

$= \left\{\left\langle \triangle, \mathcal{P}_{\delta(\varrho_)}(\triangle), \mathcal{Q}_{\delta(\varrho_)}(\triangle), \mathcal{R}_{\delta(\varrho_)}(\triangle), (1-\mathcal{P}_{\delta(\varrho_)}(\triangle))\right\rangle; \varrho_{\in}\Lambda\right\}$

$\circledast\circledcirc(\delta,\Lambda) = \circledcirc(\delta,\Lambda)$.

(ii) $\circledcirc\circledast(\delta,\Lambda) = \left\{\left\langle \triangle, (1-\mathcal{S}_{\delta(\varrho_)}(\triangle)), \mathcal{Q}_{\delta(\varrho_)}(\triangle), \mathcal{R}_{\delta(\varrho_)}(\triangle), (1-(1-\mathcal{S}_{\delta(\varrho_)}(\triangle)))\right\rangle; \varrho_{\in}\Lambda\right\}$

$= \left\{\left\langle \triangle, (1-\mathcal{S}_{\delta(\varrho_)}(\triangle)), \mathcal{Q}_{\delta(\varrho_)}(\triangle), \mathcal{R}_{\delta(\varrho_)}(\triangle), \mathcal{S}_{\delta(\varrho_)}(\triangle)\right\rangle; \varrho_{\in}\Lambda\right\}$

$\circledcirc\circledast(\delta,\Lambda) = \circledast(\delta,\Lambda)$. □

Theorem 4.6. *Let (δ_1,Θ_1) and (δ_2,Θ_2) be IQ-PNSSs over $\mathcal{U}$. Then*

(i) $\circledcirc\langle(\delta_1,\Theta_1) \wedge (\delta_2,\Theta_2)\rangle = \circledcirc(\delta_1,\Theta_1) \wedge \circledcirc(\delta_2,\Theta_2)$;

(ii) $\circledcirc\langle(\delta_1,\Theta_1) \vee (\delta_2,\Theta_2)\rangle = \circledcirc(\delta_1,\Theta_1) \vee \circledcirc(\delta_2,\Theta_2)$;

(iii) $\circledast\langle(\delta_1,\Theta_1) \wedge (\delta_2,\Theta_2)\rangle = \circledast(\delta_1,\Theta_1) \wedge \circledcirc(\delta_2,\Theta_2)$;

(iv) $\circledast\langle(\delta_1,\Theta_1) \vee (\delta_2,\Theta_2)\rangle = \circledast(\delta_1,\Theta_1) \vee \circledcirc(\delta_2,\Theta_2)$.

Proof. We provide the proofs for (i) and (iii); the proofs for (ii) and (iv) follows similarly.

(i) $\circledcirc\langle(\delta_1,\Theta_1) \wedge (\delta_2,\Theta_2)\rangle$

$= \big\{\big\langle \min(\mathcal{P}_{\delta_1(\varrho_1)}(\triangle), \mathcal{P}_{\delta_2(\varrho_2)}(\triangle)), \min(\mathcal{Q}_{\delta_1(\varrho_1)}(\triangle), \mathcal{Q}_{\delta_2(\varrho_2)}(\triangle)),$

$\min(\mathcal{R}_{\delta_1(\varrho_1)}(\triangle), \mathcal{R}_{\delta_2(\varrho_2)}(\triangle)), (1-\min(\mathcal{P}_{\delta_1(\varrho_1)}(\triangle), \mathcal{P}_{\delta_2(\varrho_2)}(\triangle)))\big\rangle, \forall(\varrho_1,\varrho_2) \in \Theta_1 \times \Theta_2\big\}$

$= \big\{\big\langle \min(\mathcal{P}_{\delta_1(\varrho_1)}(\triangle), \mathcal{P}_{\delta_2(\varrho_2)}(\triangle)), \min(\mathcal{Q}_{\delta_1(\varrho_1)}(\triangle), \mathcal{Q}_{\delta_2(\varrho_2)}(\triangle)),$

$\min(\mathcal{R}_{\delta_1(\varrho_1)}(\triangle), \mathcal{R}_{\delta_2(\varrho_2)}(\triangle)), (\max(1-(\mathcal{P}_{\delta_1(\varrho_1)}(\triangle))), (1-(\mathcal{P}_{\delta_2(\varrho_2)}(\triangle))))\big\rangle, \forall(\varrho_1,\varrho_2) \in \Theta_1 \times \Theta_2\big\}$.

Also, $\circledcirc(\delta_1,\Theta_1) = \big\{\big\langle \triangle, \mathcal{P}_{\delta_1(\varrho_1)}(\triangle), \mathcal{Q}_{\delta_1(\varrho_1)}(\triangle), \mathcal{R}_{\delta_1(\varrho_1)}(\triangle), (1-\mathcal{P}_{\delta_1(\varrho_1)}(\triangle))\big\rangle;\ \varrho_1 \in \Theta_1\big\}$,

$\circledcirc(\delta_2,\Theta_2) = \big\{\big\langle \triangle, \mathcal{P}_{\delta_2(\varrho_2)}(\triangle), \mathcal{Q}_{\delta_2(\varrho_2)}(\triangle), \mathcal{R}_{\delta_2(\varrho_2)}(\triangle), (1-\mathcal{P}_{\delta_2(\varrho_2)}(\triangle))\big\rangle;\ \varrho_2 \in \Theta_2\big\}$.

Therefore, we have

$\circledcirc(\delta_1,\Theta_1)\wedge\circledcirc(\delta_2,\Theta_2)$

$=\big\{\big\langle\min(\mathcal{P}_{\delta_1(\varrho_1)}(\triangle),\mathcal{P}_{\delta_2(\varrho_2)}(\triangle)),\min(\mathcal{Q}_{\delta_1(\varrho_1)}(\triangle),\mathcal{Q}_{\delta_2(\varrho_2)}(\triangle)),\min(\mathcal{R}_{\delta_1(\varrho_1)}(\triangle),\mathcal{R}_{\delta_2(\varrho_2)}(\triangle)),$
$(\max(1-(\mathcal{P}_{\delta_1(\varrho_1)}(\triangle))),(1-(\mathcal{P}_{\delta_2(\varrho_2)}(\triangle))))\big\rangle,\forall(\varrho_1,\varrho_2)\in\Theta_1\times\Theta_2\big\}$

$=\circledcirc\langle(\delta_1,\Theta_1)\wedge(\delta_2,\Theta_2)\rangle.$

(iii) $\circledast\langle(\delta_1,\Theta_1)\wedge(\delta_2,\Theta_2)\rangle$

$=\big\{\big\langle(1-\max(\mathcal{S}_{\delta_1(\varrho_1)}(\triangle),\mathcal{S}_{\delta_2(\varrho_2)}(\triangle))),\max(\mathcal{Q}_{\delta_1(\varrho_1)}(\triangle),\mathcal{Q}_{\delta_2(\varrho_2)}(\triangle)),$
$\max(\mathcal{R}_{\delta_1(\varrho_1)}(\triangle),\mathcal{R}_{\delta_2(\varrho_2)}(\triangle)),\max(\mathcal{S}_{\delta_1(\varrho_1)}(\triangle),\mathcal{S}_{\delta_2(\varrho_2)}(\triangle))\big\rangle,\forall(\varrho_1,\varrho_2)\in\Theta_1\times\Theta_2\big\}$

$=\big\{\big\langle(\min(1-(\mathcal{S}_{\delta_1(\varrho_1)}(\triangle))),(1-(\mathcal{S}_{\delta_2(\varrho_2)}(\triangle)))),\max(\mathcal{Q}_{\delta_1(\varrho_1)}(\triangle),\mathcal{Q}_{\delta_2(\varrho_2)}(\triangle)),$
$\max(\mathcal{R}_{\delta_1(\varrho_1)}(\triangle),\mathcal{R}_{\delta_2(\varrho_2)}(\triangle)),\max(\mathcal{S}_{\delta_1(\varrho_1)}(\triangle),\mathcal{S}_{\delta_2(\varrho_2)}(\triangle))\big\rangle,\forall(\varrho_1,\varrho_2)\in\Theta_1\times\Theta_2\big\}.$

Also, $\circledast(\delta_1,\Theta_1)=\big\{\big\langle\triangle,(1-\mathcal{S}_{\delta_1(\varrho_1)}(\triangle)),\mathcal{Q}_{\delta_1(\varrho_1)}(\triangle),\mathcal{R}_{\delta_1(\varrho_1)}(\triangle),\mathcal{S}_{\delta_1(\varrho_1)}(\triangle)\big\rangle;\ \varrho_1\in\Theta_1\big\}$,
$\circledast(\delta_2,\Theta_2)=\big\{\big\langle\triangle,(1-\mathcal{S}_{\delta_2(\varrho_2)}(\triangle)),\mathcal{Q}_{\delta_2(\varrho_2)}(\triangle),\mathcal{R}_{\delta_2(\varrho_2)}(\triangle),\mathcal{S}_{\delta_2(\varrho_2)}(\triangle)\big\rangle;\ \varrho_2\in\Theta_2\big\}.$

Therefore, we have

$\circledast(\delta_1,\Theta_1)\wedge\circledast(\delta_2,\Theta_2)$

$=\big\{\big\langle(\min(1-(\mathcal{S}_{\delta_1(\varrho_1)}(\triangle))),(1-(\mathcal{S}_{\delta_2(\varrho_2)}(\triangle)))),\max(\mathcal{Q}_{\delta_1(\varrho_1)}(\triangle),\mathcal{Q}_{\delta_2(\varrho_2)}(\triangle)),$
$\max(\mathcal{R}_{\delta_1(\varrho_1)}(\triangle),\mathcal{R}_{\delta_2(\varrho_2)}(\triangle)),\max(\mathcal{S}_{\delta_1(\varrho_1)}(\triangle),\mathcal{S}_{\delta_2(\varrho_2)}(\triangle)),\big\rangle\forall(\varrho_1,\varrho_2)\in\Theta_1\times\Theta_2\big\}$

$=\circledcirc\langle(\delta_1,\Theta_1)\wedge(\delta_2,\Theta_2)\rangle.$

□

5. $\oplus$ and $\ominus$ operators on IQ-PNSS

We present the operators ($\oplus$ and $\ominus$) on IQ-PNSS and discuss its properties.

Definition 5.1. Let (δ_1,Θ_1) and (δ_2,Θ_2) be IQ-PNSSs over $\mathcal{U}$. Then,
(i) the operator $\oplus$ is represented as $(\delta_1,\Theta_1)\oplus(\delta_2,\Theta_2)=(\delta_\oplus,\Theta_\oplus)$, where $\Theta_\oplus=\Theta_1\cup\Theta_2$. $\forall$ $\varrho_\in\Theta_\oplus$,

$$\delta_\oplus(\varrho_)=\begin{cases}\Big\{\Big\langle\triangle,(\mathcal{P}_{\delta_1(\varrho_)}(\triangle),\mathcal{Q}_{\delta_1(\varrho_)}(\triangle),\mathcal{R}_{\delta_1(\varrho_)}(\triangle),\mathcal{S}_{\delta_1(\varrho_)}(\triangle))\Big\rangle; & \text{if } \varrho_\in\Theta_1-\Theta_2\Big\},\\ \Big\{\Big\langle\triangle,(\mathcal{P}_{\delta_2(\varrho_)}(\triangle),\mathcal{Q}_{\delta_2(\varrho_)}(\triangle),\mathcal{R}_{\delta_1(\varrho_)}(\triangle),\mathcal{S}_{\delta_2(\varrho_)}(\triangle))\Big\rangle; & \text{if } \varrho_\in\Theta_2-\Theta_1\Big\},\\ \Big\{\big\langle\ \triangle,\frac{(\mathcal{P}_{\delta_1(\varrho_)}(\triangle)+\mathcal{P}_{\delta_2(\varrho_)}(\triangle))}{2},\frac{(\mathcal{Q}_{\delta_1(\varrho_)}(\triangle)+\mathcal{Q}_{\delta_2(\varrho_)}(\triangle))}{2},\\ \qquad\frac{(\mathcal{R}_{\delta_1(\varrho_)}(\triangle)+\mathcal{R}_{\delta_2(\varrho_)}(\triangle))}{2},\frac{(\mathcal{S}_{\delta_1(\varrho_)}(\triangle)+\mathcal{S}_{\delta_2(\varrho_)}(\triangle))}{2}\big\rangle; & \text{if } \varrho_\in\Theta_1\cap\Theta_2\Big\}.\end{cases}$$

(ii) the operator $\ominus$ is represented as $(\delta_1,\Theta_1)\ominus(\delta_2,\Theta_2)=(\delta_\ominus,\Theta_\ominus)$, where $\Theta_\ominus=\Theta_1\cup\Theta_2$. $\forall$ $\varrho_\in\Theta_\ominus$,

$$\delta_{\ominus}(\varrho_{)} = \begin{cases} \left\{\left\langle \triangle, (\mathcal{P}_{\delta_1(\varrho_{)}}(\triangle), \mathcal{Q}_{\delta_1(\varrho_{)}}(\triangle), \mathcal{R}_{\delta_1(\varrho_{)}}(\triangle), \mathcal{S}_{\delta_1(\varrho_{)}}(\triangle))\right\rangle; \right. & \left. \text{if } \varrho_{\in}\Theta_1 - \Theta_2\right\}, \\ \left\{\left\langle \triangle, (\mathcal{P}_{\delta_2(\varrho_{)}}(\triangle), \mathcal{Q}_{\delta_2(\varrho_{)}}(\triangle), \mathcal{R}_{\delta_1(\varrho_{)}}(\triangle), \mathcal{S}_{\delta_2(\varrho_{)}}(\triangle))\right\rangle; \right. & \left. \text{if } \varrho_{\in}\Theta_2 - \Theta_1\right\}, \\ \left\{\left\langle \triangle, \frac{2*(\mathcal{P}_{\delta_1(\varrho_{)}}(\triangle)*\mathcal{P}_{\delta_2(\varrho_{)}}(\triangle))}{\mathcal{P}_{\delta_1(\varrho_{)}}(\triangle)+\mathcal{P}_{\delta_2(\varrho_{)}}(\triangle)}, \frac{(\mathcal{Q}_{\delta_1(\varrho_{)}}(\triangle)+\mathcal{Q}_{\delta_2(\varrho_{)}}(\triangle))}{2}, \right.\right. & \\ \left.\left. \frac{(\mathcal{R}_{\delta_1(\varrho_{)}}(\triangle)+\mathcal{R}_{\delta_2(\varrho_{)}}(\triangle))}{2}, \frac{2*(\mathcal{S}_{\delta_1(\varrho_{)}}(\triangle)*\mathcal{S}_{\delta_2(\varrho_{)}}(\triangle))}{\mathcal{S}_{\delta_1(\varrho_{)}}(\triangle)+\mathcal{S}_{\delta_2(\varrho_{)}}(\triangle)}\right\rangle; \right. & \left. \text{if } \varrho_{\in}\Theta_1 \cap \Theta_2\right\}. \end{cases}$$

Example 5.2. A psychiatrist has conducted two counseling sessions with his clients, providing data in the IQ-PNSS format. The results from the first session, denoted as (δ_1, Θ_1), are shown in Table 1, while the second session's results, represented as (δ_2, Θ_2), are displayed in Table 4. To assess the combined outcomes of both sessions, we calculate $(\delta_1, \Theta_1) \oplus (\delta_2, \Theta_2)$ and $(\delta_1, \Theta_1) \ominus (\delta_2, \Theta_2)$, presenting the corresponding findings in Tables 5 and 6, respectively.

TABLE 4. Subjects with psychological disorders in IQ-PNSS (δ_2, Θ_2) form.

$\mathcal{U}$	ANX(e_1)	DEP(e_2)	MS(e_3)	DC(e_4)
$\triangle_1$	$\langle .2, .3, .5, .7\rangle$	$\langle .1, 1, 1, .8\rangle$	$\langle .4, .6, .2, .5\rangle$	$\langle .3, .6, .9, .4\rangle$
$\triangle_2$	$\langle .3, .2, .1, .6\rangle$	$\langle .7, .5, .6, .2\rangle$	$\langle .9, .8, .7, .1\rangle$	$\langle .7, .9, 1, .2\rangle$
$\triangle_3$	$\langle .2, .7, .8, .8\rangle$	$\langle .4, .7, .6, .3\rangle$	$\langle .5, .4, .8, .4\rangle$	$\langle .9, .2, .7, .1\rangle$
$\triangle_4$	$\langle .4, .3, .5, .5\rangle$	$\langle .3, .4, .5, .5\rangle$	$\langle .6, .7, .6, .3\rangle$	$\langle .7, .5, .9, .2\rangle$

(i) The IQ-PNSS $(\delta_1, \Theta_1) \oplus (\delta_2, \Theta_2)$ is shown in Table 5.

TABLE 5. Subjects with psychological disorders in IQ-PNSS $(\delta_1, \Theta_1) \oplus (\delta_2, \Theta_2)$ form.

$\mathcal{U}$	ANX(e_1)	DEP(e_2)	MS(e_3)	DC(e_4)
$\triangle_1$	$\langle .30, .25, .40, .60\rangle$	$\langle .35, .70, .65, .50\rangle$	$\langle .40, .55, .45, .45\rangle$	$\langle .20, .40, .60, .45\rangle$
$\triangle_2$	$\langle .15, .25, .25, .70\rangle$	$\langle .85, .40, .75, .10\rangle$	$\langle .65, .80, .80, .20\rangle$	$\langle .55, .75, .85, .25\rangle$
$\triangle_3$	$\langle .35, .55, .50, .65\rangle$	$\langle .35, .60, .65, .45\rangle$	$\langle .30, .35, .75, .60\rangle$	$\langle .60, .35, .80, .30\rangle$
$\triangle_4$	$\langle .50, .60, .45, .45\rangle$	$\langle .25, .40, .55, .60\rangle$	$\langle .60, .70, .70, .35\rangle$	$\langle .45, .50, .85, .45\rangle$

(ii) The IQ-PNSS $(\delta_1, \Theta_1) \ominus (\delta_2, \Theta_2)$ is shown in Table 6.

Proposition 5.3. *Let (δ_1, Θ_1) and (δ_2, Θ_2) be non-empty over $\mathcal{U}$. Then,*
(i) $(\delta_1, \Theta_1) \oplus (\delta_2, \Theta_2) = (\delta_2, \Theta_2) \oplus (\delta_1, \Theta_1)$;
(ii) $[(\delta_1, \Theta_1)^c \oplus (\delta_2, \Theta_2)^c]^c = (\delta_1, \Theta_1) \oplus (\delta_1, \Theta_1)$.

Proof. (i) Proof straightforward.

(ii) Let

$$(\delta_1, \Theta_1) = \left\{\left\langle \triangle, \mathcal{P}_{\delta_1(\varrho_{)}}(\triangle), \mathcal{Q}_{\delta_1(\varrho_{)}}(\triangle), \mathcal{R}_{\delta_1(\varrho_{)}}(\triangle), \mathcal{S}_{\delta_1(\varrho_{)}}(\triangle)\right\rangle ; \varrho_{\in}\Theta_1\right\}, \text{and}$$

TABLE 6. Subjects with psychological disorders in IQ-PNSS $(\delta_1, \Theta_1) \ominus (\delta_2, \Theta_2)$ form.

$\mathcal{U}$	ANX(e_1)	DEP(e_2)	MS(e_3)	DC(e_4)
$\triangle_1$	$\langle .27, .24, .38, .58\rangle$	$\langle .35, .70, .65, .50\rangle$	$\langle .40, .55, .45, .45\rangle$	$\langle .15, .30, .45, .44\rangle$
$\triangle_2$	$\langle .00, .24, .16, .69\rangle$	$\langle .85, .40, .75, .10\rangle$	$\langle .65, .80, .80, .20\rangle$	$\langle .51, .72, .82, .24\rangle$
$\triangle_3$	$\langle .29, .51, .32, .62\rangle$	$\langle .35, .60, .65, .45\rangle$	$\langle .30, .35, .75, .60\rangle$	$\langle .45, .29, .79, .17\rangle$
$\triangle_4$	$\langle .48, .45, .44, .44\rangle$	$\langle .25, .40, .55, .60\rangle$	$\langle .60, .70, .70, .35\rangle$	$\langle .31, .50, .85, .31\rangle$

$(\delta_2, \Theta_2) = \left\{\left\langle \triangle, \mathcal{P}_{\delta_2(\varrho)}(\triangle), \mathcal{Q}_{\delta_2(\varrho)}(\triangle), \mathcal{R}_{\delta_2(\varrho)}(\triangle), \mathcal{S}_{\delta_2(\varrho)}(\triangle)\right\rangle ; \varrho \in \Theta_2\right\}$ be IQ-PNSSs.

Then,

$[(\delta_1, \Theta_1)^c \oplus (\delta_2, \Theta_2)^c]$

$$= \begin{cases} \left\{\left\langle \triangle, (\mathcal{S}_{\delta_1(\varrho)}(\triangle), (1-\mathcal{Q}_{\delta_1(\varrho)}(\triangle)), (1-\mathcal{R}_{\delta_1(\varrho)}(\triangle)), \mathcal{P}_{\delta_1(\varrho)}(\triangle))\right\rangle ; \quad \text{if } \varrho \in \Theta_1 - \Theta_2\right\}, \\ \left\{\left\langle \triangle, (\mathcal{S}_{\delta_2(\varrho)}(\triangle), (1-\mathcal{Q}_{\delta_2(\varrho)}(\triangle)), (1-\mathcal{R}_{\delta_1(\varrho)}(\triangle)), \mathcal{P}_{\delta_2(\varrho)}(\triangle))\right\rangle ; \quad \text{if } \varrho \in \Theta_2 - \Theta_1\right\}, \\ \Big\{\big\langle \triangle, \frac{(\mathcal{S}_{\delta_1(\varrho)}(\triangle)+\mathcal{S}_{\delta_2(\varrho)}(\triangle))}{2}, \frac{((1-\mathcal{Q}_{\delta_1(\varrho)}(\triangle))+(1-\mathcal{Q}_{\delta_2(\varrho)}(\triangle)))}{2}, \\ \qquad \frac{((1-\mathcal{R}_{\delta_1(\varrho)}(\triangle))+(1-\mathcal{R}_{\delta_2(\varrho)}(\triangle)))}{2}, \frac{(\mathcal{P}_{\delta_1(\varrho)}(\triangle)+\mathcal{P}_{\delta_2(\varrho)}(\triangle))}{2}\big\rangle ; \quad \text{if } \varrho \in \Theta_1 \cap \Theta_2\Big\}. \end{cases}$$

Now consider,

$[(\delta_1, \Theta_1)^c \oplus (\delta_2, \Theta_2)^c]^c$

$$= \begin{cases} \left\{\left\langle \triangle, (\mathcal{P}_{\delta_1(\varrho)}(\triangle), \mathcal{Q}_{\delta_1(\varrho)}(\triangle), \mathcal{R}_{\delta_1(\varrho)}(\triangle), \mathcal{S}_{\delta_1(\varrho)}(\triangle))\right\rangle ; \quad \text{if } \varrho \in \Theta_1 - \Theta_2\right\}, \\ \left\{\left\langle \triangle, (\mathcal{P}_{\delta_2(\varrho)}(\triangle), \mathcal{Q}_{\delta_2(\varrho)}(\triangle), \mathcal{R}_{\delta_2(\varrho)}(\triangle), \mathcal{S}_{\delta_2(\varrho)}(\triangle))\right\rangle ; \quad \text{if } \varrho \in \Theta_2 - \Theta_1\right\}, \\ \Big\{\big\langle \triangle, \frac{(\mathcal{P}_{\delta_1(\varrho)}(\triangle)+\mathcal{P}_{\delta_2(\varrho)}(\triangle))}{2}, \frac{(\mathcal{Q}_{\delta_1(\varrho)}(\triangle)+\mathcal{Q}_{\delta_2(\varrho)}(\triangle))}{2}, \\ \qquad \frac{(\mathcal{R}_{\delta_1(\varrho)}(\triangle)+\mathcal{R}_{\delta_2(\varrho)}(\triangle))}{2}, \frac{(\mathcal{S}_{\delta_1(\varrho)}(\triangle)+\mathcal{S}_{\delta_2(\varrho)}(\triangle))}{2}\big\rangle ; \quad \text{if } \varrho \in \Theta_1 \cap \Theta_2\Big\}. \end{cases}$$

Hence, $[(\delta_1, \Theta_1)^c \oplus (\delta_2, \Theta_2)^c]^c = (\delta_1, \Theta_1) \oplus (\delta_2, \Theta_2)$. □

Proposition 5.4. *Let (δ_1, Θ_1) and (δ_2, Θ_2) be non-empty over $\mathcal{U}$. Then,*
(i) $(\delta_1, \Theta_1) \ominus (\delta_2, \Theta_2) = (\delta_2, \Theta_2) \ominus (\delta_1, \Theta_1)$*;*
(ii) $[(\delta_1, \Theta_1)^c \ominus (\delta_2, \Theta_2)^c]^c = (\delta_1, \Theta_1) \ominus (\delta_1, \Theta_1)$.

Proof. (i) Consider,

$$(\delta_1, \Theta_1) \ominus (\delta_2, \Theta_2) = \begin{cases} \left\{\left\langle \triangle, (\mathcal{P}_{\delta_1(\varrho)}(\triangle), \mathcal{Q}_{\delta_1(\varrho)}(\triangle), \mathcal{R}_{\delta_1(\varrho)}(\triangle), \mathcal{S}_{\delta_1(\varrho)}(\triangle))\right\rangle ; \quad \text{if } \varrho \in \Theta_1 - \Theta_2\right\}, \\ \left\{\left\langle \triangle, (\mathcal{P}_{\delta_2(\varrho)}(\triangle), \mathcal{Q}_{\delta_2(\varrho)}(\triangle), \mathcal{R}_{\delta_2(\varrho)}(\triangle), \mathcal{S}_{\delta_2(\varrho)}(\triangle))\right\rangle ; \quad \text{if } \varrho \in \Theta_2 - \Theta_1\right\}, \\ \Big\{\big\langle \triangle, \frac{2*(\mathcal{P}_{\delta_1(\varrho)}(\triangle)*\mathcal{P}_{\delta_2(\varrho)}(\triangle))}{\mathcal{P}_{\delta_1(\varrho)}(\triangle)+\mathcal{P}_{\delta_2(\varrho)}(\triangle)}, \frac{(\mathcal{Q}_{\delta_1(\varrho)}(\triangle)+\mathcal{Q}_{\delta_2(\varrho)}(\triangle))}{2}, \\ \qquad \frac{(\mathcal{R}_{\delta_1(\varrho)}(\triangle)+\mathcal{R}_{\delta_2(\varrho)}(\triangle))}{2}, \frac{2*(\mathcal{S}_{\delta_1(\varrho)}(\triangle)*\mathcal{S}_{\delta_2(\varrho)}(\triangle))}{\mathcal{S}_{\delta_1(\varrho)}(\triangle)+\mathcal{S}_{\delta_2(\varrho)}(\triangle)}\big\rangle ; \quad \text{if } \varrho \in \Theta_1 \cap \Theta_2\Big\}. \end{cases}$$

$$= \begin{cases} \left\{ \left\langle \triangle, (\mathcal{P}_{\delta_1(\varrho)}(\triangle), \mathcal{Q}_{\delta_1(\varrho)}(\triangle), \mathcal{R}_{\delta_1(\varrho)}(\triangle), \mathcal{S}_{\delta_1(\varrho)}(\triangle)) \right\rangle ; & \text{if } \varrho \in \Theta_1 - \Theta_2 \right\}, \\ \left\{ \left\langle \triangle, (\mathcal{P}_{\delta_2(\varrho)}(\triangle), \mathcal{Q}_{\delta_2(\varrho)}(\triangle), \mathcal{R}_{\delta_2(\varrho)}(\triangle), \mathcal{S}_{\delta_2(\varrho)}(\triangle)) \right\rangle ; & \text{if } \varrho \in \Theta_2 - \Theta_1 \right\}, \\ \left\{ \left\langle \triangle, \frac{2*(\mathcal{P}_{\delta_2(\varrho)}(\triangle)*\mathcal{P}_{\delta_1(\varrho)}(\triangle))}{\mathcal{P}_{\delta_2(\varrho)}(\triangle)+\mathcal{P}_{\delta_1(\varrho)}(\triangle)}, \frac{(\mathcal{Q}_{\delta_2(\varrho)}(\triangle)+\mathcal{Q}_{\delta_1(\varrho)}(\triangle))}{2}, \right. \right. \\ \left. \left. \frac{(\mathcal{R}_{\delta_2(\varrho)}(\triangle)+\mathcal{R}_{\delta_1(\varrho)}(\triangle))}{2}, \frac{2*(\mathcal{S}_{\delta_2(\varrho)}(\triangle)*\mathcal{S}_{\delta_1(\varrho)}(\triangle))}{\mathcal{S}_{\delta_2(\varrho)}(\triangle)+\mathcal{S}_{\delta_1(\varrho)}(\triangle)} \right\rangle ; & \text{if } \varrho \in \Theta_1 \cap \Theta_2 \right\}. \end{cases}$$

Hence, $(\delta_1, \Theta_1) \ominus (\delta_2, \Theta_2) = (\delta_2, \Theta_2) \ominus (\delta_1, \Theta_1)$.

(ii) Consider,

$(\delta_1, \Theta_1)^c \ominus (\delta_2, \Theta_2)^c$

$$= \begin{cases} \left\{ \left\langle \triangle, (\mathcal{S}_{\delta_1(\varrho)}(\triangle), (1-\mathcal{Q}_{\delta_1(\varrho)}(\triangle)), (1-\mathcal{R}_{\delta_1(\varrho)}(\triangle)), \mathcal{P}_{\delta_1(\varrho)}(\triangle)) \right\rangle ; & \text{if } \varrho \in \Theta_1 - \Theta_2 \right\}, \\ \left\{ \left\langle \triangle, (\mathcal{S}_{\delta_2(\varrho)}(\triangle), (1-\mathcal{Q}_{\delta_2(\varrho)}(\triangle)), (1-\mathcal{R}_{\delta_2(\varrho)}(\triangle)), \mathcal{P}_{\delta_2(\varrho)}(\triangle)) \right\rangle ; & \text{if } \varrho \in \Theta_2 - \Theta_1 \right\}, \\ \left\{ \left\langle \triangle, \frac{2*(\mathcal{S}_{\delta_1(\varrho)}(\triangle)*\mathcal{S}_{\delta_2(\varrho)}(\triangle))}{\mathcal{S}_{\delta_1(\varrho)}(\triangle)+\mathcal{S}_{\delta_2(\varrho)}(\triangle)}, \frac{((1-\mathcal{Q}_{\delta_1(\varrho)}(\triangle))+(1-\mathcal{Q}_{\delta_2(\varrho)}(\triangle)))}{2}, \right. \right. \\ \left. \left. \frac{((1-\mathcal{R}_{\delta_1(\varrho)}(\triangle))+(1-\mathcal{R}_{\delta_2(\varrho)}(\triangle)))}{2}, \frac{2*(\mathcal{P}_{\delta_1(\varrho)}(\triangle)*\mathcal{P}_{\delta_2(\varrho)}(\triangle))}{\mathcal{P}_{\delta_1(\varrho)}(\triangle)+\mathcal{P}_{\delta_2(\varrho)}(\triangle)} \right\rangle ; & \text{if } \varrho \in \Theta_1 \cap \Theta_2 \right\}. \end{cases}$$

Then,

$[(\delta_1, \Theta_1)^c \ominus (\delta_2, \Theta_2)^c]^c$

$$= \begin{cases} \left\{ \left\langle \triangle, (\mathcal{P}_{\delta_1(\varrho)}(\triangle), \mathcal{Q}_{\delta_1(\varrho)}(\triangle), \mathcal{R}_{\delta_1(\varrho)}(\triangle), \mathcal{S}_{\delta_1(\varrho)}(\triangle)) \right\rangle ; & \text{if } \varrho \in \Theta_1 - \Theta_2 \right\}, \\ \left\{ \left\langle \triangle, (\mathcal{P}_{\delta_2(\varrho)}(\triangle), \mathcal{Q}_{\delta_2(\varrho)}(\triangle), \mathcal{R}_{\delta_2(\varrho)}(\triangle), \mathcal{S}_{\delta_2(\varrho)}(\triangle)) \right\rangle ; & \text{if } \varrho \in \Theta_2 - \Theta_1 \right\}, \\ \left\{ \left\langle \triangle, \frac{2*(\mathcal{P}_{\delta_1(\varrho)}(\triangle)*\mathcal{P}_{\delta_2(\varrho)}(\triangle))}{\mathcal{P}_{\delta_1(\varrho)}(\triangle)+\mathcal{P}_{\delta_2(\varrho)}(\triangle)}, \frac{(\mathcal{Q}_{\delta_1(\varrho)}(\triangle)+\mathcal{Q}_{\delta_2(\varrho)}(\triangle))}{2}, \right. \right. \\ \left. \left. \frac{(\mathcal{R}_{\delta_1(\varrho)}(\triangle)+\mathcal{R}_{\delta_2(\varrho)}(\triangle))}{2}, \frac{2*(\mathcal{S}_{\delta_1(\varrho)}(\triangle)*\mathcal{S}_{\delta_2(\varrho)}(\triangle))}{\mathcal{S}_{\delta_1(\varrho)}(\triangle)+\mathcal{S}_{\delta_2(\varrho)}(\triangle)} \right\rangle ; & \text{if } \varrho \in \Theta_1 \cap \Theta_2 \right\}. \end{cases}$$

Hence, $[(\delta_1, \Theta_1)^c \ominus (\delta_2, \Theta_2)^c]^c = (\delta_1, \Theta_1) \ominus (\delta_2, \Theta_2)$. □

6. Δ_μ, $\Delta_{\mu,\nu}$ and $\phi_{\mu,\nu}$ operators on IQ-PNSS

We present the operators Δ_μ, $\Delta_{\mu,\nu}$ and $\delta_{\mu,\nu}$ on IQ-PNSS and discuss some of its properties.

Definition 6.1. Let $\mu \in [0,1]$. Then the operator $\Delta_\mu(\delta, \Lambda)$ is represented as,

$$\begin{aligned} \Delta_\mu(\delta, \Lambda) = \{ \langle \triangle, \big(\mathcal{P}_{\psi(\varrho)}(\triangle) + \mu\big(1 - \mathcal{P}_{\psi(\varrho)}(\triangle) - \mathcal{S}_{\psi(\varrho)}(\triangle)\big)\big), \mathcal{Q}_{\psi(\varrho)}(\triangle), \\ \mathcal{R}_{\psi(\varrho)}(\triangle), \big(\mathcal{S}_{\psi(\varrho)}(\triangle) + (1-\mu)\big(1 - \mathcal{P}_{\psi(\varrho)}(\triangle) - \mathcal{S}_{\psi(\varrho)}(\triangle)\big)\big)\rangle ; \varrho \in \Lambda \} \\ = \{ \langle \triangle, \big(\mathcal{P}_{\psi(\varrho)}(\triangle) + \mu\big(\pi_{\psi(\varrho)}(\triangle)\big)\big), \mathcal{Q}_{\psi(\varrho)}(\triangle), \mathcal{R}_{\psi(\varrho)}(\triangle), \\ \big(\mathcal{S}_{\psi(\varrho)}(\triangle) + (1-\mu)\big(\pi_{\psi(\varrho)}(\triangle)\big)\big)\rangle ; \varrho \in \Lambda \}, \\ \text{where } \pi_{\psi(\varrho)}(\triangle) = (1 - \mathcal{P}_{\psi(\varrho)}(\triangle) - \mathcal{S}_{\psi(\varrho)}(\triangle)). \end{aligned}$$

Proposition 6.2. *Let $\mu, \nu \in [0,1]$ and $\mu \leq \nu$. Then for every IQ-PNSS (δ, Λ) the following hold:*
(i) $\Delta_\mu(\delta,\Lambda) \Subset \Delta_\nu(\delta,\Lambda)$;
(ii) $\Delta_0(\delta,\Lambda) = \circledcirc(\delta,\Lambda)$;
(iii) $\Delta_1(\delta,\Lambda) = \circledast(\delta,\Lambda)$.

Proof. (i) $\Delta_\mu(\delta,\Lambda) = \{\langle \triangle, \big(\mathcal{P}_{\psi(\varrho)}(\triangle) + \mu\big(\pi_{\psi(\varrho)}(\triangle)\big)\big), \mathcal{Q}_{\psi(\varrho)}(\triangle), \mathcal{R}_{\psi(\varrho)}(\triangle),$
$\big(\mathcal{S}_{\psi(\varrho)}(\triangle) + (1-\mu)\big(\pi_{\psi(\varrho)}(\triangle)\big)\big)\rangle; \varrho \in \Lambda\}$,
and $\mathcal{S}_\nu(\delta,\Lambda) = \{\langle \triangle, \big(\mathcal{P}_{\psi(\varrho)}(\triangle) + \nu\big(\pi_{\psi(\varrho)}(\triangle)\big)\big), \mathcal{Q}_{\psi(\varrho)}(\triangle), \mathcal{R}_{\psi(\varrho)}(\triangle),$
$\big(\mathcal{S}_{\psi(\varrho)}(\triangle) + (1-\nu)\big(\pi_{\psi(\varrho)}(\triangle)\big)\big)\rangle; \varrho \in \Lambda\}$.

Since $\mu \leq \nu$, we have

$$(\mathcal{P}_{\psi(\varrho)}(\triangle) + \mu(\pi_{\psi(\varrho)}(\triangle))) \leq (\mathcal{P}_{\psi(\varrho)}(\triangle) + \nu(\pi_{\psi(\varrho)}(\triangle))).$$

Also, $(1-\nu) \leq (1-\mu)$, we have

$$(\mathcal{S}_{\psi(\varrho)}(\triangle) + (1-\nu)(\pi_{\psi(\varrho)}(\triangle))) \leq (\mathcal{S}_{\psi(\varrho)}(\triangle) + (1-\mu)(\pi_{\psi(\varrho)}(\triangle))).$$

Hence, $\Delta_\mu(\delta,\Lambda) \Subset \Delta_\nu(\delta,\Lambda)$.

(ii) Consider, $\mu = 0$

$$\begin{aligned}\Delta_0(\delta,\Lambda) = &\{\langle \triangle, \big(\mathcal{P}_{\psi(\varrho)}(\triangle) + 0\big(\pi_{\psi(\varrho)}(\triangle)\big)\big), \mathcal{Q}_{\psi(\varrho)}(\triangle), \mathcal{R}_{\psi(\varrho)}(\triangle),\\ &\big(\mathcal{S}_{\psi(\varrho)}(\triangle) + (1-0)\big(\pi_{\psi(\varrho)}(\triangle)\big)\big)\rangle; \varrho \in \Lambda\}\\ =&\{\langle \triangle, \big(\mathcal{P}_{\psi(\varrho)}(\triangle)\big), \big(\mathcal{Q}_{\psi(\varrho)}(\triangle)\big), \big(\mathcal{R}_{\psi(\varrho)}(\triangle)\big), \big(1 - \mathcal{P}_{\psi(\varrho)}(\triangle)\big)\rangle; \varrho \in \Lambda\}\\ =&\circledcirc(\delta,\Lambda).\end{aligned}$$

Hence, $\Delta_0(\delta,\Lambda) = \circledcirc(\delta,\Lambda)$.

(iii) Consider, $\mu = 1$

$$\begin{aligned}\Delta_1(\delta,\Lambda) = &\{\langle \triangle, \big(\mathcal{P}_{\psi(\varrho)}(\triangle) + \big(\pi_{\psi(\varrho)}(\triangle)\big)\big), \mathcal{Q}_{\psi(\varrho)}(\triangle), \mathcal{R}_{\psi(\varrho)}(\triangle),\\ &\big(\mathcal{S}_{\psi(\varrho)}(\triangle)\big)\rangle; \varrho \in \Lambda\}\\ =&\{\langle \triangle, \big(1 - \mathcal{S}_{\psi(\varrho)}(\triangle)\big), \mathcal{Q}_{\psi(\varrho)}(\triangle), \mathcal{R}_{\psi(\varrho)}(\triangle), \mathcal{S}_{\psi(\varrho)}(\triangle)\rangle; \varrho \in \Lambda\}\\ =&\circledcirc(\delta,\Lambda).\end{aligned}$$

Hence, $\Delta_1(\delta,\Lambda) = \circledast(\delta,\Lambda)$. □

Remark 6.3. Operators $\circledcirc$ and $\circledast$ are an extension of the operator Δ_μ.

Definition 6.4. Let $\mu, \nu \in [0,1]$ and $\mu + \nu \leq 1$. Then the operator $\Delta_{\mu,\nu}(\delta, \Lambda)$ is represented as,

$$\Delta_{\mu,\nu}(\delta,\Lambda) = \{\langle\ \triangle, \big(\mathcal{P}_{\psi(\varrho)}(\triangle) + \mu\big(\pi_{\psi(\varrho)}(\triangle)\big)\big), \mathcal{Q}_{\psi(\varrho)}(\triangle), \mathcal{R}_{\psi(\varrho)}(\triangle), \big(\mathcal{S}_{\psi(\varrho)}(\triangle) + \nu\big(\pi_{\psi(\varrho)}(\triangle)\big)\big)\rangle; \varrho \in \Lambda\},$$

where $\pi_{\psi(\varrho)}(\triangle) = (1 - \mathcal{P}_{\psi(\varrho)}(\triangle) - \mathcal{S}_{\psi(\varrho)}(\triangle))$.

Theorem 6.5. *Let $\mu, \nu, \chi \in [0,1]$ and $\mu + \nu \leq 1$. Then for every IQ-PNSS (δ, Λ) the following hold:*

(i) $\Delta_{\mu,\nu}(\delta, \Lambda)$ is a IQ-PNSS;

(ii) If $0 \leq \chi \leq \mu$ then $\Delta_{\chi,\nu}(\delta, \Lambda) \Subset \Delta_{\mu,\nu}(\delta, \Lambda)$;

(iii) If $0 \leq \chi \leq \nu$ then $\Delta_{\mu,\nu}(\delta, \Lambda) \Subset \mathcal{S}_{\mu,\chi}(\delta, \Lambda)$;

(iv) $\Delta_{\mu}(\delta, \Lambda) = \Delta_{\mu,(1-\mu)}(\delta, \Lambda)$;

(v) $\circledcirc(\delta, \Lambda) = \Delta_{0,1}(\delta, \Lambda)$;

(vi) $\circledast(\delta, \Lambda) = \Delta_{1,0}(\delta, \Lambda)$;

(vii) $(\mathcal{S}_{\mu,\nu}(\delta, \Lambda)^c)^c = (\mathcal{S}_{\nu,\mu}(\delta, \Lambda))$.

Proof. (i) Consider,

$$\begin{aligned}
&\frac{(\mathcal{P}_{\psi(\varrho)}(\triangle) + \mu(\pi_{\psi(\varrho)}(\triangle)))}{2} + (\mathcal{Q}_{\psi(\varrho)}(\triangle)) + (\mathcal{R}_{\psi(\varrho)}(\triangle)) + \frac{(\mathcal{S}_{\psi(\varrho)}(\triangle) + \nu(\pi_{\psi(\varrho)}(\triangle)))}{2} \\
&= \frac{(\mathcal{P}_{\psi(\varrho)}(\triangle) + \mathcal{S}_{\psi(\varrho)}(\triangle)}{2} + (\mathcal{Q}_{\psi(\varrho)}(\triangle)) + (\mathcal{R}_{\psi(\varrho)}(\triangle)) + (\mu + \nu)\frac{(\pi_{\psi(\varrho)}(\triangle))}{2} \\
&\leq \frac{(\mathcal{P}_{\psi(\varrho)}(\triangle) + \mathcal{S}_{\psi(\varrho)}(\triangle)}{2} + (\mathcal{Q}_{\psi(\varrho)}(\triangle)) + (\mathcal{R}_{\psi(\varrho)}(\triangle)) + \frac{(1 - \mathcal{P}_{\psi(\varrho)}(\triangle) - \mathcal{S}_{\psi(\varrho)}(\triangle)}{2} \\
&\leq \frac{1}{2} + 2 < 3. \qquad \text{(Since, } \mu + \nu \leq 1 \text{ and } \mathcal{Q}_{\psi(\varrho)}(\triangle) + \mathcal{R}_{\psi(\varrho)}(\triangle) \leq 2)
\end{aligned}$$

Hence, $\Delta_{\mu,\nu}(\delta, \Lambda)$ is an IQ-PNSS.

(ii) Consider,

$$\Delta_{\chi,\nu}(\delta,\Lambda) = \{\langle\ \triangle, \big(\mathcal{P}_{\psi(\varrho)}(\triangle) + \chi\big(\pi_{\psi(\varrho)}(\triangle)\big)\big), \mathcal{Q}_{\psi(\varrho)}(\triangle), \mathcal{R}_{\psi(\varrho)}(\triangle), \big(\mathcal{S}_{\psi(\varrho)}(\triangle) + \nu\big(\pi_{\psi(\varrho)}(\triangle)\big)\big)\rangle; \varrho \in \Lambda\},$$

$$\Delta_{\mu,\nu}(\delta,\Lambda) = \{\langle\ \triangle, \big(\mathcal{P}_{\psi(\varrho)}(\triangle) + \mu\big(\pi_{\psi(\varrho)}(\triangle)\big)\big), \mathcal{Q}_{\psi(\varrho)}(\triangle), \mathcal{R}_{\psi(\varrho)}(\triangle), \big(\mathcal{S}_{\psi(\varrho)}(\triangle) + \nu\big(\pi_{\psi(\varrho)}(\triangle)\big)\big)\rangle; \varrho \in \Lambda\}.$$

Now

$$(\mathcal{P}_{\psi(\varrho)}(\triangle) + \chi(\pi_{\psi(\varrho)}(\triangle)) \leq (\mathcal{P}_{\psi(\varrho)}(\triangle) + \mu(\pi_{\psi(\varrho)}(\triangle)). \qquad (\text{Since}, \chi \leq \mu)$$

Hence, $\Delta_{\chi,\nu}(\delta, \Lambda) \Subset \Delta_{\mu,\nu}(\delta, \Lambda)$.

(iii) Same like proof (ii).

(iv) Consider,

$$\begin{aligned}\Delta_{\mu,1-\mu}(\delta,\Lambda) =&\{\langle\ \triangle, \big(\mathcal{P}_{\psi(\varrho)}(\triangle)+\mu\big(\pi_{\psi(\varrho)}(\triangle)\big)\big), \mathcal{Q}_{\psi(\varrho)}(\triangle), \mathcal{R}_{\psi(\varrho)}(\triangle),\\ &\big(\mathcal{S}_{\psi(\varrho)}(\triangle)+1-\mu\big(\pi_{\psi(\varrho)}(\triangle)\big)\big)\rangle; \varrho\in\Lambda\}\\ =&\Delta_{\mu}(\delta,\Lambda).\end{aligned}$$

Hence, $\Delta_{\mu}(\delta,\Lambda)=\Delta_{\mu,(1-\mu)}(\delta,\Lambda)$.

(v) Let $\mu=0$ and $\nu=1$,

$$\begin{aligned}\Delta_{0,1}(\delta,\Lambda) =&\{\langle\ \triangle, \big(\mathcal{P}_{\psi(\varrho)}(\triangle)\big), \mathcal{Q}_{\psi(\varrho)}(\triangle), \mathcal{R}_{\psi(\varrho)}(\triangle), \big(\mathcal{S}_{\psi(\varrho)}(\triangle)+\big(\pi_{\psi(\varrho)}(\triangle)\big)\big)\rangle; \varrho\in\Lambda\}\\ =&\{\langle\ \triangle, \big(\mathcal{P}_{\psi(\varrho)}(\triangle)\big), \mathcal{Q}_{\psi(\varrho)}(\triangle), \mathcal{R}_{\psi(\varrho)}(\triangle), \big(1-\mathcal{P}_{\psi(\varrho)}(\triangle)\big)\big)\rangle; \varrho\in\Lambda\}\\ =&\circledcirc\ (\delta,\Lambda).\end{aligned}$$

Hence, $\circledcirc(\delta,\Lambda)=\Delta_{0,1}(\delta,\Lambda)$.

(vi)Let $\mu=1$ and $\nu=0$,

$$\begin{aligned}\Delta_{1,0}(\delta,\Lambda) =&\{\langle\ \triangle, \big(\mathcal{P}_{\psi(\varrho)}(\triangle)+\big(\pi_{\psi(\varrho)}(\triangle)\big)\big), \mathcal{Q}_{\psi(\varrho)}(\triangle), \mathcal{R}_{\psi(\varrho)}(\triangle), \big(\mathcal{S}_{\psi(\varrho)}(\triangle)\big)\rangle; \varrho\in\Lambda\}\\ =&\{\langle\ \triangle, \big(1-\mathcal{S}_{\psi(\varrho)}(\triangle)\big), \mathcal{Q}_{\psi(\varrho)}(\triangle), \mathcal{R}_{\psi(\varrho)}(\triangle), \big(\mathcal{S}_{\psi(\varrho)}(\triangle)\big)\big)\rangle; \varrho\in\Lambda\}\\ =&\circledast\ (\delta,\Lambda).\end{aligned}$$

Hence, $\circledast(\delta,\Lambda)=\Delta_{1,0}(\delta,\Lambda)$.

(vii) Consider,

$$\begin{aligned}\Delta_{\mu,\nu}(\delta,\Lambda)^c =& \{\langle\ \triangle, \big(\mathcal{S}_{\psi(\varrho)}(\triangle)+\mu\big(\pi_{\psi(\varrho)}(\triangle)\big)\big), (1-\mathcal{Q}_{\psi(\varrho)}(\triangle)),\\ &(1-\mathcal{R}_{\psi(\varrho)}(\triangle)), \big(\mathcal{P}_{\psi(\varrho)}(\triangle)+\nu\big(\pi_{\psi(\varrho)}(\triangle)\big)\big)\rangle; \varrho\in\Lambda\}.\\ (\mathcal{S}_{\mu,\nu}(\delta,\Lambda)^c)^c =&\{\langle\ \triangle, \big(\mathcal{P}_{\psi(\varrho)}(\triangle)+\nu\big(\pi_{\psi(\varrho)}(\triangle)\big)\big), \mathcal{Q}_{\psi(\varrho)}(\triangle), \mathcal{R}_{\psi(\varrho)}(\triangle), \big(\mathcal{S}_{\psi(\varrho)}(\triangle)+\mu\big(\pi_{\psi(\varrho)}(\triangle)\big)\big)\rangle; \varrho\in\Lambda\}\\ =&(\Delta_{\nu,\mu}(\delta,\Lambda)).\end{aligned}$$

Hence, $(\Delta_{\mu,\nu}(\delta,\Lambda)^c)^c=(\Delta_{\nu,\mu}(\delta,\Lambda))$. □

Remark 6.6. If $\mu+\nu=1$, then $\Delta_{\mu,\nu}(\delta,\Lambda)=\Delta_{\mu}(\delta,\Lambda)$.

Definition 6.7. Let $\mu,\nu\in[0,1]$ and $\mu+\nu\leq 1$. Then the operator $\phi_{\mu,\nu}(\delta,\Lambda)$ is represented as,

$$\phi_{\mu,\nu}(\delta,\Lambda)=\{\langle\ \triangle, \mu\big(\mathcal{P}_{\psi(\varrho)}(\triangle)\big), \mathcal{Q}_{\psi(\varrho)}(\triangle), \mathcal{R}_{\psi(\varrho)}(\triangle), \nu\big(\mathcal{S}_{\psi(\varrho)}(\triangle)\big)\rangle; \varrho\in\Lambda\}.$$

Theorem 6.8. *Let $\mu,\nu,\chi\in[0,1]$. Then for every IQ-PNSS (ϕ,Λ) the following hold:*
(i) $\phi_{\mu,\nu}(\delta,\Lambda)$ is an IQ-PNSS;
(ii) If $\mu\leq\chi$ then $\phi_{\mu,\nu}(\delta,\Lambda)\Subset\phi_{\chi,\nu}(\delta,\Lambda)$;
(iii) If $\nu\leq\chi$ then $\phi_{\mu,\nu}(\delta,\Lambda)\Supset\phi_{\mu,\chi}(\delta,\Lambda)$;
(iv) If $\delta\in[0,1]$ then $\phi_{\mu,\nu}(\phi_{\chi,\delta}(\delta,\Lambda))=\phi_{\mu\chi,\nu\delta}(\delta,\Lambda)=\phi_{\chi,\delta}(\mathcal{Q}_{\mu,\nu}(\delta,\Lambda))$;
(v)$(\phi_{\mu,\nu}(\delta,\Lambda)^c)^c=(\mathcal{Q}_{\nu,\mu}(\delta,\Lambda))$.

Proof. (i) Proof straightforward.

(ii) Consider,

$$\phi_{\mu,\nu}(\delta,\Lambda) = \{\langle\ \triangle, \mu\big(\mathcal{P}_{\psi(\varrho)}(\triangle)\big), \mathcal{Q}_{\psi(\varrho)}(\triangle), \mathcal{R}_{\psi(\varrho)}(\triangle), \nu\big(\mathcal{S}_{\psi(\varrho)}(\triangle)\big)\rangle; \varrho{\in}\Lambda\},$$
$$\phi_{\chi,\nu}(\delta,\Lambda) = \{\langle\ \triangle, \chi\big(\mathcal{P}_{\psi(\varrho)}(\triangle)\big), \mathcal{Q}_{\psi(\varrho)}(\triangle), \mathcal{R}_{\psi(\varrho)}(\triangle), \nu\big(\mathcal{S}_{\psi(\varrho)}(\triangle)\big)\rangle; \varrho{\in}\Lambda\}.$$

Since, $\mu \leq \chi$, $\mu\big(\mathcal{P}_{\psi(\varrho)}(\triangle)\big) \leq \chi\big(\mathcal{P}_{\psi(\varrho)}(\triangle)\big)$.

Hence, $\phi_{\mu,\nu}(\delta,\Lambda) \Subset \phi_{\chi,\nu}(\delta,\Lambda)$.

(iii) Consider,

$$\phi_{\mu,\nu}(\delta,\Lambda) = \{\langle\ \triangle, \mu\big(\mathcal{P}_{\psi(\varrho)}(\triangle)\big), \mathcal{Q}_{\psi(\varrho)}(\triangle), \mathcal{R}_{\psi(\varrho)}(\triangle), \nu\big(\mathcal{S}_{\psi(\varrho)}(\triangle)\big)\rangle; \varrho{\in}\Lambda\},$$
$$\phi_{\mu,\chi}(\delta,\Lambda) = \{\langle\ \triangle, \mu\big(\mathcal{P}_{\psi(\varrho)}(\triangle)\big), \mathcal{Q}_{\psi(\varrho)}(\triangle), \mathcal{R}_{\psi(\varrho)}(\triangle), \chi\big(\mathcal{S}_{\psi(\varrho)}(\triangle)\big)\rangle; \varrho{\in}\Lambda\}.$$

Since, $\nu \leq \chi$, $\nu\big(\mathcal{S}_{\psi(\varrho)}(\triangle)\big) \leq \chi\big(\mathcal{P}_{\psi(\varrho)}(\triangle)\big)$.

Hence, $\phi_{\mu,\nu}(\delta,\Lambda) \Supseteq \phi_{\mu,\chi}(\delta,\Lambda)$.

(iv) Consider,

$$\begin{aligned}\phi_{\mu,\nu}(\phi_{\chi,\delta}(\delta,\Lambda)) &= \{\langle\ \triangle, \mu\chi\big(\mathcal{P}_{\psi(\varrho)}(\triangle)\big), \mathcal{Q}_{\psi(\varrho)}(\triangle), \mathcal{R}_{\psi(\varrho)}(\triangle), \nu\delta\big(\mathcal{S}_{\psi(\varrho)}(\triangle)\big)\rangle; \varrho{\in}\Lambda\}\\ &= \phi_{\mu\chi,\nu\delta}(\delta,\Lambda).\\ \phi_{\chi,\delta}(\phi_{\mu,\nu}(\delta,\Lambda)) &= \{\langle\ \triangle, \chi\mu\big(\mathcal{P}_{\psi(\varrho)}(\triangle)\big), \mathcal{Q}_{\psi(\varrho)}(\triangle), \mathcal{R}_{\psi(\varrho)}(\triangle), \delta\nu\big(\mathcal{S}_{\psi(\varrho)}(\triangle)\big)\rangle; \varrho{\in}\Lambda\}\\ &= \{\langle\ \triangle, \mu\chi\big(\mathcal{P}_{\psi(\varrho)}(\triangle)\big), \mathcal{Q}_{\psi(\varrho)}(\triangle), \mathcal{R}_{\psi(\varrho)}(\triangle), \nu\delta\big(\mathcal{S}_{\psi(\varrho)}(\triangle)\big)\rangle; \varrho{\in}\Lambda\}\\ &= \phi_{\mu\chi,\nu\delta}(\delta,\Lambda)).\end{aligned}$$

Hence, $\phi_{\mu,\nu}(\phi_{\chi,\delta}(\delta,\Lambda)) = \mathcal{Q}_{\mu\chi,\nu\delta}(\delta,\Lambda) = \phi_{\chi,\delta}(\phi_{\mu,\nu}(\delta,\Lambda))$.

(v) Consider,

$$\begin{aligned}(\delta,\Lambda)^c &= \{\langle\ \triangle, \big(\mathcal{S}_{\psi(\varrho)}(\triangle)\big), (1-\mathcal{Q}_{\psi(\varrho)}(\triangle)), (1-\mathcal{R}_{\psi(\varrho)}(\triangle)), \big(\mathcal{P}_{\psi(\varrho)}(\triangle)\big)\rangle; \varrho{\in}\Lambda\}.\\ \phi_{\mu,\nu}(\delta,\Lambda)^c &= \{\langle\ \triangle, \mu\big(\mathcal{S}_{\psi(\varrho)}(\triangle)\big), (1-\mathcal{Q}_{\psi(\varrho)}(\triangle)), (1-\mathcal{R}_{\psi(\varrho)}(\triangle)), \nu\big(\mathcal{P}_{\psi(\varrho)}(\triangle)\big)\rangle; \varrho{\in}\Lambda\}.\\ (\phi_{\mu,\nu}(\delta,\Lambda)^c)^c &= \{\langle\ \triangle, \nu\big(\mathcal{P}_{\psi(\varrho)}(\triangle)\big), \mathcal{Q}_{\psi(\varrho)}(\triangle), \mathcal{R}_{\psi(\varrho)}(\triangle), \mu\big(\mathcal{S}_{\psi(\varrho)}(\triangle)\big)\rangle; \varrho{\in}\Lambda\}\\ &= \phi_{\nu,\mu}(\delta,\Lambda).\end{aligned}$$

Hence, $(\phi_{\mu,\nu}(\delta,\Lambda)^c)^c = \phi_{\nu,\mu}(\delta,\Lambda)$. □

7. Concentration ($\mathcal{CO}$) and dilation ($\mathcal{DO}$) operators on IQ-PNSS

We present ($\mathcal{CO}$) and ($\mathcal{DO}$) on IQ-PNSS and discuss its properties.

Definition 7.1. Let (δ,Λ) be an IQ-PNSS over $\mathcal{U}$. Then,

(i) the $\mathcal{CO}$ of (δ,Λ) is represented as,

$$\mathcal{C}(\delta,\Lambda) = \{\langle\ \triangle, ((\mathcal{P}_{\psi(\varrho)}(\triangle))^2, \mathcal{Q}_{\psi(\varrho)}(\triangle), \mathcal{R}_{\psi(\varrho)}(\triangle), (1-(1-\mathcal{S}_{\psi(\varrho)}(\triangle))^2)\rangle; \varrho{\in}\Lambda\};$$

(ii) the $\mathcal{DO}$ of (δ, Λ) is represented as,

$$\mathcal{D}(\delta, \Lambda) = \left\{ \left\langle \triangle, ((\mathcal{P}_{\psi(\varrho)}(\triangle),)^{\frac{1}{2}}, \mathcal{Q}_{\psi(\varrho)}(\triangle), \mathcal{R}_{\psi(\varrho)}(\triangle), (1-(1-\mathcal{S}_{\psi(\varrho)}(\triangle))^{\frac{1}{2}}) \right\rangle; \varrho \in \Lambda \right\}.$$

Proposition 7.2. *Let $\mathcal{U}$ denote a non-empty set and (δ, Λ) be an IQ-PNSS over $\mathcal{U}$.*
(i) If $\pi_{\psi(\varrho)}(\triangle) = 0$, then $\pi_{\mathcal{C}\psi(\varrho)}(\triangle) = 0$ iff $\mathcal{P}_{\psi(\varrho)}(\triangle) = 0$ or $\mathcal{P}_{\psi(\varrho)}(\triangle) = 1$;
(ii) $\circledcirc[\mathcal{C}(\psi, \Lambda)] = \mathcal{C}[\circledcirc(\psi, \Lambda)]$ iff $\mathcal{P}_{\psi(\varrho)}(\triangle) = 0$ or $\mathcal{P}_{\psi(\varrho)}(\triangle) = 1$;
(iii) $\circledast[\mathcal{C}(\psi, \Lambda)] = \mathcal{C}[\circledast(\psi, \Lambda)]$ iff $\mathcal{S}_{\psi(\varrho)}(\triangle) = 0$ or $\mathcal{S}_{\psi(\varrho)}(\triangle) = 1$.

Proof. (i) If $\pi_{\psi(\varrho)}(\triangle) = 0 \Leftrightarrow 1 - \mathcal{P}_{\psi(\varrho)}(\triangle) - \mathcal{S}_{\psi(\varrho)}(\triangle) = 0$.

$$\begin{aligned}\mathcal{C}(\delta, \Lambda) &= \left\{ \left\langle \triangle, ((\mathcal{P}_{\psi(\varrho)}(\triangle))^2, \mathcal{Q}_{\psi(\varrho)}(\triangle), \mathcal{R}_{\psi(\varrho)}(\triangle), 1-(1-\mathcal{S}_{\psi(\varrho)}(\triangle))^2) \right\rangle; \varrho \in \Lambda \right\} \\ &= \left\{ \left\langle \triangle, ((\mathcal{P}_{\psi(\varrho)}(\triangle))^2, \mathcal{Q}_{\psi(\varrho)}(\triangle), \mathcal{R}_{\psi(\varrho)}(\triangle), 1-\mathcal{P}^2_{\psi(\varrho)}(\triangle)) \right\rangle; \varrho \in \Lambda \right\}.\end{aligned}$$

If $\pi_{\mathcal{C}\psi(\varrho)}(\triangle) = 0 \Leftrightarrow 1 - \mathcal{P}_{\psi(\varrho)}(\triangle) - (1 - \mathcal{P}^2_{\psi(\varrho)}(\triangle)) = 0$.
Then $\mathcal{P}_{\psi(\varrho)}(\triangle)(1 - \mathcal{P}_{\psi(\varrho)}(\triangle)) = 0 \Leftrightarrow \mathcal{P}_{\psi(\varrho)}(\triangle) = 0$ or $\mathcal{P}_{\psi(\varrho)}(\triangle) = 1$.
(ii)

$$\circledcirc\,[\mathcal{C}(\psi, \Lambda)] = \left\{ \left\langle \triangle, (\mathcal{P}_{\psi(\varrho)}(\triangle), \mathcal{Q}_{\psi(\varrho)}(\triangle), \mathcal{R}_{\psi(\varrho)}(\triangle), 1-\mathcal{P}_{\psi(\varrho)}(\triangle)) \right\rangle; \varrho \in \Lambda \right\}. \tag{1}$$

Also,

$$\begin{aligned}\mathcal{C}[\circledcirc(\psi, \Lambda)] &= \left\{ \left\langle \triangle, ((\mathcal{P}_{\psi(\varrho)}(\triangle))^2, \mathcal{Q}_{\psi(\varrho)}(\triangle), \mathcal{R}_{\psi(\varrho)}(\triangle), 1-(1-(1-\mathcal{P}_{\psi(\varrho)}(\triangle)))^2) \right\rangle; \varrho \in \Lambda \right\} \\ &= \left\{ \left\langle \triangle, ((\mathcal{P}_{\psi(\varrho)}(\triangle))^2, \mathcal{Q}_{\psi(\varrho)}(\triangle), \mathcal{R}_{\psi(\varrho)}(\triangle), 1-\mathcal{P}^2_{\psi(\varrho)}(\triangle)) \right\rangle; \varrho \in \Lambda \right\}.\end{aligned} \tag{2}$$

By solving (1) and (2), we get

$$\begin{aligned}\circledcirc[\mathcal{C}(\psi, \Lambda)] = \mathcal{C}[\circledcirc(\psi, \Lambda)] &\Leftrightarrow 1 - \mathcal{P}_{\psi(\varrho)}(\triangle) = 1 - \mathcal{P}^2_{\psi(\varrho)}(\triangle). \\ &\Leftrightarrow \mathcal{P}_{\psi(\varrho)}(\triangle)(1 - \mathcal{P}_{\psi(\varrho)}(\triangle)) = 0. \\ &\Leftrightarrow \mathcal{P}_{\psi(\varrho)}(\triangle) = 0 \text{ or } \mathcal{P}_{\psi(\varrho)}(\triangle) = 1.\end{aligned}$$

(iii)

$$\circledast\,[\mathcal{C}(\psi, \Lambda)] = \left\{ \left\langle \triangle, (1-(1-(1-\mathcal{S}_{\psi(\varrho)}(\triangle))^2), \mathcal{Q}_{\psi(\varrho)}(\triangle), \mathcal{R}_{\psi(\varrho)}(\triangle), 1-(1-\mathcal{S}_{\psi(\varrho)}(\triangle))^2) \right\rangle; \varrho \in \Lambda \right\}. \tag{3}$$

Also,

$$\mathcal{C}[\circledast(\psi, \Lambda)] = \left\{ \left\langle \triangle, ((1-\mathcal{S}_{\psi(\varrho)}(\triangle))^2, \mathcal{Q}_{\psi(\varrho)}(\triangle), \mathcal{R}_{\psi(\varrho)}(\triangle), 1-(1-\mathcal{S}_{\psi(\varrho)}(\triangle))^2) \right\rangle; \varrho \in \Lambda \right\}. \tag{4}$$

By solving (3) and (4), we get

$$\begin{aligned}\circledast[\mathcal{C}(\psi,\Lambda)] = \mathcal{C}[\circledast(\psi,\Lambda)] &\Leftrightarrow 1-(1-(1-\mathcal{S}_{\psi(\varrho)}(\triangle))^2) = 1-\mathcal{S}_{\psi(\varrho)}(\triangle).\\ &\Leftrightarrow (1-\mathcal{S}_{\psi(\varrho)}(\triangle))^2 = 1-\mathcal{S}_{\psi(\varrho)}(\triangle).\\ &\Leftrightarrow \mathcal{S}_{\psi(\varrho)}(\triangle)(1-\mathcal{S}_{\psi(\varrho)}(\triangle)) = 0.\\ &\Leftrightarrow \mathcal{S}_{\psi(\varrho)}(\triangle) = 0 \text{ or } \mathcal{S}_{\psi(\varrho)}(\triangle) = 1.\end{aligned}$$

□

Proposition 7.3. *Let $\mathcal{U}$ denote a non-empty set and (δ,Λ) be an IQ-PNSS over $\mathcal{U}$.*
(i) If $\pi_{\psi(\varrho)}(\triangle) = 0$, then $\pi_{\mathcal{D}\psi(\varrho)}(\triangle) = 0$ iff $\mathcal{P}_{\psi(\varrho)}(\triangle) = 0$ or $\mathcal{P}_{\psi(\varrho)}(\triangle) = 1$;
(ii) $\circledcirc[\mathcal{D}(\psi,\Lambda)] = \mathcal{D}[\circledcirc(\psi,\Lambda)]$ iff $\mathcal{P}_{\psi(\varrho)}(\triangle) = 0$ or $\mathcal{P}_{\psi(\varrho)}(\triangle) = 1$;
(iii) $\circledast[\mathcal{D}(\psi,\Lambda)] = \mathcal{D}[\circledast(\psi,\Lambda)]$ iff $\mathcal{S}_{\psi(\varrho)}(\triangle) = 0$ or $\mathcal{S}_{\psi(\varrho)}(\triangle) = 1$.

Proof. If $\pi_{\psi(\varrho)}(\triangle) = 0 \Leftrightarrow 1-\mathcal{P}_{\psi(\varrho)}(\triangle)-\mathcal{S}_{\psi(\varrho)}(\triangle) = 0$.

$$\begin{aligned}\mathcal{D}(\delta,\Lambda) =&\big\{\big\langle\ \triangle, ((\mathcal{P}_{\psi(\varrho)}(\triangle))^{\frac{1}{2}}, \mathcal{Q}_{\psi(\varrho)}(\triangle), \mathcal{R}_{\psi(\varrho)}(\triangle), 1-(1-\mathcal{S}_{\psi(\varrho)}(\triangle))^{\frac{1}{2}})\big\rangle; \varrho\in\Lambda\big\}\\ =&\big\{\big\langle\ \triangle, ((\mathcal{P}_{\psi(\varrho)}(\triangle))^{\frac{1}{2}}, \mathcal{Q}_{\psi(\varrho)}(\triangle), \mathcal{R}_{\psi(\varrho)}(\triangle), 1-(\mathcal{P}_{\psi(\varrho)}(\triangle))^{\frac{1}{2}})\big\rangle; \varrho\in\Lambda\big\}.\end{aligned}$$

If $\pi_{\mathcal{D}\psi(\varrho)}(\triangle) = 0 \Leftrightarrow 1-(\mathcal{P}_{\psi(\varrho)}(\triangle))^{\frac{1}{2}} - 1-(\mathcal{P}_{\psi(\varrho)}(\triangle))^{\frac{1}{2}} = 0 \Leftrightarrow (\mathcal{P}_{\psi(\varrho)}(\triangle))^{\frac{1}{2}} = (\mathcal{P}_{\psi(\varrho)}(\triangle))^{\frac{1}{2}}$.
Then $\mathcal{P}_{\psi(\varrho)}(\triangle)(1-\mathcal{P}_{\psi(\varrho)}(\triangle)) = 0 \Leftrightarrow \mathcal{P}_{\psi(\varrho)}(\triangle) = 0$ or $\mathcal{P}_{\psi(\varrho)}(\triangle) = 1$.

(ii)

$$\circledcirc\,[\mathcal{D}(\psi,\Lambda)] = \big\{\big\langle\ \triangle, (\mathcal{P}_{\psi(\varrho)}(\triangle))^{\frac{1}{2}}, \mathcal{Q}_{\psi(\varrho)}(\triangle), \mathcal{R}_{\psi(\varrho)}(\triangle), 1-(\mathcal{P}_{\psi(\varrho)}(\triangle))^{\frac{1}{2}})\big\rangle; \varrho\in\Lambda\big\}. \tag{5}$$

Also,

$$\begin{aligned}\mathcal{D}[\circledcirc(\psi,\Lambda)] &= \big\{\big\langle\ \triangle, ((\mathcal{P}_{\psi(\varrho)}(\triangle))^{\frac{1}{2}}, \mathcal{Q}_{\psi(\varrho)}(\triangle), \mathcal{R}_{\psi(\varrho)}(\triangle), 1-(1-(1-\mathcal{P}_{\psi(\varrho)}(\triangle)))^{\frac{1}{2}})\big\rangle; \varrho\in\Lambda\big\}\\ &= \big\{\big\langle\ \triangle, ((\mathcal{P}_{\psi(\varrho)}(\triangle))^{\frac{1}{2}}, \mathcal{Q}_{\psi(\varrho)}(\triangle), \mathcal{R}_{\psi(\varrho)}(\triangle), 1-(\mathcal{P}_{\psi(\varrho)}(\triangle))^{\frac{1}{2}})\big\rangle; \varrho\in\Lambda\big\}.\end{aligned} \tag{6}$$

By solving (5) and (6), we get

$$\begin{aligned}\circledcirc[\mathcal{D}(\psi,\Lambda)] = \mathcal{D}[\circledcirc(\psi,\Lambda)] &\Leftrightarrow 1-(\mathcal{P}_{\psi(\varrho)}(\triangle))^{\frac{1}{2}} = 1-(\mathcal{P}_{\psi(\varrho)}(\triangle))^{\frac{1}{2}}.\\ &\Leftrightarrow \mathcal{P}_{\psi(\varrho)}(\triangle)(1-\mathcal{P}_{\psi(\varrho)}(\triangle)) = 0.\\ &\Leftrightarrow \mathcal{P}_{\psi(\varrho)}(\triangle) = 0 \text{ or } \mathcal{P}_{\psi(\varrho)}(\triangle) = 1.\end{aligned}$$

(iii)

$$\circledast\,[\mathcal{D}(\psi,\Lambda)] = \big\{\big\langle\ \triangle, (1-(1-(1-\mathcal{S}_{\psi(\varrho)}(\triangle)))^{\frac{1}{2}}, \mathcal{Q}_{\psi(\varrho)}(\triangle), \mathcal{R}_{\psi(\varrho)}(\triangle), 1-(1-\mathcal{S}_{\psi(\varrho)}(\triangle))^{\frac{1}{2}})\big\rangle; \varrho\in\Lambda\big\}. \tag{7}$$

Also,

$$\mathcal{D}[\circledast(\psi,\Lambda)] = \big\{\big\langle\ \triangle, (1-\mathcal{S}_{\psi(\varrho)}(\triangle)^{\frac{1}{2}}, \mathcal{Q}_{\psi(\varrho)}(\triangle), \mathcal{R}_{\psi(\varrho)}(\triangle), 1-(1-\mathcal{S}_{\psi(\varrho)}(\triangle))^{\frac{1}{2}})\big\rangle; \varrho\in\Lambda\big\}. \tag{8}$$

By solving (7) and (8), we get

$$\circledast[\mathcal{D}(\psi,\Lambda)] = \mathcal{D}[\circledast(\psi,\Lambda)] \Leftrightarrow 1-(\mathcal{S}_{\psi(\varrho)}(\triangle))^{\frac{1}{2}} = 1-(\mathcal{S}_{\psi(\varrho)}(\triangle))^{\frac{1}{2}}.$$
$$\Leftrightarrow \mathcal{S}_{\psi(\varrho)}(\triangle)(1-\mathcal{S}_{\psi(\varrho)}(\triangle)) = 0.$$
$$\Leftrightarrow \mathcal{S}_{\psi(\varrho)}(\triangle) = 0 \text{ or } \mathcal{S}_{\psi(\varrho)}(\triangle) = 1.$$

□

Proposition 7.4. *For any IQ-PNSS* (ψ,Λ), $\mathcal{C}(\psi,\Lambda) \Subset (\psi,\Lambda) \Subset \mathcal{D}(\psi,\Lambda)$.

Proof. We have

$$(\psi,\Lambda) = \left\{\left\langle \triangle, \mathcal{P}_{\delta(\varrho)}(\triangle), \mathcal{Q}_{\delta(\varrho)}(\triangle), \mathcal{R}_{\delta(\varrho)}(\triangle), \mathcal{S}_{\delta(\varrho)}(\triangle)\right\rangle;\ \varrho\in\Lambda\right\}.$$

$$\mathcal{C}(\delta,\Lambda) = \left\{\left\langle\ \triangle, (\mathcal{P}_{\psi(\varrho)}(\triangle), \mathcal{Q}_{\psi(\varrho)}(\triangle), \mathcal{R}_{\delta(\varrho)}(\triangle), (1-(1-\mathcal{S}_{\psi(\varrho)}(\triangle))^2))\right\rangle; \varrho\in\Lambda\right\}.$$

Since, $\mathcal{S}_{\delta(\varrho)}(\triangle) \in [0,1], (1-(1-\mathcal{S}_{\psi(\varrho)}(\triangle))^2) \geq \mathcal{S}_{\delta(\varrho)}(\triangle)$.

$$\text{Hence, } \mathcal{C}(\delta,\Lambda) \Subset (\psi,\Lambda). \tag{9}$$

$$\mathcal{D}(\delta,\Lambda) = \left\{\left\langle\ \triangle, ((\mathcal{P}_{\psi(\varrho)}(\triangle),)^{\frac{1}{2}}, \mathcal{Q}_{\psi(\varrho)}(\triangle), \mathcal{R}_{\psi(\varrho)}(\triangle), (1-(1-\mathcal{S}_{\psi(\varrho)}(\triangle))^{\frac{1}{2}}))\right\rangle; \varrho\in\Lambda\right\}.$$

Since, $\mathcal{P}_{\psi(\varrho)}(\triangle), \mathcal{S}_{\psi(\varrho)}(\triangle) \in [0,1]$,
$\mathcal{P}_{\psi(\varrho)}(\triangle) \leq (\mathcal{P}_{\psi(\varrho)}(\triangle),)^{\frac{1}{2}}$, $\mathcal{S}_{\psi(\varrho)}(\triangle) \geq (1-(1-\mathcal{S}_{\psi(\varrho)}(\triangle))^{\frac{1}{2}})$.

$$\text{Hence, } (\psi,\Lambda) \Subset \mathcal{D}(\delta,\Lambda). \tag{10}$$

By solving (9) and (10), we get $\mathcal{C}(\psi,\Lambda) \Subset (\psi,\Lambda) \Subset \mathcal{D}(\psi,\Lambda)$. □

8. **Conclusion**

We have discussed the concepts of Intuitionistic Quadri-Partitioned Neutrosophic Sets (IQ-PNS) and Intuitionistic Quadri-Partitioned Neutrosophic Soft Sets (IQ-PNSS), thoroughly examining their essential properties. Our primary aim has been to tackle the challenges that experts encounter when managing truth, indeterminacy, and falsity values under defined constraints. The framework we propose not only offers a robust approach to addressing these complexities but also holds potential for future applications involving hypersoft sets, paving the way for further research and exploration in this area.

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Enhanced Neutrosophic Set and Machine Learning Approach for Kidney Disease Prediction

Humam M Al-Doori[1], Tareef S Alkellezli[2], Ahmed Abdelhafeez[3*,4], Mohamed Eassaa[3,4], Mohamed S. Sawah[5], Ahmed A El-Douh[3,6]

[1]Cybersecurity Sciences Department, College of Science, Ashur University, Baghdad, Iraq
[2]Cybersecurity Engineering Department, College of Engineering, Ashur University, Baghdad, Iraq
[3]Computer Science Department, Faculty of Information System and Computer Science, October 6 University, Giza, 12585, Egypt
[4]Applied Science Research Center. Applied Science Private University, Amman, Jordan
[5]Department of Computer Science, Faculty of Information Technology, Ajloun National University P.O.43, Ajloun-26810, Jordan
[6]Cybersecurity Technology Engineering Department, College of Engineering Technology, Ashur University, Baghdad, Iraq
*Correspondence: aahafeez.scis@o6u.edu.eg

Abstract:
Kidney disease (KD) is a gradually increasing global health concern. It is a chronic illness linked to higher rates of morbidity and mortality, a higher risk of cardiovascular disease and numerous other illnesses, and expensive medical expenses. The machine learning (ML) models are applied for KD prediction with higher accuracy and precision. The KD dataset has uncertainty and vague information, so, we used the neutrosophic set (NS) to deal with vague and uncertainty information in the KD dataset. The KD dataset is converted into the N-KD dataset with three membership functions: truth, indeterminacy, and falsity. Three ML models are used in this study such as logistic regression (LR), support vector machine (SVM), and k-nearest neighbor (KNN). These ML models are applied to the N-KD dataset. The results show the LR has higher accuracy and precision on the N-KD dataset than the original KD dataset.

Keywords: Neutrosophic Sets; Machine Learning Models; Kidney Disease; Uncertainty Models; Logistic Regression.

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1. Introduction

Kidney disease (KD) is a gradually increasing global health concern. It is a chronic illness linked to higher rates of morbidity and mortality, a higher risk of cardiovascular disease and numerous other illnesses, and expensive medical expenses. Just 10% of people who require treatment to survive may be represented by the more than two million people who undergo dialysis or kidney transplants worldwide. Just five wealthy nations, which account for 12% of the world's population, are home to the majority of the two million renal failure patients who receive therapy. In contrast, barely 20% of the world's population is treated in the roughly 100 developing nations that make up about half of the world's population.[1], [2].

Due to the prohibitive cost of dialysis or kidney transplantation, over a million people in 112 lower-income countries pass away from untreated renal failure each year. The early identification, management, and control of the condition are therefore crucial.[3], [4]. Due to patient heterogeneity and the dynamic and hidden nature of KD in its initial stages, it is crucial to forecast its progression with a decent degree of precision. Stages of severity are frequently used to define KD. The stage, whether a patient is progressing, and the rate of progression all affect clinical judgments.[2], [5]. Determining the disease stage is also especially important because it provides several indicators that help determine the necessary interventions and therapies.

In the healthcare industry, machine learning (ML) algorithms have been employed for classification and prediction. The Support Vector Machine Algorithm (SVM) has been utilized by Yu et al. [6] to categorize and predict patients with diabetes and pre-diabetes. The findings indicate that SVM is helpful in classifying patients with common diseases. Like this, Magnin et al. [7] Used a Support Vector Machine (SVM) to classify Alzheimer's disease by analyzing whole-brain anatomical magnetic resonance imaging (MRI) for a group of patients. The findings indicate that SVM is a promising method for early Alzheimer's disease identification.[8], [9]. The KD dataset has uncaring information. This uncertainty has a negative effect on the ML models. So, in this study, we used the uncertainty framework to overcome the uncertainty data in the KD dataset.

1.1 Neutrosophic Sets

As an extension of intuitionistic and fuzzy logic, neutrosophic logic was presented. The expansion of fuzzy logic known as intuitionistic logic involves two degrees of membership: the degree of Falseness (non-membership) and the degree of Truth (membership). However, intuitionistic logic may not be enough to address the inherent inconsistency or indeterminacy levels that are frequently present in fuzzy systems because it can only accommodate incomplete information.[10], [11]. Neutrosophic sets, which are represented by degrees of membership for Truth, Indeterminacy, and Falseness, were developed to get around this limitation. The total of those three independent values falls between 0 and 3.

Based on a relatively recent area of philosophy called neutrosophic sets, these sets can simulate human knowledge, preferences, and evaluation schemes by addressing the inherent ambiguities, inconsistencies, and indeterminacies in each set of data.[12], [13].

Because natural systems are complex, decision-makers frequently face several uncertainties when making choices based on imprecise, unclear, and incomplete information. In recent years, the theory of fuzzy sets (FSs) and its numerous extensions have become popular approaches for dealing with incomplete data[14], [15]. However, the existing structures need to be better able to handle the ambiguous and inconsistent data that is frequently present in natural systems.

To alleviate this specific limitation, Smarandache put out the notion of Neutrosophic Sets (NSs). NS theory is a useful approach for handling data that has inconsistencies, ambiguity, and flaws. It can be thought of as an extension of the FS and intuitionistic fuzzy set. NSs offers a unique and useful paradigm for dealing with ambiguity and uncertainty in a variety of domains[16], [17]. The main benefit of NSs is their capacity to accurately portray real-world uncertainties by capturing a three-component representation: truth, indeterminacy, and falsehood.

1.2 Neutrosophic Set with ML Models

The NS can be used with ML models to improve accuracy and precision score. The NS can convert the original dataset into the neutrosophic dataset to overcome uncertainty and vague information in the training ML models.

Ahmed et al. [18] used the combination of NS with the ML models to reduce the subjectivity pertaining to expert driven decisions and produce a reliable ordering of the hospital building assets. Their model can aid in building consistent, unbiased and automated ranking for decision making issues.

Khan and Alghamdi [19] used the neutrosophic set to evaluate the IIoT devices trust score with the ML models. They used the neutrosophic clustering to classify the extracted features.

Kaya and Dengiz [20] used the neutrosophic set and ML models to detect soil quality index and then assess the performance of the ML models. They obtained higher performance based on neutrosophic set.

Saqlain et al. [21] used the neutrosophic set to solve the decision-making problems with the ML models. They used the ML models to rank the selection of sites for a new store.

Ashika et al. [22] used the neutrosophic set with the ML models for breast cancer prediction. They used the neutrosophic set for capturing uncertainty in original dataset. They converted their original dataset into a neutrosophic set.

The main contributions of this study are:

I. Converting the KD dataset into the N-KD dataset.
II. Training three ML models such as LR, KNN, and SVC.
III. Evaluating these ML models into accuracy, precision, recall, and f1 score.
IV. Comparing three ML models on the N-KD dataset and KD original dataset.

The rest of this study is organized as follows: Section 2 shows the materials and methods of this study with the NS to overcome the vague information. Section 3 shows the results and discussion of the three ML models on the N-KD dataset. Section 4 shows the comparison analysis between the three ML models on the N-KD dataset and the original dataset. Section 5 shows the conclusions of this study.

2. Materials and Methods

The materials and methods in this study are presented in this section by several steps such as:

2.1 Proposed methodology

The aim of this study is to present the ML models with NS for the prediction of kidney disease (KD). The dataset was gathered from the Kaggle website and then preprocessed to ensure its quality and consistency. The KD dataset is converted to the N-dataset and each data point is featured with three values instead of one value such as truth T, indeterminacy I, and falsity F. This approach can overcome the uncertainty and vague information on the KD dataset.

We can convert the KD dataset into the N-dataset such as:

$KD_N = f_{T,I,F}(KD)$

We can split the N-KD dataset into training and testing such as:

$KD_N = s(f_{KD_N}) = s(X_{train}, X_{test}, y_{train}, y_{test})$

We can normalize the N-KD dataset using the Standard Scaler method.

$KD_N = f_{Nor}\left(KD_{N_{(s)}}\right)$

The normalized N-KD dataset is trained using different ML models such as Logistic Regression, SVC, and KNN. These models are chosen based on their ability to deal with intricate features interacting and identify subtle patterns to obtain accurate results in KD prediction.

$$KD_{N_{(ML)}} = f_{ML}\left(KD_{N_{(Nor)}}\right) = KD_{ML}\left(LR_{KD_{N_{(Nor)}}}, SVC_{KD_{N_{(Nor)}}}, KNN_{KD_{N_{(Nor)}}}\right)$$

The ML models are validated using accuracy, precision, recall, and f1 score.

$$KD_{N_{(Metrics)}} = KD_{N(ML)}\left(Metrics_{Acc,Pre,Recall,F1\ score}\right)$$

Finally, this study performed a comparative analysis between the N-KD dataset and the original dataset to show the accuracy between them.

$$KD_{Comp} = Comp(KD_N, KD)$$

The flowchart of this study is shown in Figure 1.

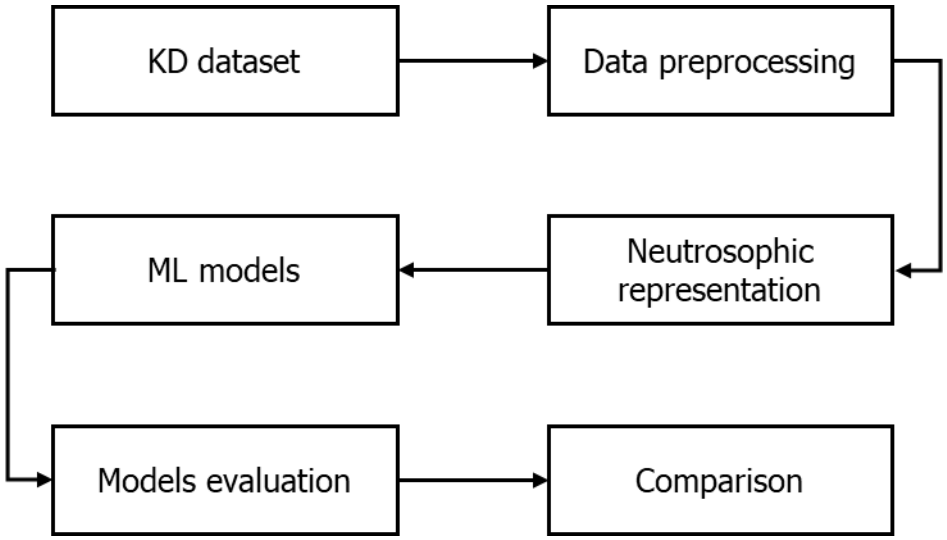

Figure 1. The flowchart of KD dataset prediction.

2.2 Dataset description

KD is used to predict kidney disease. It has four hundred rows and twenty-six columns. Table 1 shows the description of the KD dataset with some statistical methods to show the details of the dataset. Then we plot the distribution of the points in the dataset as shown in Figure 2. Figure 3 shows the box plot of each feature. Then we compute the count of each class in each feature as shown in Figure 4. Then we obtain the correlation matrix as shown in Figure 5.

We show the normal class has more rows than abnormal in the red blood cells. We show the normal class has more rows more than abnormal in the pus cells. We show the not present class has more rows more than present in the pus cell clumps cells. We show the present class has more rows than present in the bacteria cells. We show the no class has more rows than yes in the hypertension cells. We show the no class has more rows than yes in the diabetes mellitus cells. We show the no class has more rows than yes in the coronary artery disease cells. We show the good class has more rows than the poor in the appetite cells. We show the no class has more rows than yes in the pedal edema cells. We show the no class has more rows than yes in the anemia cells. We show the has disease class has more rows more than has not disease in the target class.

Table 1. Description of the dataset

	count	mean	std	Min	25%	50%	75%	Max
Id	400	199.5	115.6143	0	99.75	199.5	299.25	399
Age	391	51.48338	17.16971	2	42	55	64.5	90

blood pressure	388	76.46907	13.68364	50	70	80	80	180
specific gravity	353	1.017408	0.005717	1.005	1.01	1.02	1.02	1.025
Albumin	354	1.016949	1.352679	0	0	0	2	5
Sugar	351	0.450142	1.099191	0	0	0	0	5
blood_glucose_random	356	148.0365	79.28171	22	99	121	163	490
blood urea	381	57.42572	50.50301	1.5	27	42	66	391
serum creatinine	383	3.072454	5.741126	0.4	0.9	1.3	2.8	76
Sodium	313	137.5288	10.40875	4.5	135	138	142	163
Potassium	312	4.627244	3.193904	2.5	3.8	4.4	4.9	47
Hemoglobin	348	12.52644	2.912587	3.1	10.3	12.65	15	17.8
packed_cell_volume	329	38.8845	8.990105	9	32	40	45	54
white_blood_cell_count	294	8406.122	2944.474	2200	6500	8000	9800	26400
red_blood_cell_count	269	4.707435	1.025323	2.1	3.9	4.8	5.4	8

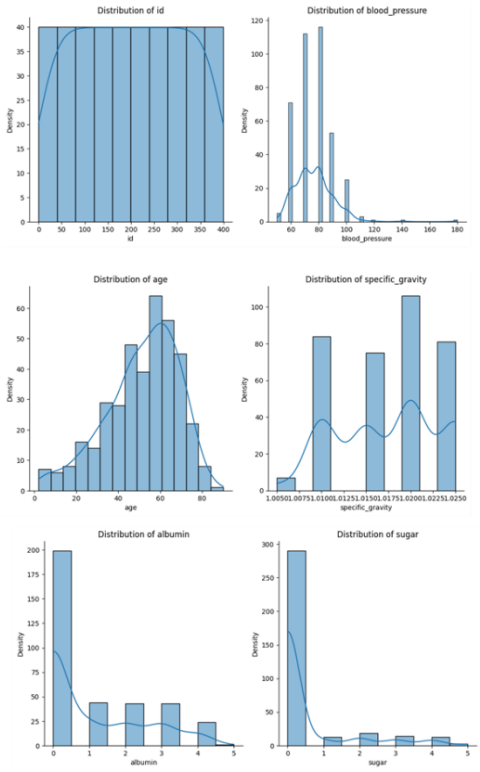

Figure 2. The distribution of the dataset.

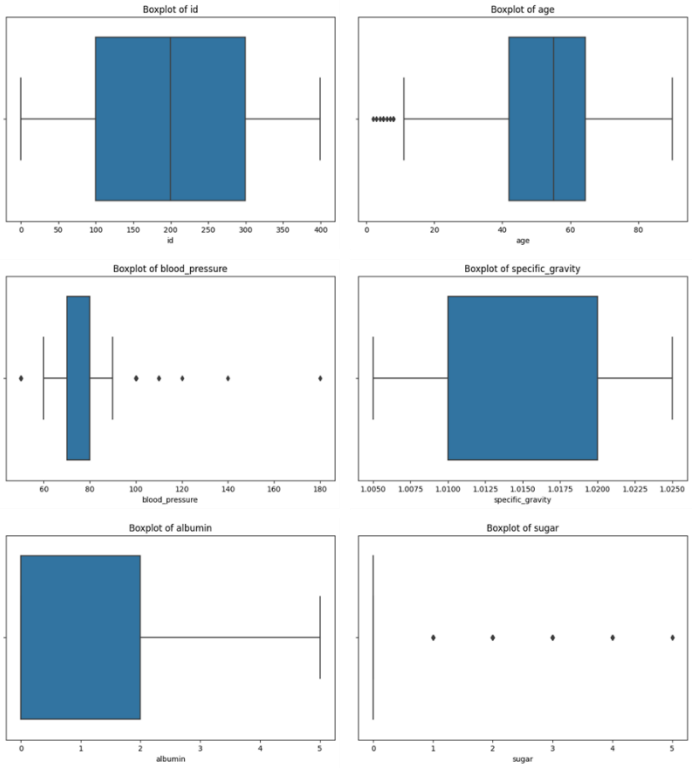

Figure 3. The box plot of the dataset.

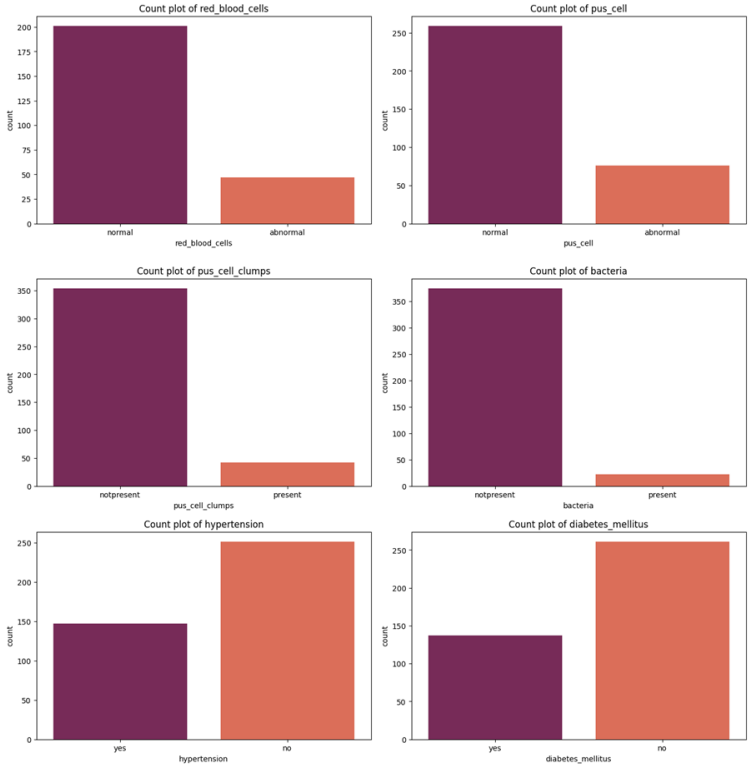

Figure 4. The Count of each point in each feature.

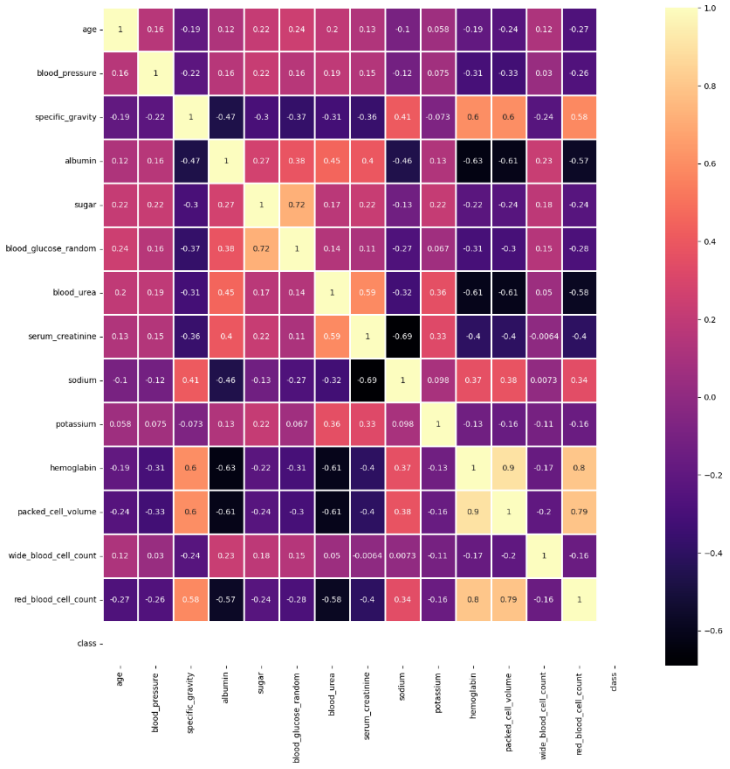

Figure 5. The Correlation matrix.

2.3 Data Preprocessing

The N-KD dataset was processed to obtain higher accuracy from the ML models. We divide the features with diseases and without diseases. Then this target class is encoded into 0 and 1. Then all features are normalized with the standard scaler method. These steps can improve the ML model evaluation on the N-KD dataset.

2.4 Neutrosophic Sets (NS)

The NS has three membership functions Truth, indeterminacy, and Falsity.[23], [24]. Where $X: T, I, F$ These values can be obtained with functions such as:

$$T_A(X), I_A(X), F_A(X)$$
$$T_A \; maps \; X \; to \; the \; interva \;]0^-, 1^+[$$
$$I_A \; maps \; X \; to \; the \; interva \;]0^-, 1^+[$$
$$F_A \; maps \; X \; to \; the \; interva \;]0^-, 1^+[$$

The sum of these functions is between 0 and 3. This flexibility can enable the NS to deal with uncertainty and vague data. The N-KD dataset contains the three elements of NS.

Definition 1

NS has three membership functions such as:

$$A_N: X \rightarrow [0^-, 1^+]$$
$$B_N: X \rightarrow [0^-, 1^+]$$
$$C_N: X \rightarrow [0^-, 1^+]$$
$$0^- \leq A_N(x) + B_N(x) + C_N(x) \leq 3^+$$

Definition 2

Let N and M are different NS. N can be considered a subset of M, if and only if conditions meet such as:

$$Inf\ A_N(x) \le Inf\ A_M(x)$$
$$Sup\ A_N(x) \le Sup\ A_M(x)$$

$$Inf\ B_N(x) \le Inf\ B_M(x)$$
$$Sup\ B_N(x) \le Sup\ B_M(x)$$

$$Inf\ C_N(x) \le Inf\ C_M(x)$$
$$Sup\ C_N(x) \le Sup\ C_M(x)$$

Definition 3

Considered two neutrosophic numbers such as $x = (A_1, B_1, C_1)\ and\ y = (A_2, B_2, C_2)$

$$x \oplus y = \begin{pmatrix} A_1(x) + A_2(x) - A_1(x)A_2(x) \\ B_1(x)B_2(x), \\ C_1(x)C_2(x) \end{pmatrix}$$

$$x \otimes y = \begin{pmatrix} A_1(x)A_2(x), \\ B_1(x) + B_2(x) - B_1(x)B_2(x), \\ C_1(x) + C_2(x) - C_1(x)C_2(x) \end{pmatrix}$$

$$\sigma x = \begin{pmatrix} 1 - \left(1 - A_1(x)\right)^{\sigma}, \\ B_1^{\sigma}(x), \\ C_1^{\sigma}(x) \end{pmatrix}$$

$$x^{\sigma} = \begin{pmatrix} A_1(x)^{\sigma}, \\ 1 - \left(1 - B_1(x)\right)^{\sigma}, \\ 1 - \left(1 - C_1(x)\right)^{\sigma} \end{pmatrix}$$

2.5 Neutrosophic Dataset Formation

To overcome the uncertainty in the KD dataset we convert it into the N-KD dataset.[22].

$X = \{x_1, x_2, \dots, x_n\}$

$\forall x(t, i, f) \in < T_A, I_A, F_A >$

We added the N-components into the original dataset to solve the vague and uncertainty information. We obtain the three functions such as:

$$T = 1 - \frac{\|x - U^+\|}{\max(\|x_{train} - U^+\|)}$$
$$T = 1 - \frac{\|x - U^{all}\|}{\max(\|x_{train} - U^{all}\|)}$$
$$T = 1 - \frac{\|x - U^-\|}{\max(\|x_{train} - U^-\|)}$$
$$U^+ = \sum_{n^+} x$$
$$U^{all} = \sum_{n^{all}} x$$
$$U^- = \sum_{n^-} x$$

2.6 ML Models

Logistic Regression (LR)

A supervised machine learning approach called logistic regression predicts the likelihood of an outcome, occurrence, or observation to complete binary classification tasks[25].

Different examples of classification problems

KNN

The k-nearest neighbor algorithm, or KNN, is a machine learning method that compares a single data point with a set of data it has learned and memorized to generate predictions based on proximity[26].

SVC

By carrying out optimal data transformations that establish boundaries between data points based on predefined classes, labels, or outputs, supervised learning models enable support vector machines (SVMs), a type of machine learning algorithm, to solve challenging classification, regression, and outlier detection problems [27].

3. Results and Discussion

This section shows the results of the proposed approach under the N-KD dataset.

3.1 Experimental Setup

Three ML models are used to predict the KD dataset. The dataset is converted to the N-KD dataset to improve the accuracy of the ML models and deal with uncertainty and vague information. Standard metrics are employed in this study to evaluate the ML models. We compared it with the original KD dataset to show the effectiveness of the NS in dealing with vague information.

3.2 Experimental Results

Figure 6 shows the results of the ML models under the N-KD dataset. We show the LR has higher accuracy, followed by the KNN, and SVC. We obtained accuracy, precision, recall, and f1 score.

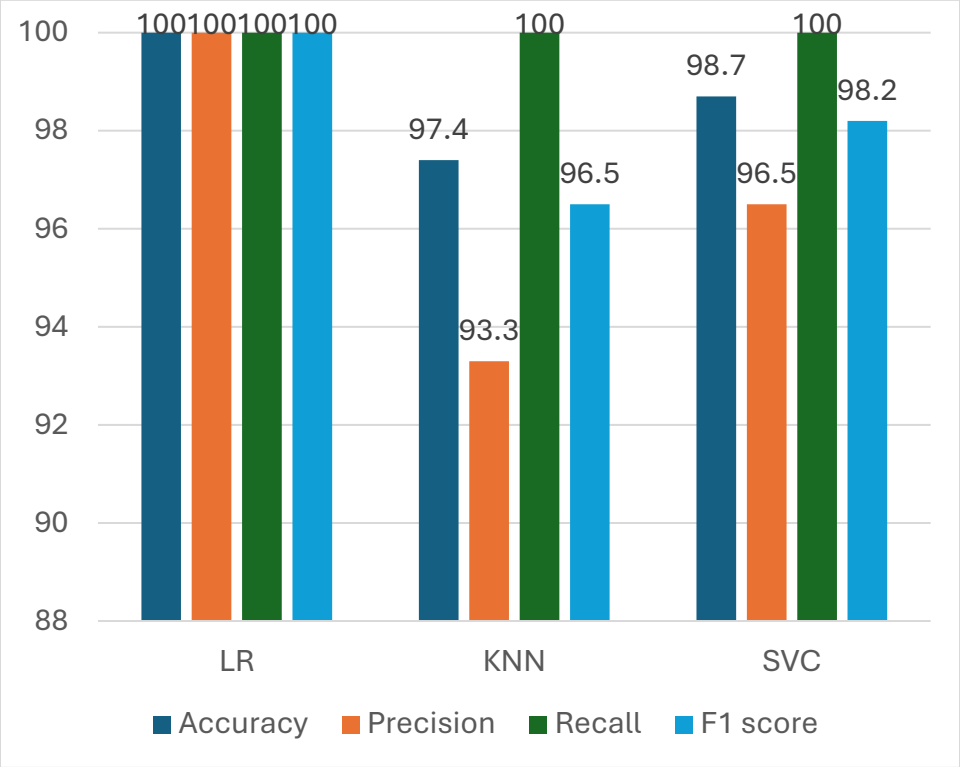

Figure 6. The ML models result under the N-KD dataset.

4. Comparison Analysis

We compare the ML models on the N-KD dataset and the original dataset. We show the accuracy, precision, recall, and f1 score as a classification report, as shown in Figure 7. Then, we obtained the confusion matrix

before and after the N-KD dataset, as shown in Figure 8. We show that the ML models obtained higher accuracy scores on the N-KD dataset than the original due to the NS and overcoming the uncertainty and vague information.

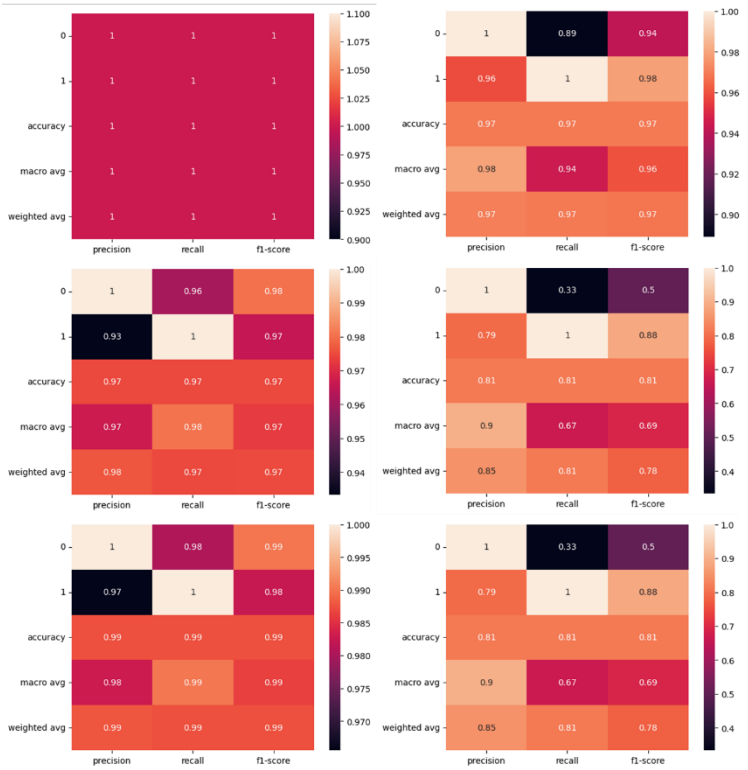

Figure 7. The classification report on the N-KD dataset and original dataset.

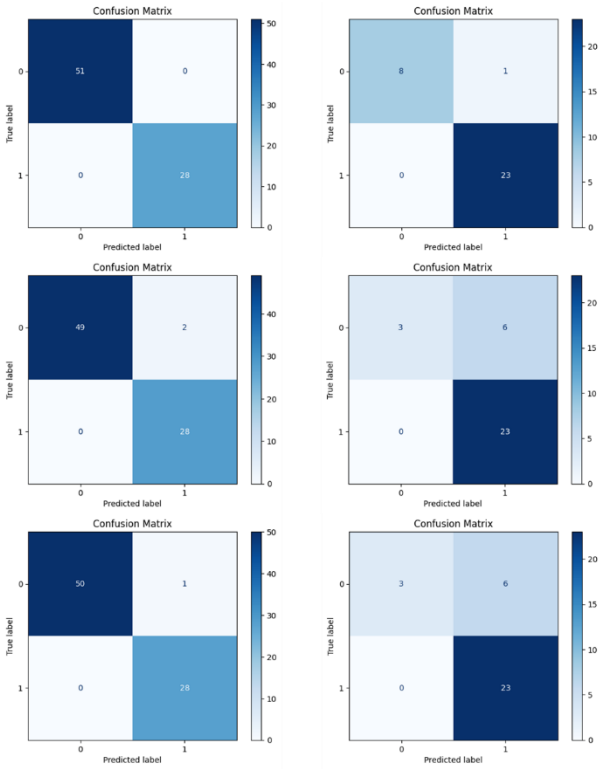

Figure 8. The confusion matrix on the N-KD dataset and original dataset.

5. Conclusions

This study integrated the ML models with the neutrosophic set to predict the KD dataset with uncertainty and vague data in the KD dataset. We trained three ML models such as LR, KNN, and SVC to predict the KD dataset. We converted the original KD dataset into the N-KD dataset with three values: truth, indeterminacy, and falsity. We applied the dataset preprocessing steps to improve the quality of the dataset. The results show the LR has higher accuracy than two other ML models on the N-KD dataset. We compare three ML models on the N-KD dataset and the original KD dataset. We show the ML models obtained higher accuracy than the original KD dataset. The three values of NS can deal with uncertainty and improve the accuracy of the ML models. We obtained the classification report between the ML models and the confusion matrix to show the true class and predicted class. In the future study, the proposed model can be applied to different prediction and classification problems to obtain higher accuracy and precision.

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Finite-precision stabilization for neutrosophic linear systems using a descriptor model approach

Rongyao Ling[1,2], Songquan Shi[1,*], Wenhua Cui[1] and Yiming Fang[1]

[1]College of Mechanical and Electrical Engineering, Shaoxing University, P. R. China; sqshi@usx.edu.cn

[2]College of Mechanical and Electrical Engineering, Shihezi University, P. R. China; lingrongyao@usx.edu.cn

[*]Correspondence: sqshi@usx.edu.cn; Tel.: +8613735274150

Abstract. This paper focuses on the stabilization problem of neutrosophic linear systems using finite-precision digital controllers. The main challenge lies in decoupling and processing the indeterminacies inherent in such systems. To tackle this, a standardization method that unifies the bounds of parameter indeterminacies is first proposed. Additionally, a fixed-point representation scheme is introduced to define both the precision and representation of the controller parameters. To handle multiple uncertainties in the system, a descriptor model approach is introduced, which transforms the closed-loop system into a singular system, allowing for more effective decoupling of uncertainties. Using singular system theory, a novel solvability condition is derived for system stabilization and a parametric approach for controller design is introduced, accompanied by a numerical example to validate the proposed methods.

Keywords: Neutrosophic linear systems; Finite word length; Descriptor models; System stabilization

1. Introduction

Parameter uncertainty is common in model-based analysis and control problems, including military, electricity, medical, etc [1–3]. The presence of uncertainty in system parameters can lead to significant performance degradation and adversely affect the system's stability, as the controller must be designed to operate under the worst-case scenario. Throughout the years, various methods have been developed to tackle uncertainty in control systems, including robust control techniques [4,5], fuzzy logic systems [6,7] and filter-based approach [8,9]. These approaches aim to mitigate the impact of uncertainty on system performance by modeling the uncertainty and ensuring system robustness against it.

A more recent and innovative approach to managing parameter uncertainty is the use of neutrosophic numbers. This method has been applied to both transfer function cases [10] and state-space model cases [11]. Meanwhile, the neutrosophic theory has also been successfully applied to control engineering, e.g., neutrosophic estimation [12], decision making [13], teaching quality evaluation [14] and clustering analysis [15]. Due to the unique structure of neutrosophic numbers, which comprises both determinate and indeterminate components, it is natural appropriate for representing the control systems with parameter uncertainties. With the above-mentioned characteristics, the neutrosophic linear systems is established in [11] to describe parameter uncertain systems and the state feedback stabilization for neutrosophic linear systems is also investigated for the SISO case. However, further research is needed to extend the stabilization problem of MIMO neutrosophic linear systems with more practical constraints being taken into consideration.

In the context of digital controller design, a critical challenge arises due to the finite precision of digital representations. Controller parameters must be quantized to fit within a finite word length (FWL), introducing representation errors that can significantly degrade system performance or even cause instability [16, 17]. The influence of finite word length on system performance has been a subject of extensive study with various stability measures proposed [18, 19]. However, most existing studies focus on FWL effects in isolation, without considering the combined impact of other uncertainties, such as parameter indeterminacies and model inaccuracies. Furthermore, existing approaches are often limited to state-space representations, which are not general enough to address more complex system architectures or the practical constraints of controller realization [20, 21]. Given these challenges, there is a clear need for novel modeling and controller design methods that integrate both parameter uncertainty and finite word length effects. Such methods would provide greater flexibility and accuracy in system analysis and control design, especially for systems with multiple uncertainties and practical implementation constraints.

The stabilization of MIMO neutrosophic linear systems under realistic digital implementation conditions is a complex and underexplored problem in control theory. Existing literature on parameter uncertainty focuses primarily on SISO systems, leaving the stabilization of more complex MIMO systems with parameter uncertainties, especially under finite-precision digital controller constraints, as a critical research gap. This paper proposes an innovative approach that combines neutrosophic modeling with FWL stability analysis, offering a comprehensive solution to the stabilization of MIMO neutrosophic linear systems under realistic digital implementation conditions. To the best of the authors' knowledge, there is a lack of systematic studies in the literature that specifically address the stabilization problem for MIMO neutrosophic linear systems. Although considerable attention has been devoted to the finite word

length (FWL) stability analysis of control systems, there is a notable lack of information concerning finite-precision digital controllers design for parameter uncertainty systems.

The aforementioned discussion motivated this study, where the stabilization problem subject to the FWL effect is considered for neutrosophic linear systems. Without requiring additional constraints or complex analysis methods, a novel descriptor model approach is proposed in this paper to describe the closed-loop system with multiple uncertainties. Meanwhile, a systematic procedure is provided on how to separate the coupled uncertainties of plant and controller gains. And the parameterization of the feedback gain is therefore accomplished. This paper presents several novel contributions: 1) a new descriptor model is introduced to represent the closed-loop system with multiple uncertainties, enabling more effective decoupling of plant and controller uncertainties without requiring additional constraints or complex analysis; 2) separation of Coupled Uncertainties: A systematic procedure is proposed to separate the coupled uncertainties in the plant and controller gains, simplifying the stabilization process; 3) a novel method for parameterizing the feedback gain is presented, ensuring more effective and practical digital controller design under finite precision.

Notation

The definitions presented below are used consistently throughout the paper. $\mathbb{R}^n$ represents the n-dimensional Euclidean space, while $\|\cdot\|$ represents the Euclidean norm. For a square matrix, the superscripts-1 and T indicate the inverse and transpose, respectively. Additionally, $\mathbb{Z}$ refers to the set of all neutrosophic numbers, while $\mathbb{Z}^{n\times m}$ denotes the collection of $n\times n$ neutrosophic matrices. $\mathbb{N}$ represents the set of all integers and $\mathbb{N}^+$ denotes the set of all positive integral numbers. The space $l_2[0,\infty)$ consists of square-summable infinite sequences. The symbol $\odot$ and $\otimes$ denote the Hadamard product and Kronecker product, respectively. U_n denotes the identity matrix with $n\times n$ dimensions. The notation 0_n refers to an $n\times n$ zero matrix, and $0_{n\times m}$ denotes a zero matrix of dimensions $n\times m$. $A = diag(a_1,\ a_2,\ \cdots,\ a_n)$ is a diagonal matrix with diagonal elements a_i. The symbol $*$ is used in some matrix expressions to represent the symmetric terms. Moreover, $E\{\cdot\}$, $\rho\{\cdot\}$, $det\{\cdot\}$ and $deg\{\cdot\}$ stand for mathematical expectation, spectral radius, matrix determinant, and polynomial degree, respectively.

2. **Preliminary**

This paper focuses on the stabilization problem for neutrosophic linear systems, where the uncertainty in the system parameters is modeled using neutrosophic numbers. The feedback gain is implemented digitally, with its parameters represented using a floating-point scheme.

To address the coupled constraints inherent in this stabilization problem, a descriptor model-based approach is proposed. This section provides a brief introduction to relevant basic concepts for neutrosophic number, floating-point scheme and descriptor model to facilitate further discussion.

2.1. *Neutrosophic number*

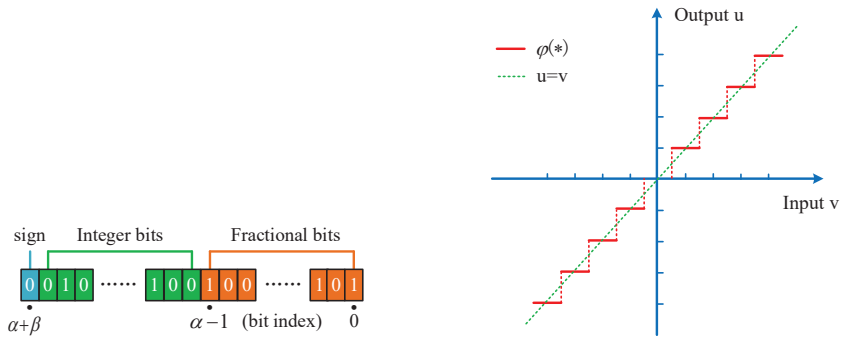
Neutrosophic numbers extend traditional numbers by allowing for more flexibility in representing uncertainty, imprecision, and contradiction. These numbers are often used in areas like decision-making, fuzzy logic, and artificial intelligence, especially when dealing with systems that have conflicting, uncertain, or incomplete information. In uncertain scenarios, a neutrosophic number z is expressed in the standard form as [22]: $z = a + b\mathcal{I} \in \mathbb{Z}$, where a represents the determinate term and $b\mathcal{I}$ represents the indeterminate term. Here, $a,\ b,\ \mathcal{I} \in \mathbb{R}$ with $\mathcal{I}$ lying within the interval $[\mathcal{I}^L,\ \mathcal{I}^U]$. For two neutrosophic numbers with specific forms of $z_1 = a_1 + b_1\mathcal{I}$, $z_2 = a_2 + b_2\mathcal{I}$, the operations are defined as follows [23]:

$$
\begin{aligned}
z_1 + z_2 &= (a_1 + b_1\mathcal{I}) + (a_2 + b_2\mathcal{I}) = (a_1 + a_2) + (b_1 + b_2)\mathcal{I}, \\
z_1 - z_2 &= (a_1 + b_1\mathcal{I}) - (a_2 + b_2\mathcal{I}) = (a_1 - a_2) + (b_1 - b_2)\mathcal{I}, \\
z_1 \times z_2 &= (a_1 + b_1\mathcal{I}) \times (a_2 + b_2\mathcal{I}) = a_1a_2 + (a_1b_2 + a_2b_1)\mathcal{I} + b_1b_2\mathcal{I}^2, \\
\frac{z_1}{z_2} &= \frac{a_1 + b_1\mathcal{I}}{a_2 + b_2\mathcal{I}}.
\end{aligned}
$$

2.2. *Parameter representation*

A representation scheme refers to the method or structure used to represent data, information, or numbers in a specific format within a system or framework. Choosing a suitable representation scheme is essential because it determines how information is represented, manipulated, and interpreted by computers, algorithms, or other systems. When approximating a real number $a \in \mathbb{R}$ digitally, the accuracy of its representation is influenced by the chosen structure and word length. This paper introduces a fixed-point representation approach. scheme [25]. to represent the parameters of the feedback gain. The fixed-point representation scheme utilizes a overall bit length of $\gamma = \alpha + \beta + 1$, as shown in Figure 1(a). Here, 1 bit is allocated for the sign, α bits is reserved for the integer part, and β bits for the fractional part. This representation allows for accurate representation of the integral part of a given real number a without overflow, assuming the word length is sufficiently large, specifically $\alpha = \log_2\lfloor a \rfloor$. However, the fractional part cannot be exactly represented, and the error in representation of a is determined only by the word length $\beta \in \mathbb{N}^+$. In particular, after representing a in fixed-point form, it will be perturbed by this error as:

$$\varphi(a) = a + \Delta_a,\ |\Delta_a| < 2^{-(\beta+1)} \in \mathbb{R}, \tag{1}$$



(a) (b)

FIGURE 1. Fixed-point representation: (a) Storage model; (b) Input-output characteristic.

where $\varphi(*)$ represents the mapping to the fixed-point form with the input-output characteristic being shown in Figure 1(b).

2.3. *Descriptor model*

Singular system refers to system of differential equations that include both differential equations and algebraic constraints. These systems are often more complex to solve because they can have singularities that make traditional analysis methods inapplicable. Unlike state-space models, singular systems described by the descriptor models must address regularity and causality, in addition to stability. In a discrete-time singular system:

$$\mathcal{E}\S(k+1) = \mathcal{A}\S(k),$$

where $\S(k) \in \mathbb{R}^{n_\S}$ denotes the state vector, $\mathcal{E},\ \mathcal{A} \in \mathbb{R}^{n_\S \times n_\S}$ are system parameter matrices with known numerical values and $\mathcal{E}$ maybe singular, i.e. $rank(\mathcal{E}) \leq n_\S$, the following definitions are introduced [24]:

- The singular system is regular if $det(z\mathcal{E} - \mathcal{A})$ is not identically zero;
- The singular system is causal if $deg[det(z\mathcal{E} - \mathcal{A})] = rank(\mathcal{E})$;
- The singular system is stable if $\rho(\mathcal{E},\ \mathcal{A}) < 1$;
- The singular system is admissible if it is regular, causal and stable.

3. **Problem Formulation**

Consider systems with unknown parameters described by the following discrete-time neutrosophic state-space model:

$$\begin{cases} x(k+1) = A(I)x(k) + B(I)u(k), \\ \quad y(k) = C(I)x(k), \end{cases} \tag{2}$$

where $x(k) \in \mathbb{R}^n$, $u(k) \in \mathbb{R}^m$ and $y(k) \in \mathbb{R}^p$ are the state, the control input and the controlled output, respectively. $A(I) \in \mathbb{Z}^{n\times n}$, $B(I) \in \mathbb{Z}^{n\times m}$ and $C(I) \in \mathbb{Z}^{p\times n}$ in (2) are neutrosophic matrices with the specific forms given by:

$$A(I) = \tilde{A} + \check{A}I,\ B(I) = \tilde{B} + \check{B}I,\ C(I) = \tilde{C} + \check{C}I, \tag{3}$$

where $\tilde{A} \in \mathbb{R}^{n\times n}$, $\check{A} \in \mathbb{R}^{n\times n}$, $\tilde{B} \in \mathbb{R}^{n\times m}$, $\check{B} \in \mathbb{R}^{n\times m}$, $\tilde{C} \in \mathbb{R}^{p\times n}$, $\check{C} \in \mathbb{R}^{p\times n}$, $I \in [\check{I},\ \hat{I}]$ with $\check{I}$ and $\hat{I}\ \in \mathbb{R}$. In Eq. (3), the indeterminate term I is introduced to express the uncertainty of system parameters caused by manufacturing tolerance, external environment change and other possible factors.

Without loss of generality, assuming the system described by (2) is completely state-controllable [11], and for further stabilization, the state-feedback control law is required to be designed in the following form:

$$v(k) = \begin{bmatrix} v_1(k) & v_2(k) & \cdots & v_m(k) \end{bmatrix}^T = Kx(k), \tag{4}$$

where $K \in \mathbb{R}^{m\times n}$ is the state-feedback gain to be determined. Due to resource limitations and to ensure implementation feasibility, the feedback gain K in (4) must be digitally implemented with finite word length, and the fixed-point representation scheme described in Section 2.2 is adopted. The integer parts of all coefficients in K are assumed to be represented exactly without overflow, whereas the fractional parts are represented with a finite bit length β. Thus, the coefficient representation depends only on β, and the integer word length α is omitted in the rest of this paper.

Based on the discussion in Section 2.2, the representation of a real matrix can be further examined. For a given feedback gain K, let $D^K \in \mathbb{R}^{m\times n}$ be a real matrix with the same dimensions as K, and its elements are determined as follows:

$$D_{ij}^K = \begin{cases} 0,\ if\ K_{ij} \in \mathbb{N}, \\ 1,\ if\ K_{ij} \notin \mathbb{N}, \end{cases}$$

where D_{ij}^K represents the element of D^K in the i-th row and j-th column, while K_{ij} denotes the element in the i-th row and j-th column of K.

In general, controller parameters require careful design and tuning, with their values typically needing to be moderate. High-gain controller parameters can lead to issues such as controller saturation, amplification of measurement errors, and system oscillations. To address these concerns, it is reasonable to assume that the parameters in the feedback gain K are moderate. Specifically, the integer part of these parameters can be accurately represented using a fixed-point representation with a sufficient number of bits α. Therefore, in the remainder of this paper, the discussion will focus exclusively on the effects of finite word length on the fractional part of the controller parameters. Using a fixed-point representation with β bits

for the fractional part and considering the analysis of the representation error provided in (1), the feedback gain is perturbed to

$$\varphi(K) = K + \Delta K \in \mathbb{R}^{m\times n}, \tag{5}$$

where

$$\Delta K = \begin{bmatrix} D_{11}^K \Delta_{11}^K & \cdots & D_{1n}^K \Delta_{1n}^K \\ \vdots & \ddots & \vdots \\ D_{m1}^K \Delta_{m1}^K & \cdots & D_{mn}^K \Delta_{mn}^K \end{bmatrix}, \ |\Delta_{ij}^K| < 2^{-(\beta+1)} \in \mathbb{R}.$$

Based on the above discussion, the control input $u(k)$ in (2) can be obtained as:

$$u(k) = \varphi(K)x(k) = (K + \Delta K)x(k). \tag{6}$$

Combining the plant (2) and its control input (6), the closed-loop system is obtained as:

$$\begin{cases} x(k+1) = [(\tilde{A} + \breve{A}I) + (\tilde{B} + \breve{B}I)(K + \Delta K)]x(k), \\ y(k) = (\tilde{C} + \breve{C}I)x(k). \end{cases} \tag{7}$$

The control problem addressed in this paper is described as follows.

Problem 1. Design the state-feedback gain K to guarantee the asymptotic stability of the closed-loop system in (7), subject to the unknown indeterminacy I and ΔK.

Noting the unknown indeterminacies I and ΔK present in the closed-loop system (7), Problem 1 is identified as a stabilization problem with multiple coupled unknown coefficients, which makes it challenging to parameterize the feedback gain K. Therefore, a descriptor model approach is proposed in the following section to address these challenges, along with the corresponding solvability conditions for Problem 1.

4. Main Results

In this section, a comprehensive solution method for solving Problem 1 is detailed. More specifically, a descriptor model approach is proposed in subsection 4.1 to rewrite the closed-loop system (7). Based on the obtained equivalent model for (7), a stability analysis condition is deduced in subsection 4.2, along with a method for designing the feedback gain. And Figure 2 illustrates the key steps of the proposed method, providing a clear overview of the entire process from system modeling to stabilization.

To aid in deriving the main results, the following lemmas are presented first.

Lemma 4.1. *For a neutrosophic number $z = a + bI$ with $a,\ b,\ I \in \mathbb{R}$ and $\mathcal{I} \in [\check{I},\ \hat{I}]$, it can be rewritten as:*

$$z = \bar{a} + \bar{b}\bar{I}, \tag{8}$$

where $\bar{a} = a + b\frac{\check{I}+\hat{I}}{2}$, $\bar{b} = b\frac{\check{I}-\hat{I}}{2}$, $\bar{I} \in [-1, 1]$.

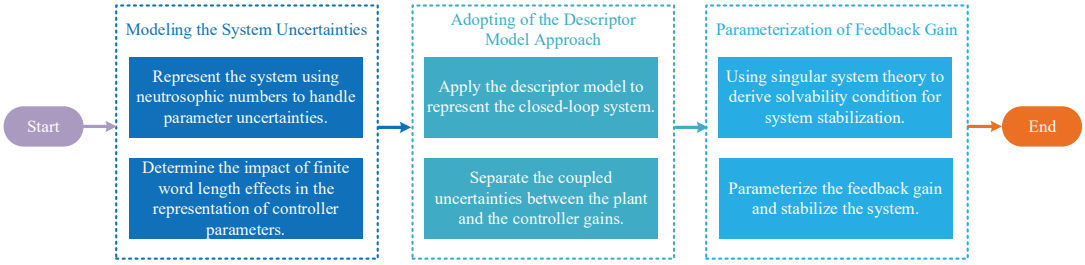

Start
Modeling the System Uncertainties
Represent the system using neutrosophic numbers to handle parameter uncertainties.
Determine the impact of finite word length effects in the representation of controller parameters.
Adopting of the Descriptor Model Approach
Apply the descriptor model to represent the closed-loop system.
Separate the coupled uncertainties between the plant and the controller gains.
Parameterization of Feedback Gain
Using singular system theory to derive solvability condition for system stabilization.
Parameterize the feedback gain and stabilize the system.
End

FIGURE 2. Flowchart of the proposed method.

Proof. By submitting $\bar{a} = a + b\frac{\check{I}+\hat{I}}{2}$ and $\bar{b} = b\frac{\check{I}-\hat{I}}{2}$ into Eq. (8), it is obtained that

$$z = a + b\frac{\check{I}+\hat{I}}{2} + b\frac{\check{I}-\hat{I}}{2}\bar{I}.$$

Noting that $\bar{I} = \frac{2}{\check{I}-\hat{I}}(I - \frac{\check{I}+\hat{I}}{2}) \in [-1, 1]$, the above equation can be expressed as:

$$z = a + b\frac{\check{I}+\hat{I}}{2} + b\frac{\check{I}-\hat{I}}{2}\frac{2}{\check{I}-\hat{I}}(I - \frac{\check{I}+\hat{I}}{2}) = a + b\frac{\check{I}+\hat{I}}{2} + b(I - \frac{\check{I}+\hat{I}}{2}).$$

Then it is observed that $z = \bar{a} + \bar{b}\bar{I} = a + bI$. □

Lemma 4.2. *[26] For matrices Φ_1, Φ_2 and Φ_3 with appropriate dimensions with Φ_1 satisfying $\Phi_1 = \Phi_1^T$, then*

$$\Phi_1 + \Phi_2\Phi_4(k)\Phi_3^T + \Phi_3\Phi_4^T(k)\Phi_2^T < 0$$

holds for all $\Phi_4^T(k)\Phi_4(k) < U$ if and only if a scalar $\epsilon > 0$ exists such that

$$\Phi_1 + \epsilon\Phi_2\Phi_2^T + \epsilon^{-1}\Phi_3\Phi_3^T < 0.$$

Lemma 4.3. *[24] For real matrices Φ_{11}, Φ_{12}, Φ_{21} and Φ_{22} with appropriate dimensions, if $\Phi = \begin{bmatrix} \Phi_{11} & \Phi_{12} \\ \Phi_{21} & \Phi_{22} \end{bmatrix}$ is invertible and satisfies $\Phi + \Phi^T < 0$. There holds the following inequality:*

$$\Phi_{11} + \Phi_{11}^T - \Phi_{12}\Phi_{22}^{-1}\Phi_{21} - \Phi_{21}^T\Phi_{22}^{-T}\Phi_{12}^T < 0.$$

4.1. *System remodeling*

This subsection aims to introduce a new approach for remodeling the closed-loop system (7). This framework will enable the proposal of an appropriate analysis and design methodology that can effectively handle the coupled unknown parameters. To this end, the unknown indeterminacy I and the corresponding system parameters in (2) are standardized by employing the method outlined in Lemma 4.1.

By denoting $\bar{I} \in [-1, 1]$, the standardization can be obtained for parameters in (2) as:

$$A_1 = \tilde{A} + \frac{\hat{I} + \check{I}}{2}\breve{A},\ A_2 = \tilde{A} + \frac{\hat{I} - \check{I}}{2}\breve{A},\ B_1 = \tilde{B} + \frac{\hat{I} + \check{I}}{2}\breve{B},$$
$$B_2 = \tilde{B} + \frac{\hat{I} - \check{I}}{2}\breve{B},\ C_1 = \tilde{C} + \frac{\hat{I} + \check{I}}{2}\breve{C},\ C_2 = \tilde{C} + \frac{\hat{I} - \check{I}}{2}\breve{C},$$
$$A_1 \in \mathbb{R}^{n\times n}, A_2 \in \mathbb{R}^{n\times n},\ B_1 \in \mathbb{R}^{n\times m}\ B_2 \in \mathbb{R}^{n\times m},\ C_1 \in \mathbb{R}^{p\times n},\ C_2 \in \mathbb{R}^{p\times n}.$$

With the above standardization, the closed-loop system described by (7) can be reformulated as:

$$\begin{cases} x(k+1) = [(A_1 + A_2\bar{I}) + (B_1 + B_2\bar{I})(K + \Delta K)]x(k), \\ \quad y(k) = (C_1 + C_2\bar{I})x(k). \end{cases} \tag{9}$$

To further address the coupling between $\bar{I}$ and ΔK, an intermediate variable $\zeta(k) \in \mathbb{R}^m$ is introduced, and the closed-loop system (9) is reformulated in the descriptor model form as follows:

$$\begin{cases} \hat{E}\hat{x}(k+1) = (\hat{A}_1 + \hat{A}_2\bar{I} + \hat{B}\hat{K}_1 + \hat{B}\hat{K}_2)\hat{x}(k), \\ \quad y(k) = (\hat{C}_1 + \hat{C}_2\bar{I})\hat{x}(k), \end{cases} \tag{10}$$

where

$$\hat{x} = \begin{bmatrix} x(k) \\ \zeta(k) \end{bmatrix},\ \hat{E} = \begin{bmatrix} U_n & 0_{n\times m} \\ 0_{m\times n} & 0_m \end{bmatrix},\ \hat{A}_1 = \begin{bmatrix} A_1 & B_1 \\ 0_{m\times n} & -U_m \end{bmatrix},\ \hat{A}_2 = \begin{bmatrix} A_2 & B_2 \\ 0_{m\times n} & 0_m \end{bmatrix},$$
$$\hat{B} = \begin{bmatrix} 0_{n\times m} \\ U_m \end{bmatrix},\ \hat{K}_1 = \begin{bmatrix} K^T \\ 0_m \end{bmatrix}^T,\ \hat{K}_2 = \begin{bmatrix} \Delta K^T \\ 0_m \end{bmatrix}^T\ \hat{C}_1 = \begin{bmatrix} C_1^T \\ 0_{m\times p} \end{bmatrix}^T,\ \hat{C}_2 = \begin{bmatrix} C_2^T \\ 0_{m\times p} \end{bmatrix}^T,$$

and $\zeta(k) \in \mathbb{R}^m$ is the intermediate vector introduced.

4.2. *Stability analysis and stabilization*

Based on the reformulated descriptor model (10), this subsection derives a stability condition for Problem 1 and proposes a technique for feedback gain design.

Theorem 4.4. *For a given scalar $\beta \in \mathbb{N}^+$, the descriptor system* (10) *is admissible provided that matrices $Q \in \mathbb{R}^{n\times n}$, $R \in \mathbb{R}^{n\times m}$, $S \in \mathbb{R}^{m\times m}$, $P \in \mathbb{R}^{n\times n}$ (with $P > 0$), and scalars $\epsilon > 0$, $\epsilon_i > 0$ for $i = 1,\ 2,\ \cdots,\ m$ exist, such that the following inequality holds:*

$$\begin{bmatrix} -\frac{1}{2}Q - \frac{1}{2}Q^T & \Theta_{12} & P - Q^T - \frac{1}{2}Q & \Theta_{41} \\ * & \Theta_{22} & \Theta_{23} & \Theta_{42} \\ * & * & -Q - Q^T & \Theta_{43} \\ * & * & * & \Theta_{44} \end{bmatrix} < 0, \tag{11}$$

where

$$\Theta_{12}=\Phi_1(\hat{A}_1+\hat{B}\hat{K}_1)^T,\ \Theta_{23}=\Theta_{12}^T,\ \Theta_{22}=\Phi_2(\hat{A}_1+\hat{B}\hat{K}_1)^T+(\hat{A}_1+\hat{B}\hat{K}_1)\Phi_2^T-\Phi_3+\Phi_4,$$

$$\Theta_{41}=\Theta_{43}=\underbrace{\left[\begin{array}{cccc}\Phi_1 & \Phi_1 & \cdots & \Phi_1\end{array}\right]}_{m+1},\ \Theta_{42}=\underbrace{\left[\begin{array}{cccc}\Phi_2 & \Phi_2 & \cdots & \Phi_2\end{array}\right]}_{m+1},\ \Theta_{44}=E_\epsilon\otimes U_{(m+1)(n+m)},$$

$$\Phi_1=\left[\begin{array}{cc}Q & R\end{array}\right],\ \Phi_2=\left[\begin{array}{cc}0_n & 0_{n\times m}\\ 0_{m\times n} & S\end{array}\right],\ \Phi_3=\left[\begin{array}{cc}P & 0_{n\times m}\\ 0_{m\times n} & 0_m\end{array}\right],\ \Phi_4=\left[\begin{array}{cc}\Phi_{41} & 0_{n\times m}\\ 0_{m\times n} & \Phi_{42}\end{array}\right],$$

$$E_\epsilon=diag([\epsilon\ \epsilon_1\ \epsilon_2\ \cdots\ \epsilon_m]),\ \Delta_\beta=2^{-(\beta+1)},\ \Phi_{41}=\epsilon(A_2A_2^T+B_2B_2^T),\ \Phi_{42}=\Delta_\beta^2E_\epsilon\odot(D_KD_K^T),$$

$$D_K=\left[\begin{array}{ccc}D_{11}^K & \cdots & D_{1n}^K\\ \vdots & \ddots & \vdots\\ D_{m1}^K & \cdots & D_{mn}^K\end{array}\right].$$

Proof of Theorem 4.4. Assuming that the inequality (11) holds, one can apply Schur complement to obtain

$$\bar{\Phi}_4+\epsilon^{-1}\bar{\Phi}_1\bar{\Phi}_1^T+\epsilon\bar{\Phi}_2\bar{\Phi}_2^T+\sum_{i=1}^{m}(\epsilon_i^{-1}\bar{\Phi}_1\bar{\Phi}_1^T)+\bar{\Phi}_3<0,$$

where

$$\bar{\Phi}_4=\left[\begin{array}{ccc}-\frac{1}{2}Q-\frac{1}{2}Q^T & \Theta_{12} & P-Q^T-\frac{1}{2}Q\\ * & \bar{\Theta}_{22} & \Theta_{23}\\ * & * & -Q-Q^T\end{array}\right],\ \bar{\Phi}_1=\left[\begin{array}{c}\Phi_1\\ \Phi_2\\ \Phi_1\end{array}\right],\ \bar{\Phi}_2=\left[\begin{array}{c}0_{n\times(n+m)}\\ \hat{A}_2\\ 0_{n\times(n+m)}\end{array}\right],$$

$$\bar{\Phi}_3=\left[\begin{array}{ccc}0_n & 0_n & 0_{n\times m}\\ * & 0_n & 0_{n\times m}\\ * & * & \Phi_4\end{array}\right],\ \bar{\Theta}_{22}=\Phi_2(\hat{A}_1+\hat{B}\hat{K}_1)^T+(\hat{A}_1+\hat{B}\hat{K}_1)\Phi_2^T-\Phi_3.$$

Noting that

$$\bar{\Phi}_3=\sum_{i=1}^{m}(\Delta_\beta^2\epsilon_i\bar{\Phi}_{3i}\bar{\Phi}_{3i}^T)$$

where

$$\bar{\Phi}_{3i}=\left[\begin{array}{c}0_{n\times(n+m)}\\ \bar{D}_{Ki}\\ 0_{n\times(n+m)}\end{array}\right],\ \bar{D}_{Ki}=\left[\begin{array}{cc}0_n & 0_{n\times m}\\ 0_{m\times n} & D_{Ki}\end{array}\right],\ D_{Ki}=\left[\begin{array}{c}0_{(i-1)\times n}\\ [D_{i1}^K\ D_{i2}^K\ \cdots\ D_{in}^K]\\ 0_{(m-i)\times n}\end{array}\right],$$

there holds

$$\bar{\Phi}_4+\bar{\Phi}_4+\epsilon^{-1}\bar{\Phi}_1\bar{\Phi}_1^T+\epsilon\bar{\Phi}_2\bar{\Phi}_2^T+\sum_{i=1}^{m}(\epsilon_i^{-1}\bar{\Phi}_1\bar{\Phi}_1^T+\Delta_\beta^2\epsilon_i\bar{\Phi}_{3i}\bar{\Phi}_{3i}^T)<0.$$

Based on the above inequality, applying Lemma 4.2 yields:

$$\bar{\Phi}_5=\bar{\Phi}_4+\bar{\Phi}_1\bar{I}\bar{\Phi}_2^T+\bar{\Phi}_2\bar{I}\bar{\Phi}_1^T+\sum_{i=1}^{m}(\Delta_\beta\bar{\Phi}_1\frac{\bar{\Delta}_i}{\Delta_\beta}\bar{\Phi}_{3i}^T\bar{+}\Delta_\beta\bar{\Phi}_{3i}\frac{\bar{\Delta}_i}{\Delta_\beta}\bar{\Phi}_1^T)<0,$$

where

$$\bar{\Phi}_5 = \begin{bmatrix} -\frac{1}{2}Q - \frac{1}{2}Q^T & \Phi_1\hat{A}^T & P - Q^T - \frac{1}{2}Q \\ * & \Phi_2\hat{A}^T + \hat{A}\Phi_2^T - \Phi_3 & \hat{A}\Phi_1^T \\ * & * & -Q - Q^T \end{bmatrix},\ \bar{\Delta}_i = \begin{bmatrix} \Delta_i & 0_{n\times m} \\ 0_{m\times n} & 0_m \end{bmatrix},$$

$$\hat{A} = (\hat{A}_1 + \hat{A}_2\bar{I} + \hat{B}\hat{K}_1 + \hat{B}\hat{K}_2) = \begin{bmatrix} \hat{A}_{11} & \hat{A}_{12} \\ \hat{A}_{21} & \hat{A}_{22} \end{bmatrix} = \begin{bmatrix} A_1 + A_2\bar{I} & B_1 + B_2\bar{I} \\ K + \Delta K & -U_m \end{bmatrix},$$

$$\Delta_i = diag(\begin{bmatrix}\Delta_{i1}^K\ \Delta_{i2}^K\ \cdots\ \Delta_{in}^K\end{bmatrix}).$$

Noting the decompositions of $\hat{A}$, the above inequality can be rewritten as:

$$\bar{\Phi}_5 = \begin{bmatrix} -\frac{1}{2}Q - \frac{1}{2}Q^T & Q\hat{A}_{11}^T + R\hat{A}_{12}^T & Q\hat{A}_{21}^T + R\hat{A}_{22}^T & P - Q^T - \frac{1}{2}Q \\ * & -P & \hat{A}_{12}S^T & \hat{A}_{11}Q^T + \hat{A}_{12}R^T \\ * & * & S\hat{A}_{22}^T + \hat{A}_{22}S^T & \hat{A}_{21}Q^T + \hat{A}_{22}R^T \\ * & * & * & -Q - Q^T \end{bmatrix} < 0.$$

Multiply the above inequality on the left and right by the following matrix:

$$\begin{bmatrix} U_n & 0_n & 0_{n\times m} & 0_n \\ 0_n & U_n & 0_{n\times m} & 0_n \\ 0_n & 0_n & 0_{n\times m} & U_n \\ 0_{m\times n} & 0_{m\times n} & U_m & 0_{m\times n} \end{bmatrix}$$

and its transpose, respectively, results in

$$\Lambda + \Lambda^T < 0, \tag{12}$$

where

$$\Lambda = \begin{bmatrix} -\frac{1}{2}Q & Q\hat{A}_{11}^T + R\hat{A}_{12}^T & P - Q^T - \frac{1}{2}Q & Q\hat{A}_{21}^T + R\hat{A}_{22}^T \\ 0_n & -\frac{1}{2}P & 0_n & 0_{n\times m} \\ 0_n & Q\hat{A}_{11}^T + R\hat{A}_{12}^T & -Q & Q\hat{A}_{21}^T + R\hat{A}_{22}^T \\ 0_{m\times n} & S\hat{A}_{12}^T & 0_{m\times n} & S\hat{A}_{22}^T \end{bmatrix}.$$

Note that within Λ, the condition $S\hat{A}_{22}^T + \hat{A}_{22}S^T < 0$ holds. Furthermore, based on the matrix norm properties [27], both $\hat{A}_{22}$ and S are nonsingular matrices. Consequently, the singular system (10) is regular and causal [28], and can therefore be reduced to a state-space representation:

$$x(k+1) = \bar{A}x(k),$$

where $\bar{A} = \hat{A}_{11} - \hat{A}_{12}\hat{A}_{22}^{-1}\hat{A}_{21}$.

The above simplified state-space system can be determined to be asymptotic stable if a matrix $P > 0$ can be found such that $\bar{A}^T P\bar{A} - P < 0$. Applying the Schur complement, this condition can be equivalently rewritten as:

$$\begin{bmatrix} -P & P\bar{A} \\ * & -P \end{bmatrix} < 0,$$

which can be reformulated as $\Xi_\Gamma^T \Xi \Xi_\Gamma < 0$, where

$$\Xi = \begin{bmatrix} 0_n & 0_n & P \\ 0_n & -P & 0_n \\ P & 0_n & 0_n \end{bmatrix}, \ \Xi_\Gamma = \begin{bmatrix} U_n & 0_n \\ 0_n & U_n \\ -\frac{1}{2}U_n & \bar{A} \end{bmatrix}.$$

Given that $P > 0$, a direct constraint can be introduced as:

$$\begin{bmatrix} -P & 0_n \\ 0_n & -2P \end{bmatrix} < 0,$$

which can be reformulated as $\Xi_\Psi^T \Xi \Xi_\Psi < 0$, where

$$\Xi_\Psi^T = \begin{bmatrix} 0_n & U_n & 0_n \\ -U_n & 0_n & U_n \end{bmatrix}.$$

Noting that the inequalities $\Xi_\Gamma^T \Xi \Xi_\Gamma < 0$ hold, the application of the Projection Lemma [29] yields

$$\Xi + \Gamma^T Q^T \Psi + \Psi^T Q \Gamma < 0, \tag{13}$$

where

$$\Psi = \begin{bmatrix} U_n & 0_n & U_n \end{bmatrix}, \ \Gamma = \begin{bmatrix} -\frac{1}{2}U_n & \bar{A} & -U_n \end{bmatrix},$$

with $\Psi \Xi_\Psi = 0_{n \times 2n}$, $\Gamma \Xi_\Gamma = 0_{n \times 2n}$.

Applying Lemma 4.3 to inequality (12) results in

$$\begin{bmatrix} -\frac{1}{2}Q - \frac{1}{2}Q^T & Q\bar{A} & P - Q^T - \frac{1}{2}Q \\ * & -P & \bar{A}^T Q^T \\ * & * & -Q - Q^T \end{bmatrix} < 0,$$

which simplifies to inequality (13). Consequently, since $\bar{A}$ is stable, the singular system (10) is admissible.

Remark 4.5. In Theorem 4.4, the variable $\Delta_\beta = 2^{-(\beta+1)}$ represents the maximum error bound for the parameters in the state-feedback gain K in (4) when using fixed-point representation, as discussed in Section 2.2. Here, $\beta \in \mathbb{N}^+$ represents the word length utilized for expressing the fractional component of the parameters. Thus, the parameter representation error can be controlled by selecting an appropriate value for β. Especially, if the feedback matrix K contains only integer parameters, it will be unaffected by representation errors and can be accurately represented. To illustrate this, Theorem 4.4 introduces the matrix D_K. An element of D_K is 0 if the corresponding parameter in K is an integer; otherwise, the element in D_K is 1. For a given constant state-feedback gain K, D_K can be determined according to these rules. Conversely, if K is a matrix to be designed and is not predetermined, D_K can be set to a matrix with all elements equal to 1 if there is no additional available information.

In Theorem 4.4, determining the state-feedback gain K is challenging due to the coupling between the gain parameters in K and the unknown matrix Q. To address this issue, Theorem 4.6 provides a framework that facilitates the practical design of the feedback gain.

Theorem 4.6. *For a given scalar $\beta \in \mathbb{N}^+$, the descriptor system (10) is admissible provided that matrices $Q \in \mathbb{R}^{n\times n}$, $R \in \mathbb{R}^{n\times m}$, $S \in \mathbb{R}^{m\times m}$, $P \in \mathbb{R}^{n\times n}$ (with $P > 0$), and scalars $\epsilon > 0$, $\epsilon_i > 0$ for $i = 1,\ 2,\ \cdots,\ m$ exist, such that the following inequality holds:*

$$\begin{bmatrix} -\frac{1}{2}Q - \frac{1}{2}Q^T & \bar{\Theta}_{12} & P - Q^T - \frac{1}{2}Q & \Theta_{41} \\ * & \Theta_{22} & \bar{\Theta}_{23} & \Theta_{42} \\ * & * & -Q - Q^T & \Theta_{43} \\ * & * & * & \Theta_{44} \end{bmatrix} < 0, \tag{14}$$

where $\bar{\Theta}_{12} = \begin{bmatrix} QA_1^T + RB_1^T & Z - R \end{bmatrix}$, $\bar{\Theta}_{23} = \bar{\Theta}_{12}^T$. Furthermore, the determination of the state-feedback gain K is carried out as follows:

$$K = Z^T Q^{-T}. \tag{15}$$

Proof of Theorem 4.6. By substituting $\bar{Q}K^T$ with Z in (11), (14) can be obtained. Therefore, (10) is admissible provided that (14) is satisfied.

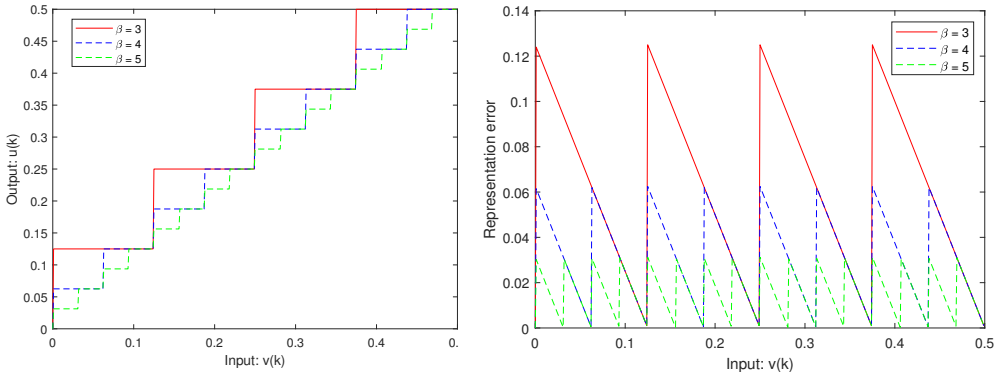
5. Example illustration

In the following discussion, a simulation case is given to demonstrate the application of the proposed design method and to validate its effectiveness. In other words, the finite-precision stabilization problem for discrete-time neutrosophic systems, as illustrated in (2), needs to be addressed. As discussed earlier, based on Lemma 4.1, any system of the form shown in (2) can be standardized and ultimately rewritten as the closed-loop system depicted in (9). Based on these results, the standardized system presented in (9) is adopted as the plant, with the known parameters specified as follows:

$$A_1 = \begin{bmatrix} 1.5000 & 0.1000 & 0.3000 \\ 0.7000 & 1.2000 & 0.8000 \\ 0.5000 & 0.8000 & 1.0000 \end{bmatrix},\ B_1 = \begin{bmatrix} 0.5000 & 0.4000 \\ 0.6000 & 0.3000 \\ 0.4000 & 0.2000 \end{bmatrix},$$

$$A_2 = \begin{bmatrix} 0.0300 & 0.0600 & 0.0900 \\ 0.0600 & 0.0900 & 0.0600 \\ 0.0600 & 0.0900 & 0.0300 \end{bmatrix},\ B_2 = \begin{bmatrix} 0.0600 & 0.0300 \\ 0.0900 & 0.0600 \\ 0.0600 & 0.0600 \end{bmatrix},\ \bar{I} \in [-1, 1].$$

For the system outlined above, the design of the state feedback gain K is required, and the effects of fixed-point representation on its parameters must be considered. The matrix K will be perturbed as shown in (5). The parameters $D_{ij}^K,\ i = 1,\ 2,\ \cdots,\ m,\ j = 1,\ 2,\ \cdots,\ n$ for



(a) (b)

FIGURE 3. (a) Input-output characteristics of the fixed-point representation scheme proposed in Algorithm 1; (b) Representation error of the fixed-point representation scheme.

ΔK in (5) are set as follows:

$$D_K = \begin{bmatrix} D_{11}^K & D_{12}^K & D_{13}^K \\ D_{21}^K & D_{22}^K & D_{23}^K \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}.$$

Then, the bit length used to represent the fractional part is set to $\beta = 4$, which implies an upper bound on the representation error as $\Delta_\beta = 0.0625$. Based on the parameters of the illustrative example, the coefficient matrix in Theorem 4.6 is constructed, and the inequality (14) is solved to obtain:

$$\begin{aligned} K = Z^T Q^{-T} &= \begin{bmatrix} -3.8027 & -3.7431 \\ -3.3534 & -0.5345 \\ -2.6144 & -0.7285 \end{bmatrix}^T \begin{bmatrix} 2.9995 & 1.1979 & -0.0964 \\ 1.1979 & 3.3472 & -2.5932 \\ -0.0964 & -2.5932 & 4.8143 \end{bmatrix}^{-T} \\ &= \begin{bmatrix} -0.5000 & -2.1875 & -1.7500 \\ -1.4375 & 0.3750 & 0.0625 \end{bmatrix}. \end{aligned} \tag{16}$$

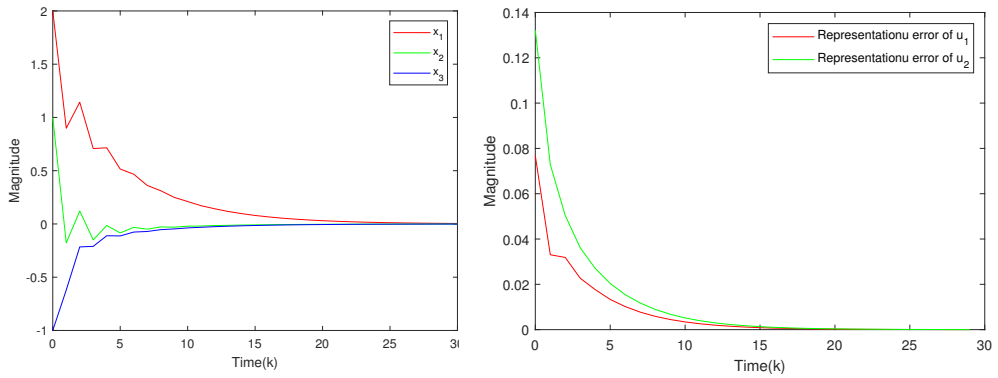
To implement the fixed-point representation scheme proposed in this paper, Algorithm 1 is presented for representing the fractional part of the feedback gain parameters K in (16). The input-output characteristics of this fixed-point representation scheme are illustrated in Figure 3(a) for different bit length values β and the corresponding representation error is shown in Figure 3(b). As observed, with an increase in bit length β, the representation error decreases. The state evolution of the closed-loop system described by (9), using the parameters outlined in this section, is presented in Figure 4(a). Figure 4(b) shows the error introduced in the control signal $u(k)$ by the fixed-point representation scheme. These results validate the effectiveness of the proposed design method.

Algorithm 1 Fixed-point representation scheme

Input: $\beta \in \mathbb{N}^+$ and $-1 < v(k) < 1$;

Output: $u(k)$;

if $v(k) >= 0$ **then**
  for all $i = 1,\ 2,\ \cdots,\ 2^{\beta}$ **do**
    if $v(k) >= (k-1) \times 2^{-\beta}$ & $v(k) < k \times 2^{-\beta}$ **then**
      $u(k) = k \times 2^{-\beta}$, break;
    end if
  end for
else
  for all $i = 1,\ 2,\ \cdots,\ 2^{\beta}$ **do**
    if $-v(k) >= (k-1) \times 2^{-\beta}$ & $-v(k) < k \times 2^{-\beta}$ **then**
      $u(k) = -k \times 2^{-\beta}$, break;
    end if
  end for
end if
return $u(k)$.



(a) (b)

FIGURE 4. (a) State evolution of the closed-loop system; (b) Error introduced in the control signal $u(k)$ by the fixed-point representation scheme.

Table 1 illustrates the feasibility of the inequality (14) under parameter uncertainties of varying magnitudes and different bit lengths. As shown in Table 1, the feasibility of the controller solvability decreases as the system parameter uncertainties increase. Similarly, as the bit length used to represent the fractional part of the control gain decreases, the feasibility of the controller solvability also diminishes.

TABLE 1. Feasibility of the inequality (14) with parameter uncertainties of varying magnitudes and different bit lengths.

	$\beta = 2$	$\beta = 3$	$\beta = 4$	$\beta = 5$
$\bar{I} \in [-0.5, 0.5]$	infeasible	feasible	feasible	feasible
$\bar{I} \in [-1, 1]$	infeasible	infeasible	feasible	feasible
$\bar{I} \in [-2, 2]$	infeasible	infeasible	infeasible	feasible

TABLE 2. Convergence performance of the system under different bit lengths.

Bit lengths for representation	$\beta = 3$	$\beta = 4$	$\beta = 5$
Convergence performance J	8.3064	8.2121	8.0395

To compare the convergence performance of the system under different bit lengths, the following performance index is defined:

$$J = \sum_{k=0}^{\infty} \begin{bmatrix} x_1(k) & x_2(k) & x_3(k) \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1(k) & x_2(k) & x_3(k) \end{bmatrix}^T.$$

The results are presented in Table 2, where it can be observed that as the bit length increases, the convergence performance of system improves.

6. **Discussion**

The stabilization of MIMO neutrosophic linear systems under finite-precision digital control is a critical but underexplored area. Existing research on FWL effects and parameter uncertainty in control systems often focuses on simplified models or single-input systems, leaving MIMO systems with complex uncertainties largely unaddressed. This work addresses these challenges by proposing an integrated approach that combines neutrosophic modeling with FWL stability analysis, a unique contribution to the field. Through the proposed descriptor model approach, the closed-loop system is successfully reformulated into a singular system form. This transformation allowed us to apply stability analysis techniques suitable for singular systems, leading to a comprehensive understanding of how parameter uncertainties and fixed-point representations impact the stability and solvability of the controller. While the proposed method effectively addresses the stabilization of MIMO neutrosophic linear systems under finite word length effects, the model does not account for time-varying uncertainties or nonlinearities that may exist in some practical applications. These limitations may limit the applicability of the method in highly dynamic or nonlinear systems, which remains an area for future research.

7. Conclusions

This paper addresses the finite-precision stabilization problem for neutrosophic linear systems, where system parameter uncertainties are represented using neutrosophic numbers and controller parameters are modeled using a fixed-point scheme. A descriptor model approach is proposed to handle the coupling between parameter uncertainties and representation errors, reformulating the closed-loop system into a singular system form. Stability analysis and controller parameterization are achieved through analytical and design methods for singular systems. While the current study focuses on state feedback control, the approach and techniques outlined could be extended to address other types of control, such as dynamic output feedback stabilization, which remains a promising direction for future research. Furthermore, the investigation of distributed control strategies for neutrosophic linear systems under finite-precision conditions presents another exciting avenue for further exploration.

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Neutrosophic Analysis of the Cultural Impact of the Pinkullo in Llata: Integrating ChatGPT and fsQCA

[1] Fredy Romulo Marcellini Morales, [2] Betty Leon Trujillo, [3] Esio Ocana Igarza , [4] Roberto Carlos Cardenas Viviano

[1] Daniel Alomía Robles National University , fmarcellini@undar.edu.pe

[2] Daniel Alomía Robles National University , bleon@undar.edu.pe

[3] Daniel Alomia Oaks National University , eocana@undar.edu

[4] Daniel Alomía Robles National University , rcardenas@undar.edu .

Abstract

This study explores the cultural impact of the Pinkullo, a traditional Andean aerophone, on the community of Llata, Peru, through an innovative approach that integrates neutrosophic set theory with computational methods. By employing fuzzy-set Qualitative Comparative Analysis (fsQCA) and a novel application of ChatGPT for neutrosophic sentiment analysis, the research investigates the complex interplay between the Pinkullo and various aspects of community life, including cultural participation, social impact, and cultural preservation. Textual data collected from surveys was analyzed using the "ChatGPT Constructor" add-on in Orange Data Mining to generate neutrosophic scores, representing the truth, indeterminacy, and falsity of expressed sentiments towards the Pinkullo. These scores were then fuzzified and incorporated into the fsQCA to identify the key configurations of factors that contribute to a high degree of cultural participation. The findings reveal that a positive sentiment towards the Pinkullo, combined with its perceived social impact and role in cultural preservation, constitutes a crucial pathway for fostering cultural engagement. This research not only provides valuable insights into the enduring significance of the Pinkullo in Llata but also demonstrates the potential of integrating large language models and neutrosophic logic to analyze complex cultural phenomena, paving the way for new interdisciplinary approaches in the study of cultural heritage..

Keywords: Pinkullo, Neutrosophic Analysis. Cultural Impact, ChatGPT, fsQCA

1. Introduction

Traditional music serves as a powerful expression of cultural identity, with indigenous musical instruments playing a crucial role in preserving this heritage. Among these, the Pinkullo, an Andean aerophone characteristic of Peru's mountainous regions, stands as more than just an instrument; it embodies a profound spiritual, social, and cultural connection [1]. Particularly in the district of Llata, the Pinkullo holds a central position in festivities, rituals, and community practices. However, its multifaceted influence and deeper meaning have yet to be thoroughly explored from a rigorous scientific perspective. This study proposes to analyze Pinkullo's cultural and social impact in Llata using an innovative approach based on neutrosophic set theory, thereby bridging the gap between traditional cultural studies and quantitative analysis.

In recent years, there's been a growing interest in documenting and understanding traditional instruments, driven by a need to comprehend cultural dynamics in an increasingly globalized world that often threatens to homogenize local expressions [2]. While the Pinkullo has traditionally been studied from ethnographic and musicological perspectives [3], the application of mathematical and theoretical models to its analysis remains largely unexplored. This research seeks to fill that void by integrating qualitative and quantitative approaches to capture the complexity of Pinkullo's influence on the community of Llata.

The central challenge guiding this research is the lack of adequate methodologies to analyze complex cultural phenomena that involve multiple dimensions of meaning and perception, such as the impact of the Pinkullo.

A key question driving this study is: How can the influence of the Pinkullo on a community be assessed using tools that effectively handle the inherent ambiguity, uncertainty, and subjectivity of cultural systems? This challenge demands an approach that transcends conventional methods. In the existing literature, dominant approaches to the analysis of traditional music tend to be descriptive and lack models that integrate indeterminacy and subjectivity [5]. Recent studies on Andean instruments like the Pinkullo have highlighted the need to consider not only their musical dimension but also their role as catalysts for social cohesion and community resilience [6]. This work responds to these limitations by adopting neutrosophic set theory, a mathematical model capable of representing and analyzing phenomena involving uncertainty, contradiction, and subjectivity [7, 8], as its core analytical framework.

To achieve this, qualitative data collection techniques, including interviews and participatory observations, were combined with quantitative tools incorporating fuzzy variables, all within the framework of neutrosophic set theory. This methodological approach provides a more comprehensive and rigorous view of Pinkullo's impact in Llata [9, 10]. The primary objective of this study is to analyze the influence of the Pinkullo on the cultural and social dynamics of the Llata district using this innovative method. Furthermore, this research aims to generate a replicable methodological framework applicable to the analysis of other complex cultural phenomena.

Ultimately, this work seeks to illuminate the intricate relationship between music, culture, and community, demonstrating how an ancient instrument like the Pinkullo continues to resonate and shape social dynamics in the present day. By employing this innovative approach, the research not only aims to quantify the multifaceted impact of the Pinkullo on cultural participation but also to offer a methodological framework adaptable to other complex cultural phenomena. The findings are anticipated to contribute significantly to the understanding and appreciation of Andean cultural heritage, providing valuable tools for its preservation in an increasingly interconnected world [11]. Moreover, this study bridges the analytical and the humanistic, demonstrating that the intersection between these perspectives can generate deep and transformative knowledge, enriching the field of cultural studies and showcasing the power of integrating mathematical models into traditionally qualitative disciplines.

2. Preliminaries

2.1Complexity theory and causality and neutrosophic sets.

Relationships Understanding the relationships between variables is rarely straightforward. Complexity theory, which emphasizes non-linear interactions, provides a robust framework for such analysis. It posits that the same cause can yield vastly different outcomes depending on the context, highlighting principles like conjunction, equifinality, and causal asymmetry. Neutrosophy enhances this perspective by explicitly incorporating indeterminacy and uncertainty, elements often inherent in social phenomena[12, 13].

- Conjunction: Outcomes result from the interaction of multiple antecedent conditions, not from isolated factors. For example, a restaurant's success depends not solely on food quality but also on its location, service, and ambiance. A combination of average food, excellent service, and a prime location can be as successful as outstanding cuisine alone.

- Equifinality: A system can achieve the same outcome through different initial conditions and pathways. Multiple strategies, considering various combinations of factors, can lead to restaurant success. This applies broadly across social, political, and economic contexts.

- Causal Asymmetry: The factors leading to an outcome's presence are not necessarily the opposite of those leading to its absence. A successful restaurant might thrive due to high-quality food, but the absence of high-quality food doesn't automatically doom a different restaurant. This challenges deterministic approaches.

Neutrosophy offers a more nuanced approach by acknowledging inherent ambiguity and incomplete information. It utilizes neutrosophic sets to represent each variable with three components[14, 15]:

T: Truth (degree of membership)

I: Indeterminacy (degree of uncertainty)

F: Falsity (degree of non-membership)

This framework acknowledges that uncertainty is not a system flaw but an intrinsic characteristic of complex phenomena.

To assess the validity of neutrosophic configurations, we use consistency and coverage [16, 17].

Consistency measures the reliability of the set of causal conditions in producing the outcome:

$$Consistency\ (Y_i \leq\ X_i) = \frac{\sum \min\ (X_i, Y_i)}{\sum Y_i} \quad (1)$$

Coverage indicates the extent to which the causal conditions explain the outcome:

$$Coverage\ (Y_i \leq\ X_i) = \frac{\sum min\ (X_i, Y_i)}{\sum X_i} \quad (2)$$

Where:
$X_i$is the membership value of case i in the set of causal conditions.
$Y_i$is the membership value of case i in the result set.

Values greater than 0.8 for both measures generally signify a strong relationship uncertainty [18].

2.2. Neutrosophic Liker scales

Surveys using neutrosophic Likert scales [19, 20] effectively measure the diversity of opinions and their influence on public policies and social discourse, capturing areas of consensus, disagreement, and ambivalence.

Below we present the fundamental definitions and concepts related to neutrosophic sets and single-valued neutrosophic sets.

Definition 1 ([21]). Let U be a discurse universe. N ={ (x , T (x), I (x), F (x)): $x \in U$ } is a neutrosophic set, denoted by a truth membership function, $T : U \rightarrow]0-, 1+[$; an indeterminacy membership function, $I : U \rightarrow]0-, 1+[$; and a falsity membership function, $F : U \rightarrow]0-, 1+[$.

Single-valued neutrosophic sets provide a way to represent and analyze possible elements in the universe of discourse U

Definition 2 ([22]). Let U be a universe of discourse. A single-valued neutrosophic set is defined as N ={ (x , T (x), I (x), F (x)): $x \in U$ }, which is identified by a truth membership function, $TN : U \rightarrow [0, 1]$; indeterminacy membership function, $IN : U \rightarrow [0, 1]$; and falsity membership function, $FN : U \rightarrow [0, 1]$, with $0 \leq TN\ (x) + IN\ (x) + FN\ (x) \leq 3$

Using neutrosophic scales with single-valued neutrosophic sets, responses are classified according to the total of the True, Indeterminate, and False components as follows[23]:

- T+I+F<1: Incomplete
- T+I+F=1: Complete
- T+I+F>1: Contradictory

These values are obtained because, in many cases, the opinions are incomplete or contradictory. This classification is one of the advantages of using neutrosophic methods, as it allows a more nuanced understanding of the different degrees of truth, indeterminacy, and falsity in the responses.

3. Material and Methods

This study investigated the influence of the Pinkullo, a traditional musical instrument, on the cultural fabric of Llata, Peru. The research employed a mixed-methods approach, combining quantitative data analysis using fuzzy-set Qualitative Comparative Analysis (fsQCA)[24] with qualitative sentiment analysis.

Participants:

A purposive sample of 20 inhabitants of Llata was surveyed. Participants included artisans, musicians, and community leaders recognized for their extensive knowledge of the Pinkullo cultural tradition.

Variables:

The study also considered the following variables:

Cultural Participation (CP): Measured by the frequency and quality of participation in events featuring the Pinkullo, such as traditional festivals and ceremonies.

Social Impact (SI): Assessed the role of the Pinkullo in fostering community interaction and strengthening social ties among Llata's inhabitants.

Cultural Preservation (CP): Evaluated the contribution of the Pinkullo to preserving ancestral traditions and knowledge. Note this is the same acronym as Cultural Participation but they represent different factors. Consider changing to avoid confusion.

Perceived Sentiment towards Pinkullo (PSP): Captured the emotions and attitudes evoked by the Pinkullo within the community, categorized as positive, neutral, or negative. This variable was derived from open-ended textual responses using sentiment analysis.

Data Collection:

Data on PI, CP, SI, and Cultural Preservation were collected through a survey employing Likert scales. Textual data for PSP was gathered via an open-ended question in the same survey.

Sentiment Analysis:

Textual responses related to PSP were processed using a sentiment analysis pipeline implemented in Orange Data Mining software, as illustrated in Figure 1 [25].

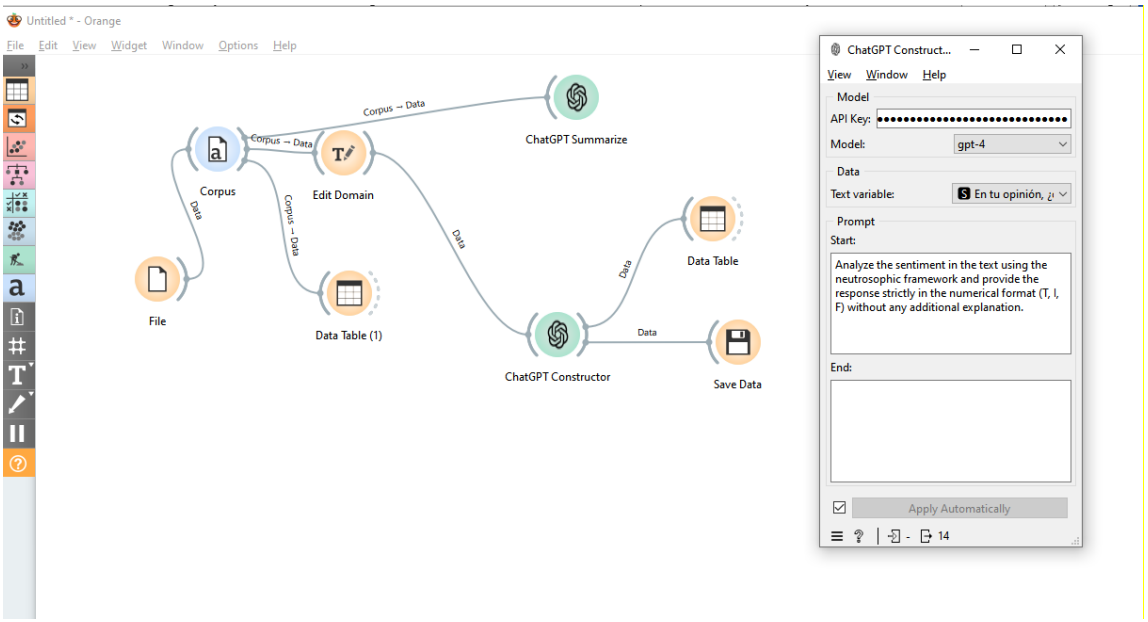

Figure 1: Orange Data Mining workflow for neutrosophic sentiment analysis using the "ChatGPT Constructor" widget.

A pivotal aspect of our methodology involved the innovative application of a Large Language Model, specifically ChatGPT [26, 27], for conducting neutrosophic sentiment analysis of the textual data gathered from the open-ended survey responses. This was achieved by leveraging the "ChatGPT Constructor" add-on within the Orange Data Mining software (see Figure 1), enabling seamless integration of the language model into our analytical workflow[28]. The prompt, "Analyze the sentiment in the text using the neutrosophic framework and provide the response strictly in the numerical format (T, I, F) without any additional explanation," was used to elicit neutrosophic scores—representing truth, indeterminacy, and falsity—directly from ChatGPT. These scores were subsequently transformed into fuzzy sets and served as input for the fuzzy-set Qualitative Comparative Analysis (fsQCA) [29, 30]. This novel integration of ChatGPT-powered, neutrosophic sentiment analysis [31] with fsQCA allowed for a nuanced exploration of the complex interplay between perceived sentiment towards the Pinkullo and the identified causal conditions influencing cultural participation. This approach represents a methodological advancement, demonstrating how AI-driven tools can be effectively combined with established social science methods to analyze qualitative data quantitatively.

Fuzzification:

The neutrosophic sets obtained are transformed into equivalent fuzzy sets, following the procedure described in [32]. This step is essential for the subsequent analysis, allowing to handle the uncertainty and ambiguity inherent in the collected data. $AN = \{x, (TA(x), IA(x), FA(x)) : x \in X\}$an NS. Its equivalent fuzzy membership set is defined as $AF = \{(x, \mu A(x)) : x \in X\}$,where $\mu A(x) = s\left((TA(x), IA(x), FA(x)), (1,0,0)\right)$. Then, using the similarity equation proposed in,

$$\mu A(x) = 1 - \frac{1}{2}[(1 - T_A(x)) + \max\{I_A(x), F_A(x)\}] \quad (3)$$

Since the range of the similarity measure function is the unit interval [0,1], μA (x) ∈ [0,1] for all x ∈ X. Therefore, the membership function of the derived fuzzy set belongs to [0, 1] and hence satisfies the property of a fuzzy set (FS) membership function.

Data Analysis:

Fuzzy-set Qualitative Comparative Analysis (fsQCA) was employed to identify combinations of conditions (CP, SI, Cultural Preservation, PSP) associated with varying degrees of the outcome (PI). Data processing was conducted using the fsQCA 4.0 software for Windows [33].

4. Results.

Cultural Participation (CP) will be selected as the outcome for the fsQCA analysis. This is because it represents the frequency and quality of participation in events where the Pinkullo is used, such as traditional festivals and ceremonies, making it a key indicator of cultural engagement.

The other variables, such as Social Impact (SI), Cultural Preservation (CP), and Perceived Sentiment towards Pinkullo (PSP), will be treated as causal conditions that influence the level of cultural participation.

A survey was conducted with a group of 20 inhabitants of Llata, including artisans, musicians, and community leaders with extensive knowledge of the Pinkullo cultural tradition (see Table 1).

Table 1. Survey data for the 20 participants

Participant	PC	IS	CP	SP
1	(0.9, 0.8, 0.1)	(0.6, 1.0, 0.6)	(0.3, 0.7, 0.3)	(0.8, 0.6, 0.1)
2	(0.6, 0.6, 0.6)	(1.0, 1.0, 1.0)	(0.6, 0.1, 0.6)	(0.6, 0.6, 0.2)
3	(0.8, 0.7, 0.4)	(0.7, 0.9, 0.6)	(0.8, 0.6, 0.6)	(0.8, 0.6, 0.3)
4	(1.0, 1.0, 0.0)	(0.8, 0.8, 0.0)	(1.0, 0.9, 0.3)	(0.7, 1.0, 0.4)
5	(1.0, 0.6, 0.0)	(1.0, 0.6, 1.0)	(1.0, 0.6, 0.1)	(0.9, 0.6, 0.1)
6	(0.9, 0.9, 0.9)	(0.9, 0.9, 0.9)	(0.9, 0.9, 0.2)	(0.9, 0.9, 0.1)
7	(0.1, 0.6, 0.8)	(1.0, 0.0, 0.0)	(0.6, 0.6, 0.6)	(0.8, 0.6, 0.1)
8	(1.0, 0.9, 0.1)	(0.9, 0.9, 0.1)	(0.9, 0.9, 0.1)	(0.9, 0.9, 0.1)
9	(1.0, 1.0, 0.0)	(0.8, 0.8, 0.0)	(1.0, 0.0, 0.0)	(0.9, 0.0, 0.0)
10	(0.7, 1.0, 0.1)	(0.9, 0.4, 0.0)	(0.6, 0.9, 0.1)	(1.0, 0.0, 0.0)
11	(0.4, 0.7, 0.1)	(0.3, 0.9, 0.4)	(0.8, 0.4, 0.6)	(0.4, 0.8, 0.3)
12	(0.6, 1.0, 0.6)	(0.6, 0.6, 0.1)	(0.1, 0.6, 0.7)	(1.0, 0.1, 1.0)
13	(0.7, 0.8, 0.4)	(0.8, 1.0, 0.5)	(0.7, 0.5, 0.4)	(0.7, 0.7, 0.5)
14	(0.5, 0.7, 0.2)	(0.6, 0.8, 0.6)	(0.5, 0.6, 0.5)	(0.6, 0.6, 0.4)
15	(0.9, 0.7, 0.3)	(0.7, 1.0, 0.4)	(0.6, 0.8, 0.4)	(0.8, 0.7, 0.3)
16	(0.8, 0.6, 0.5)	(0.9, 0.7, 0.3)	(0.7, 0.5, 0.6)	(0.8, 0.6, 0.4)
17	(0.6, 0.8, 0.4)	(0.7, 0.8, 0.2)	(0.5, 0.7, 0.5)	(0.7, 0.8, 0.3)
18	(0.7, 0.9, 0.3)	(0.6, 0.9, 0.2)	(0.7, 0.6, 0.6)	(0.8, 0.7, 0.3)
19	(0.6, 0.8, 0.5)	(0.8, 1.0, 0.3)	(0.6, 0.7, 0.4)	(0.7, 0.8, 0.4)
20	(0.8, 0.7, 0.4)	(0.9, 0.8, 0.2)	(0.7, 0.6, 0.5)	(0.8, 0.7, 0.4)

The fuzzification process is developed using the equation (3)

Table 2: Fuzzy values .

Participant	PC	IS	CP	SP
1	0,55	0,3	0,3	0,6
2	0,5	0,5	0,5	0,5
3	0,55	0,4	0,6	0,6
4	0,5	0,5	0,55	0,35
5	0,7	0,5	0,7	0,65
6	0,5	0,5	0,5	0,5
7	0,15	1	0,5	0,6
8	0,55	0,5	0,5	0,5

9	0,5	0,5	1	0,95
10	0,35	0,75	0,35	1
11	0,35	0,2	0,6	0,3
12	0,3	0,5	0,2	0,5
13	0,45	0,4	0,6	0,5
14	0,4	0,4	0,45	0,5
15	0,6	0,35	0,4	0,55
16	0,6	0,6	0,55	0,6
17	0,4	0,45	0,4	0,45
18	0,4	0,35	0,55	0,55
19	0,4	0,4	0,45	0,45
20	0,55	0,55	0,55	0,55

Figure 2 illustrates the relationship between Cultural Participation (CP) and Perceived Sentiment towards Pinkullo (PSP). The data points reveal a moderate positive association, indicating that higher levels of cultural participation tend to align with more favorable perceptions of Pinkullo. The consistency values—0.799 for $X \leq Y$ and 0.962 for $X \geq Y$—suggest a strong alignment between the two variables, with minor deviations observed in specific cases. This relationship underscores the importance of positive sentiment in fostering active cultural engagement.

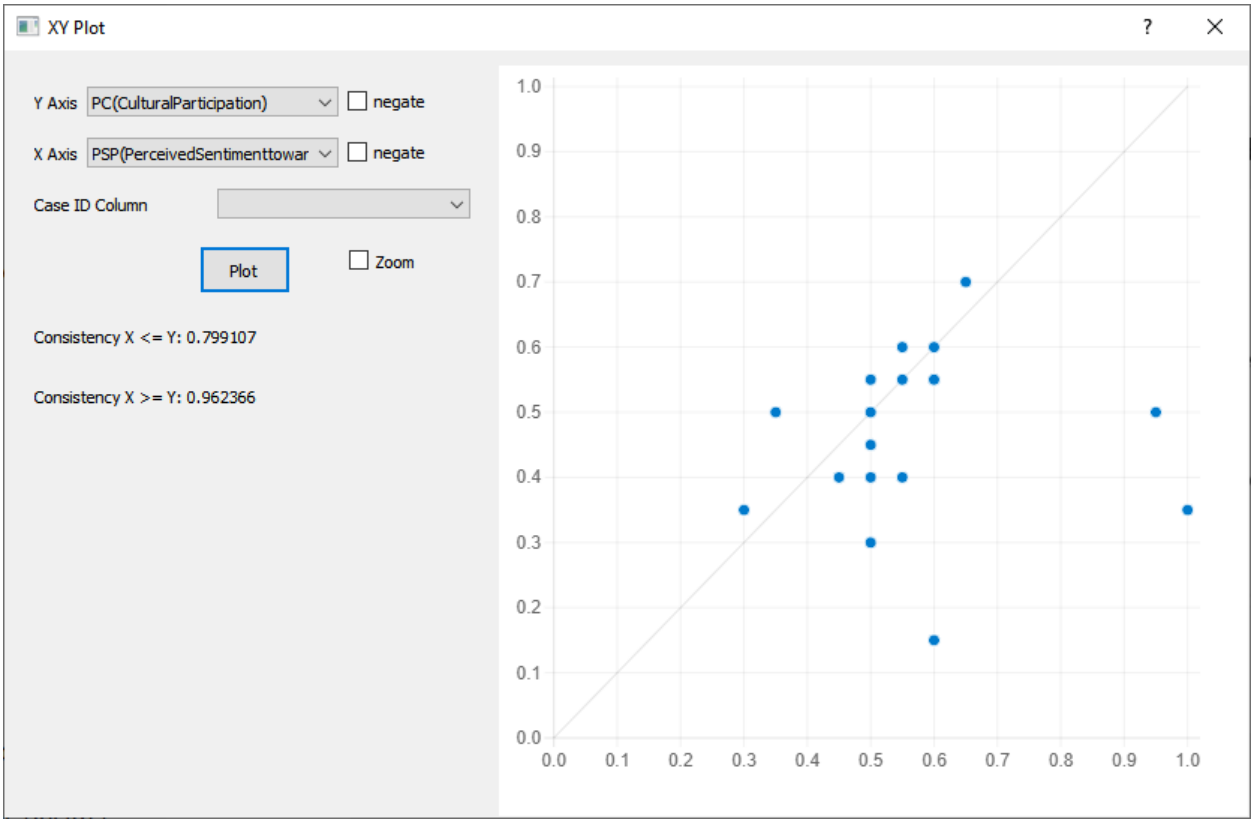

Figure 2. Relationship between Cultural Participation and Perceived Sentiment towards Pinkullo

. Figure 3 presents a subset/superset analysis exploring the relationships between key factors such as Social Impact (IS), Cultural Preservation (CP), and Perceived Sentiment towards Pinkullo (PSP). It provides three metrics for each term or combination of terms: consistency, indicating the degree to which the

condition supports the outcome; coverage, showing how much of the outcome is explained by the condition; and combined, a summary measure reflecting overall relevance. The highest consistency (0.957317) is observed in the combination of IS, CP, and PSP, emphasizing their collective influence on the cultural and social impact associated with Pinkullo.

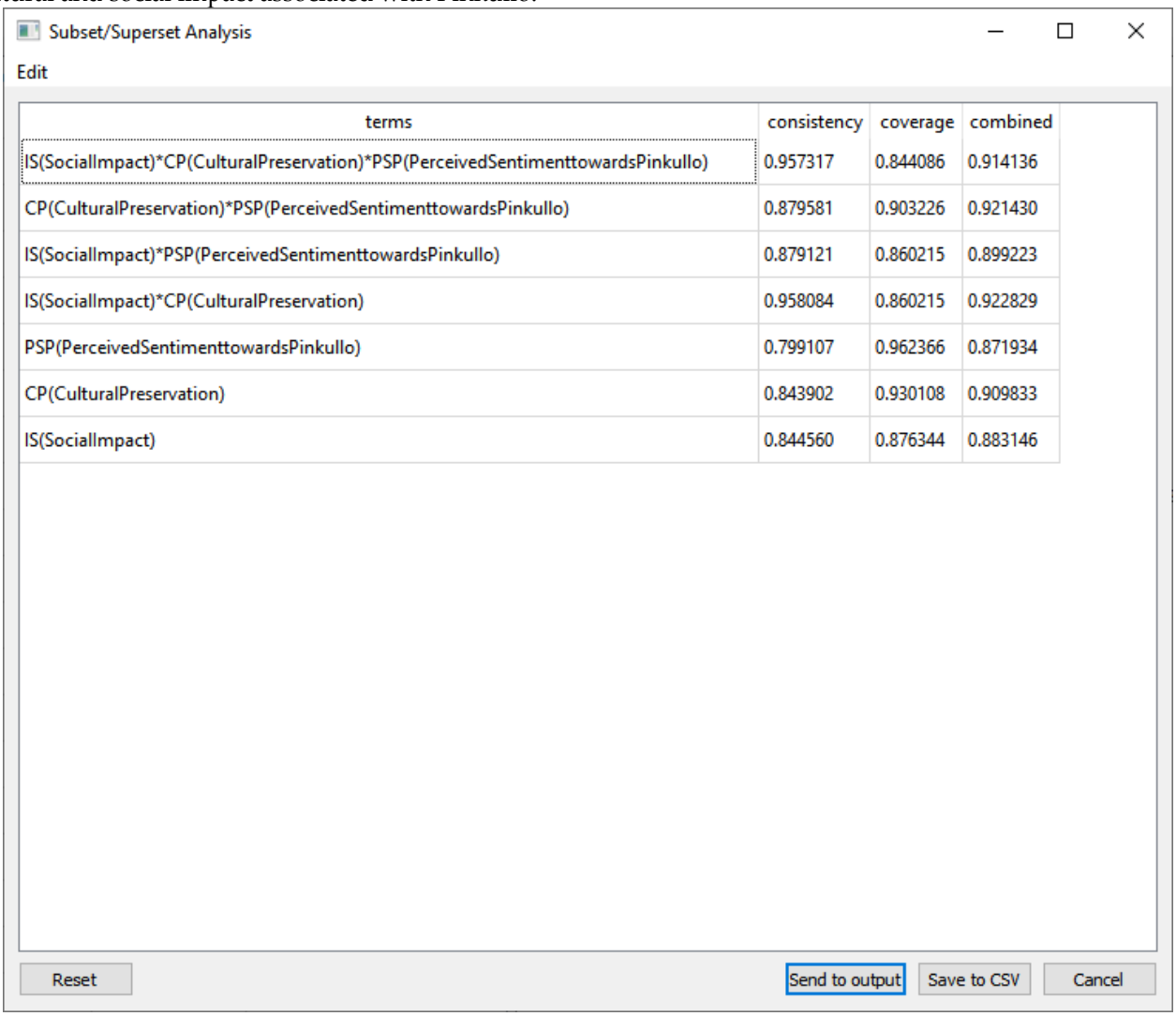

Figure 3. Subset/Superset Analysis of Cultural and Social Factors Related to Pinkullo

The results of the fuzzy-set Qualitative Comparative Analysis (fsQCA), with Cultural Participation (CP) as the outcome, highlight the complex interplay of factors contributing to engagement with the Pinkullo tradition in Llata. The moderate positive association observed between Cultural Participation (CP) and Perceived Sentiment towards Pinkullo (PSP), illustrated in Figure 2, suggests that a positive emotional connection to Pinkullo is a significant driver of participation in related cultural events. This is further supported by the high consistency values (0.799 for $X \leq Y$ and 0.962 for $X \geq Y$), indicating that positive sentiment is often a necessary, though perhaps not always sufficient, condition for active participation. These findings align with broader theories of cultural sociology, which emphasize the role of affect and emotion in motivating collective action and cultural preservation. The minor deviations observed in certain cases, where high participation occurs despite less positive sentiment, could be attributed to other factors like social obligation or a sense of duty towards preserving cultural heritage, suggesting avenues for further qualitative investigation.

Furthermore, the subset/superset analysis presented in Figure 3 provides crucial insights into the configurations of conditions that foster cultural engagement. The particularly high consistency value (0.957317) for the combined influence of Social Impact (SI), Cultural Preservation (CP), and Perceived Sentiment (PSP) underscores the synergistic effect of these factors. This suggests that Pinkullo's influence on cultural participation is maximized when it is perceived as having a positive social impact, contributing to the preservation of cultural heritage, and eliciting positive emotions. This configuration (SICPPSP) implies that initiatives aimed at promoting cultural participation should focus not only on showcasing Pinkullo's artistic value but also on emphasizing its role in strengthening social bonds and preserving cultural identity. This integrative approach, which considers the interplay of social, cultural, and emotional dimensions, offers a more holistic understanding of cultural participation than analyzing each factor in isolation. The relatively lower coverage values across the table suggest that while these factors are important, other unmeasured variables or alternative pathways may also contribute to cultural participation, highlighting the inherent complexity of cultural dynamics within the community.

5. Conclusions

This research investigated the influence of the Pinkullo on cultural participation in Llata, Peru, employing a novel approach that integrates fuzzy-set Qualitative Comparative Analysis (fsQCA) with neutrosophic sentiment analysis. A key innovation of this study was the use of a Large Language Model, specifically ChatGPT with the "ChatGPT Constructor" add-on in Orange Data Mining, to analyze open-ended textual responses and generate neutrosophic sentiment scores. This application of ChatGPT to derive structured neutrosophic data from qualitative text represents a pioneering methodological advancement in the field of cultural studies. To the best of our knowledge, this is the first study to utilize a Large Language Model for this specific purpose, demonstrating the potential of artificial intelligence to enhance the analysis of complex cultural phenomena.

The findings underscore the multifaceted relationship between the Pinkullo and cultural engagement. A positive emotional connection to the instrument quantified through neutrosophic sentiment analysis, was found to be a crucial driver of participation. The synergy between Social Impact (SI), Cultural Preservation (CP), and Perceived Sentiment (PSP) further highlighted the Pinkullo's significance, not just as a musical instrument, but as a cornerstone of social cohesion and cultural identity.

While the study acknowledges limitations, such as the potential influence of unmeasured factors suggested by the relatively low coverage values in the fsQCA, the adopted methodology incorporating neutrosophic logic and ChatGPT proved valuable in handling the ambiguity inherent in cultural data. This research contributes significantly to our understanding of Andean cultural heritage, specifically the enduring significance of the Pinkullo in Llata.

In conclusion, this study provides a novel methodological framework, leveraging the capabilities of a Large Language Model (ChatGPT) for neutrosophic sentiment analysis, to investigate the complex interplay between a traditional musical instrument and its community. The successful implementation of this approach paves the way for future research to explore other complex cultural phenomena using similar AI-enhanced methodologies, potentially revolutionizing the integration of quantitative analysis in traditionally qualitative disciplines.

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Division of 3-Refined Neutrosophic Numbers by using the Identification Method and the 3-Refined AH-Isometry

Yaser Ahmad Alhasan[1,*], Mohamed Elghazali Ali Mohieldin Mohamed[2], Qusay Ahassan[3] and Raja Abdullah Abdulfatah[4]

[1]Deanship of the Preparatory Year, Prince Sattam bin Abdulaziz University, Alkharj, Saudi Arabia.; y.alhasan@psau.edu.sa
[2]Deanship the Preparatory Year, Prince Sattam bin Abdulaziz University, Alkharj Saudi Arabia.;mo.mohammed@psau.edu.sa
[3]Faculty of Informatics Engineering, Cordoba Private University, Syria; qusayalhassan77@gmail.com
[4]Deanship the Preparatory Year, Prince Sattam bin Abdulaziz University, Alkharj, Saudi Arabia.; r.abdulfatah@psau.edu.sa

[*]Corresponding author: y.alhasan@psau.edu.sa

Abstract: This paper aims to study the division of 3-refined neutrosophic numbers, where it was discussed through two methods, the first is the identification method, and the second is the 3-refined AH-isometry. In addition to presenting special cases related to division of 3-refined neutrosophic numbers. The most important idea in this paper is that we were able to move from the formula we obtained in the 3-refined AH-isometry to the formula we found in the identification method and vice versa, hence this proves the validity of both formulas.

Keywords: 3-refined AH-Isometry; 3-refined, neutrosophic numbers; division.

1. Introduction and Preliminaries

Smarandache made refined neutrosophic numbers available in the following form: $(a, b_1I_1, b_2I_2, \dots, b_nI_n)$ where $a, b_1, b_2, \dots, b_n \in R \text{ or } C$ [1].

Agboola introduced the concept of refined neutrosophic algebraic structures [2]. In addition, the refined neutrosophic rings I was studied in paper [3], where it assumed that I splits into two indeterminacies I_1 [contradiction (true (T) and false (F))] and I_2 [ignorance (true (T) or false (F))]. Abobala presented the papers on some special substructures of refined neutrosophic rings and a study of ah-substructures in n-refined neutrosophic vector spaces [6-7].

Alhasan.Y and Abdulfatah. R also presented a study on the division of refined neutrosophic number. [8].

Finally, there are papers presented in n-Valued Refined Neutrosophic Logic and Its Applications in Physics, Neutrosophic Rings I [4-5].

2. Main Discussion

2.1. Division of 3-refined neutrosophic numbers by using the Identification Method

Let m_1, m_2 are two 3-refined neutrosophic numbers, where:

$$m_1 = a_1 + b_1I_1 + c_1I_2 + d_1I_3 \ , \qquad m_2 = a_2 + b_2I_1 + c_2I_2 + d_2I_3$$

To find $\frac{m_1}{m_2}$, we write:

$$\frac{a_1 + b_1I_1 + c_1I_2 + d_1I_3}{a_2 + b_2I_1 + c_2I_2 + d_2I_3} \equiv x + yI_1 + zI_2 + \mathrm{w}I_3$$

where x , y and z are real unknowns.

$$a_1 + b_1I_1 + c_1I_2 + d_1I_3 \equiv (a_2 + b_2I_1 + c_2I_2 + d_2I_3)(x + yI_1 + zI_2 + \mathrm{w}I_3)$$

$$\begin{aligned} a_1 + b_1I_1 + c_1I_2 + d_1I_3 & \\ \equiv\ & a_2x + a_2yI_1 + a_2zI_2 + a_2\mathrm{w}I_3 + b_2xI_1 + b_2yI_1 + b_2zI_1 + b_2\mathrm{w}I_1 + c_2xI_2 + c_2yI_1 \\ & + c_2zI_2 + c_2\mathrm{w}I_2 + d_2xI_3 + d_2yI_1 + d_2zI_2 + d_2\mathrm{w}I_3 \end{aligned}$$

$$\begin{aligned} a_1 + b_1I_1 + c_1I_2 + d_1I_3 & \\ \equiv\ & a_2x + [b_2x + (a_2 + b_2 + c_2 + d_2)y + b_2z + b_2\mathrm{w}]I_1 \\ & + [c_2x + (a_2 + c_2 + d_2)z + c_2\mathrm{w}]I_2 + +[d_2x + (a_2 + d_2)\mathrm{w}]I_3 \end{aligned}$$

Whereas: $I_1I_2 = I_2I_1 = I_1$

By identifying:

$$a_2x = a_1$$
$$b_2x + (a_2 + b_2 + c_2 + d_2)y + b_2z + b_2\mathrm{w} = b_1 \quad (1)$$
$$c_2x + (a_2 + c_2 + d_2)z + c_2\mathrm{w} = c_1 \quad (2)$$
$$d_2x + (a_2 + d_2)\mathrm{w} = d_1 \quad (3)$$

We obtain unique one solution only, provided that:

$$\begin{vmatrix} a_2 & 0 & 0 & 0 \\ b_2 & (a_2 + b_2 + c_2 + d_2) & b_2 & b_2 \\ c_2 & 0 & (a_2 + c_2 + d_2) & c_2 \\ d_2 & 0 & 0 & (a_2 + d_2) \end{vmatrix} \neq 0$$

$$\Rightarrow \quad a_2(a_2 + d_2)(a_2 + c_2 + d_2)(a_2 + b_2 + c_2 + d_2) \neq 0$$

where:

$$a_2 \neq 0 \ , a_2 \neq d_2 \ , a_2 \neq -c_2 - d_2 \ , a_2 \neq -b_2 - c_2 - d_2$$

then:

$$x = \frac{a_1}{a_2}$$

by substituting in (3):

$$\frac{a_1d_2}{a_2} + (a_2 + d_2)\mathrm{w} = d_1$$

$$w = \frac{a_2 d_1 - a_1 d_2}{a_2(a_2 + d_2)}$$

by substituting in (2):

$$\frac{a_1 c_2}{a_2} + (a_2 + c_2 + d_2)z + \frac{a_2 d_1 c_2 - a_1 d_2 c_2}{a_2(a_2 + d_2)} = c_1$$

$$\frac{a_1 c_2 a_2 + a_1 c_2 d_2}{a_2(a_2 + d_2)} + (a_2 + c_2 + d_2)z + \frac{a_2 d_1 c_2 - a_1 d_2 c_2}{a_2(a_2 + d_2)} = c_1$$

$$(a_2 + c_2 + d_2)z = \frac{c_1 a_2(a_2 + d_2) - a_1 c_2 a_2 - a_2 d_1 c_2}{a_2(a_2 + d_2)}$$

$$(a_2 + c_2 + d_2)z = \frac{c_1(a_2 + d_2) - c_2(a_1 + d_1)}{(a_2 + d_2)}$$

$$z = \frac{c_1(a_2 + d_2) - c_2(a_1 + d_1)}{(a_2 + d_2)(a_2 + c_2 + d_2)}$$

by substituting in (1):

$$\frac{a_1 b_2}{a_2} + (a_2 + b_2 + c_2 + d_2)y + b_2\left(\frac{c_1(a_2 + d_2) - c_2(a_1 + d_1)}{(a_2 + d_2)(a_2 + c_2 + d_2)}\right) + b_2\left(\frac{a_2 d_1 - a_1 d_2}{a_2(a_2 + d_2)}\right) = b_1$$

$$\frac{a_1 b_2}{a_2} + (a_2 + b_2 + c_2 + d_2)y + \frac{c_1 b_2(a_2 + d_2) - c_2 b_2(a_1 + d_1)}{(a_2 + d_2)(a_2 + c_2 + d_2)} + \frac{a_2 b_2 d_1 - a_1 b_2 d_2}{a_2(a_2 + d_2)} = b_1$$

$$(a_2 + b_2 + c_2 + d_2)y + \frac{b_2\big(a_1(a_2 + d_2)(a_2 + c_2 + d_2) + c_1 a_2(a_2 + d_2) - c_2 a_2(a_1 + d_1) + (a_2 d_1 - a_1 d_2)(a_2 + c_2 + d_2)\big)}{a_2(a_2 + d_2)(a_2 + c_2 + d_2)} = b_1$$

$$(a_2 + b_2 + c_2 + d_2)y + b_2\left(\frac{(a_2 + d_2)(a_2 a_1 + c_2 a_1 + d_2 a_1)}{a_2(a_2 + d_2)(a_2 + c_2 + d_2)}\right) + b_2\left(\frac{c_1 a_2(a_2 + d_2) - c_2 a_1 a_2 - c_2 d_1 a_2}{a_2(a_2 + d_2)(a_2 + c_2 + d_2)}\right) + b_2\left(\frac{{a_2}^2 d_1 - a_2 a_1 d_2 + c_2 a_2 d_1 - a_1 d_2 c_2 + a_2 d_1 d_2 - a_1 {d_2}^2}{a_2(a_2 + d_2)(a_2 + c_2 + d_2)}\right) = b_1$$

$$(a_2+b_2+c_2+d_2)y+b_2\left(\frac{(a_2+d_2)(a_2a_1+c_2a_1+d_2a_1)}{a_2(a_2+d_2)(a_2+c_2+d_2)}\right)$$
$$+b_2\left(\frac{c_1a_2(a_2+d_2)-c_2a_1a_2+{a_2}^2d_1-a_2a_1d_2-a_1d_2c_2+a_2d_1d_2-a_1{d_2}^2}{a_2(a_2+d_2)(a_2+c_2+d_2)}\right)$$
$$=b_1$$

$$(a_2+b_2+c_2+d_2)y+b_2\left(\frac{(a_2+d_2)(a_2a_1+c_2a_1+d_2a_1)}{a_2(a_2+d_2)(a_2+c_2+d_2)}\right)$$
$$+b_2\left(\frac{c_1a_2(a_2+d_2)-c_2a_1(a_2+d_2)-a_1d_2(a_2+d_2)+a_2d_1(a_2+d_2)}{a_2(a_2+d_2)(a_2+c_2+d_2)}\right)=b_1$$

$$(a_2+b_2+c_2+d_2)y+b_2\left(\frac{a_2a_1+c_2a_1+d_2a_1}{a_2(a_2+c_2+d_2)}\right)+b_2\left(\frac{c_1a_2-c_2a_1-a_1d_2+a_2d_1}{a_2(a_2+c_2+d_2)}\right)=b_1$$

$$(a_2+b_2+c_2+d_2)y=b_1-b_2\left(\frac{a_2a_1+c_2a_1+d_2a_1}{a_2(a_2+c_2+d_2)}\right)-b_2\left(\frac{c_1a_2-c_2a_1-a_1d_2+a_2d_1}{a_2(a_2+c_2+d_2)}\right)$$

$$(a_2+b_2+c_2+d_2)y=\frac{b_1a_2(a_2+c_2+d_2)}{a_2(a_2+c_2+d_2)}-b_2\left(\frac{a_2a_1+c_1a_2+a_2d_1}{a_2(a_2+c_2+d_2)}\right)$$

$$y=\frac{b_1(a_2+c_2+d_2)-b_2(a_1+c_1+d_1)}{(a_2+c_2+d_2)(a_2+b_2+c_2+d_2)}$$

hence:

$$\frac{a_1+b_1I_1+c_1I_2+d_1I_3}{a_2+b_2I_1+c_2I_2+d_2I_3}$$

$$\equiv\frac{a_1}{a_2}+\left[\frac{b_1(a_2+c_2+d_2)-b_2(a_1+c_1+d_1)}{(a_2+c_2+d_2)(a_2+b_2+c_2+d_2)}\right]I_1+\left[\frac{c_1(a_2+d_2)-c_2(a_1+d_1)}{(a_2+d_2)(a_2+c_2+d_2)}\right]I_2+\left[\frac{a_2d_1-a_1d_2}{a_2(a_2+d_2)}\right]I_3$$

Example 1

$$\frac{2+I_1+I_2+I_3}{2+I_1+2I_2+3I_3}=1+\frac{3}{56}I_1-\frac{1}{35}I_2-\frac{2}{5}I_3$$

Let's check the answer:

$$(2+I_1+I_2+I_3)\left(1+\tfrac{3}{56}I_1-\tfrac{1}{35}I_2-\tfrac{2}{5}I_3\right)=2+\left[\tfrac{3}{56}(8)+1-\tfrac{1}{35}-\tfrac{2}{5}\right]I_1+\left[-\tfrac{1}{35}(7)+2-\tfrac{4}{5}\right]I_2+$$

$$\left[-\tfrac{2}{5}(5)+3\right]I_3=2+I_1+2I_2+3I_3\quad\text{(True)}$$

As consequences, we have:

$$1)\ \frac{a_1+b_1I_1+c_1I_2+d_1I_3}{k(a_1+b_1I_1+c_1I_2+d_1I_3)}=\frac{1}{k}\ ;\ \ a_1\neq 0\ ,a_1\neq d_1\ ,a_1\neq -c_1-d_1\ ,a_1\neq -b_1-c_1-d_1$$

2) $$\frac{I_1}{a_2+b_2I_1+c_2I_2+d_2I_3}=\frac{b_1}{a_2+b_2+c_2+d_2}I_1$$

Example 2

$$\frac{I_1}{4-2I_1+5I_2+4I_3}=\frac{1}{11}I_1$$

Let's check the answer:

$$(4-2I_1+5I_2+4I_3)\left(\tfrac{1}{11}I_1\right)=I_1 \quad \text{(True)}$$

3) $$\frac{I_2}{a_2+b_2I_1+c_2I_2+d_2I_3}=\left[\frac{-b_2c_1}{(a_2+c_2+d_2)(a_2+b_2+c_2+d_2)}\right]I_1+\left[\frac{c_1}{a_2+c_2+d_2}\right]I_2$$

Example 3

$$\frac{I_2}{1-I_1+3I_2+7I_3}=\frac{1}{110}I_1+\frac{1}{11}I_2$$

Let's check the answer:

$$(1-I_1+3I_2+7I_3)\left(\tfrac{1}{110}I_1+\tfrac{1}{11}I_2\right)=I_2 \quad \text{(True)}$$

4) $$\frac{I_3}{a_2+b_2I_1+c_2I_2+d_2I_3}$$

$$=\left[\frac{-b_2d_1}{(a_2+c_2+d_2)(a_2+b_2+c_2+d_2)}\right]I_1+\left[\frac{-c_2d_1}{(a_2+d_2)(a_2+c_2+d_2)}\right]I_2+\left[\frac{d_1}{a_2+d_2}\right]I_3$$

Example 4

$$\frac{I_3}{2-3I_1+4I_2+5I_3}=\frac{3}{88}I_1-\frac{4}{77}I_2+\frac{1}{7}I_3$$

Let's check the answer:

$$(2-3I_1+4I_2+5I_3)\left(\tfrac{3}{88}I_1-\tfrac{4}{77}I_2+\tfrac{1}{7}I_3\right)=I_3 \quad \text{(True)}$$

5) $$\frac{a_1+b_1I_1+c_1I_2+d_1I_3}{k}=\frac{a_1}{k}+\frac{b_1}{k}I_1+\frac{c_1}{k}I_2+\frac{d_1}{k}I_3\,;k\neq 0$$

6) $$\frac{k}{a_2+b_2I_1+c_2I_2+d_2I_3}$$

$$\equiv\frac{k}{a_2}+\left[\frac{-kb_2}{(a_2+c_2+d_2)(a_2+b_2+c_2+d_2)}\right]I_1+\left[\frac{-kc_2}{(a_2+d_2)(a_2+c_2+d_2)}\right]I_2+\left[\frac{-kd_2}{a_2(a_2+d_2)}\right]I_3$$

In particular:

* $$\frac{a_1+b_1I_1+c_1I_2+d_1I_3}{I_1+I_2+I_3}=undefined$$

* $$\frac{a_1+b_1I_1+c_1I_2+d_1I_3}{I_1+I_2}=undefined$$

* $$\frac{a_1+b_1I_1+c_1I_2+d_1I_3}{I_2+I_3}=undefined$$

$* \dfrac{a_1 + b_1I_1 + c_1I_2 + d_1I_3}{I_1 + I_3} = undefined$

$* \dfrac{a_1 + b_1I_1 + c_1I_2 + d_1I_3}{I_1} = undefined$

$* \dfrac{a_1 + b_1I_1 + c_1I_2 + d_1I_3}{I_2} = undefined$

$* \dfrac{a_1 + b_1I_1 + c_1I_2 + d_1I_3}{I_3} = undefined$

$* \dfrac{a_1 + b_1I_1 + c_1I_2 + d_1I_3}{k(I_1 + I_2 + I_3)} = undefined\ ;\ k \neq 0$

2.2. Division of 3-refined neutrosophic numbers by using the 3-Refined AH-Isometry

Let $f: R[I_1, I_2, I_3] \times R[I_1, I_2, I_3] \longrightarrow R[I_1, I_2, I_3]$ be a literal 3-refined neutrosophic function of three variables, then:

$$f(a_1 + b_1I_1 + c_1I_2 + d_1I_3, a_2 + b_2I_1 + c_2I_2 + d_2I_3)$$

$$\begin{aligned} &= f(a_1, a_2) + [f(a_1 + b_1 + c_1 + d_1, a_2 + b_2 + c_2 + d_2) - f(a_2 + c_2 + d_2, a_1 + c_1 + d_1)]I_1 \\ &\quad + [f(a_2 + c_2 + d_2, a_1 + c_1 + d_1) - f(a_2 + d_2, a_1 + d_1)]I_2 \\ &\quad + [f(a_2 + d_2, a_1 + d_1) - f(a_1, a_2)]I_3 \end{aligned}$$

$$\equiv \frac{a_1}{a_2} + \left[\frac{a_1 + b_1 + c_1 + d_1}{a_2 + b_2 + c_2 + d_2} - \frac{a_1 + c_1 + d_1}{a_2 + c_2 + d_2}\right]I_1 + \left[\frac{a_1 + c_1 + d_1}{a_2 + c_2 + d_2} - \frac{a_1 + d_1}{a_2 + d_2}\right]I_2 + \left[\frac{a_1 + d_1}{a_2 + d_2} - \frac{a_1}{a_2}\right]I_3$$

hence:

$$\frac{a_1 + b_1I_1 + c_1I_2 + d_1I_3}{a_2 + b_2I_1 + c_2I_2 + d_2I_3}$$

$$\equiv \frac{a_1}{a_2} + \left[\frac{a_1 + b_1 + c_1 + d_1}{a_2 + b_2 + c_2 + d_2} - \frac{a_1 + c_1 + d_1}{a_2 + c_2 + d_2}\right]I_1 + \left[\frac{a_1 + c_1 + d_1}{a_2 + c_2 + d_2} - \frac{a_1 + d_1}{a_2 + d_2}\right]I_2 + \left[\frac{a_1 + d_1}{a_2 + d_2} - \frac{a_1}{a_2}\right]I_3$$

Whereas: $a_2 \neq 0\ , a_2 \neq d_2\ , a_2 \neq -c_2 - d_2\ , a_2 \neq -b_2 - c_2 - d_2$

Example 6

$$\frac{2 + I_1 + I_2 + I_3}{2 + I_1 + 2I_2 + 3I_3} = 1 + \left[\frac{5}{8} - \frac{4}{7}\right]I_1 + \left[\frac{4}{7} - \frac{3}{5}\right]I_2 + \left[\frac{3}{5} - 1\right]I_3 = 1 + \frac{3}{56}I_1 - \frac{1}{35}I_2 - \frac{2}{5}I_3$$

(The same result that we got using the Identification Method in example 1)

Example 7

$$\frac{-5 + I_1 + 4I_2 - 2I_3}{25 + 2I_1 + I_2 + 4I_3} = \frac{-1}{5} + \left[\frac{-1}{16} + \frac{1}{10}\right]I_1 + \left[\frac{-1}{10} + \frac{7}{29}\right]I_2 + \left[\frac{-7}{29} + \frac{1}{5}\right]I_3 = \frac{-1}{5} + \frac{3}{80}I_1 + \frac{41}{290}I_2 - \frac{6}{145}I_3$$

Let's check the answer:

$$(25 + 2I_1 + I_2 + 4I_3)\left(\tfrac{-1}{5} + \tfrac{3}{80}I_1 + \tfrac{41}{290}I_2 - \tfrac{6}{145}I_3\right) = -5 + I_1 + 4I_2 - 2I_3 \quad \text{(True)}$$

3. The relationship between the identification method and the 3-refined ah-isometry

3.1 Moving from identification method to 3-refined AH-Isometry

We have:

$$\frac{a_1 + b_1 I_1 + c_1 I_2 + d_1 I_3}{a_2 + b_2 I_1 + c_2 I_2 + d_2 I_3}$$

$$\equiv \frac{a_1}{a_2} + \left[\frac{b_1(a_2 + c_2 + d_2) - b_2(a_1 + c_1 + d_1)}{(a_2 + c_2 + d_2)(a_2 + b_2 + c_2 + d_2)}\right] I_1 + \left[\frac{c_1(a_2 + d_2) - c_2(a_1 + d_1)}{(a_2 + d_2)(a_2 + c_2 + d_2)}\right] I_2 + \left[\frac{a_2 d_1 - a_1 d_2}{a_2(a_2 + d_2)}\right] I_3$$

$$\equiv \frac{a_1}{a_2}$$
$$+ \left[\frac{(a_2 a_1 + a_2 b_1 + a_2 c_1 + a_2 d_1) + (c_2 a_1 + c_2 b_1 + c_2 c_1 + c_2 d_1) + (d_2 a_1 + d_2 b_1 + d_2 c_1 + d_2 d_1)}{(a_2 + c_2 + d_2)(a_2 + b_2 + c_2 + d_2)}\right.$$
$$\left. + \frac{-a_2 a_1 - a_2 c_1 - a_2 d_1 - c_2 a_1 - c_2 c_1 - c_2 d_1 - d_2 a_1 - d_2 c_1 - d_2 d_1 - b_2 a_1 - b_2 c_1 - b_2 d_1}{(a_2 + c_2 + d_2)(a_2 + b_2 + c_2 + d_2)}\right] I_1$$
$$+ \left[\frac{a_2 a_1 + c_1 a_2 + a_2 d_1 + a_1 d_2 + c_1 d_2 + d_1 d_2 - a_2 a_1 - c_2 a_1 - a_1 d_2 - a_2 d_1 - c_2 d_1 - d_1 d_2}{(a_2 + d_2)(a_2 + c_2 + d_2)}\right] I_2$$
$$+ \left[\frac{a_2 a_1 + a_2 d_1 - a_2 a_1 - a_1 d_2}{a_2(a_2 + d_2)}\right] I_3$$

$$\equiv \frac{a_1}{a_2} + \left[\frac{a_2(a_1 + b_1 + c_1 + d_1) + c_2(a_1 + b_1 + c_1 + d_1) + d_2(a_1 + b_1 + c_1 + d_1)}{(a_2 + c_2 + d_2)(a_2 + b_2 + c_2 + d_2)}\right.$$
$$\left. + \frac{-a_1(a_2 + b_2 + c_2 + d_2) - c_1(a_2 + b_2 + c_2 + d_2) - d_1(a_2 + b_2 + c_2 + d_2)}{(a_2 + c_2 + d_2)(a_2 + b_2 + c_2 + d_2)}\right] I_1$$
$$+ \left[\frac{a_2(a_1 + c_1 + d_1) + d_2(a_1 + c_1 + d_1) - a_1(a_2 + c_2 + d_2) - d_1(a_2 + c_2 + d_1)}{(a_2 + d_2)(a_2 + c_2 + d_2)}\right] I_2$$
$$+ \left[\frac{a_2(a_1 + d_1) - a_1(a_2 + d_2)}{a_2(a_2 + d_2)}\right] I_3$$

$$\equiv \frac{a_1}{a_2} + \left[\frac{(a_2 + c_2 + d_2)(a_1 + b_1 + c_1 + d_1)}{(a_2 + c_2 + d_2)(a_2 + b_2 + c_2 + d_2)} - \frac{(a_1 + c_1 + d_1)(a_2 + b_2 + c_2 + d_2)}{(a_2 + c_2 + d_2)(a_2 + b_2 + c_2 + d_2)}\right] I_1$$
$$+ \left[\frac{(a_2 + d_2)(a_1 + c_1 + d_1) - (a_1 + d_1)(a_2 + c_2 + d_1)}{(a_2 + d_2)(a_2 + c_2 + d_2)}\right] I_2 + \left[\frac{a_1 + d_1}{a_2 + d_2} - \frac{a_1}{a_2}\right] I_3$$

Hence, we find the form of the method of 3-Refined AH-Isometry:

$$\equiv \frac{a_1}{a_2} + \left[\frac{a_1 + b_1 + c_1 + d_1}{a_2 + b_2 + c_2 + d_2} - \frac{a_1 + c_1 + d_1}{a_2 + c_2 + d_2}\right] I_1 + \left[\frac{a_1 + c_1 + d_1}{a_2 + c_2 + d_2} - \frac{a_1 + d_1}{a_2 + d_2}\right] I_2 + \left[\frac{a_1 + d_1}{a_2 + d_2} - \frac{a_1}{a_2}\right] I_3$$

3.2 Moving from 3-refined AH-Isometry to identification method

We have:

$$\frac{a_1 + b_1 I_1 + c_1 I_2 + d_1 I_3}{a_2 + b_2 I_1 + c_2 I_2 + d_2 I_3}$$

$$\equiv \frac{a_1}{a_2} + \left[\frac{a_1 + b_1 + c_1 + d_1}{a_2 + b_2 + c_2 + d_2} - \frac{a_1 + c_1 + d_1}{a_2 + c_2 + d_2}\right] I_1 + \left[\frac{a_1 + c_1 + d_1}{a_2 + c_2 + d_2} - \frac{a_1 + d_1}{a_2 + d_2}\right] I_2 + \left[\frac{a_1 + d_1}{a_2 + d_2} - \frac{a_1}{a_2}\right] I_3$$

then:

$$\equiv \frac{a_1}{a_2}$$
$$+ \left[\frac{a_2(a_1 + c_1 + d_1) + c_2(a_1 + c_1 + d_1) + d_2(a_1 + c_1 + d_1) + b_1(a_2 + c_2 + d_2)}{(a_2 + c_2 + d_2)(a_2 + b_2 + c_2 + d_2)}\right.$$
$$\left. - \frac{a_2(a_1 + c_1 + d_1) - b_2(a_1 + c_1 + d_1) - c_2(a_1 + c_1 + d_1) - d_2(a_1 + c_1 + d_1)}{(a_2 + c_2 + d_2)(a_2 + b_2 + c_2 + d_2)}\right] I_1$$
$$+ \left[\frac{a_2(a_1 + d_1) + d_2(a_1 + d_1) + c_1(a_2 + d_2) - a_2(a_1 + d_1) - c_2(a_1 + d_1) - d_2(a_1 + d_1)}{(a_2 + d_2)(a_2 + c_2 + d_2)}\right] I_2$$
$$+ \left[\frac{a_1 a_2 + d_1 a_2 - a_1 a_2 - a_1 d_2}{a_2(a_2 + d_2)}\right] I_3$$

Hence, we find the form of the Identification method:

$$\frac{a_1 + b_1 I_1 + c_1 I_2 + d_1 I_3}{a_2 + b_2 I_1 + c_2 I_2 + d_2 I_3}$$

$$\equiv \frac{a_1}{a_2} + \left[\frac{b_1(a_2 + c_2 + d_2) - b_2(a_1 + c_1 + d_1)}{(a_2 + c_2 + d_2)(a_2 + b_2 + c_2 + d_2)}\right] I_1 + \left[\frac{c_1(a_2 + d_2) - c_2(a_1 + d_1)}{(a_2 + d_2)(a_2 + c_2 + d_2)}\right] I_2 + \left[\frac{a_2 d_1 - a_1 d_2}{a_2(a_2 + d_2)}\right] I_3$$

4. Conclusions

This paper deals with the division of 3-refined neutrosophic numbers using two methods, the identification method and the 3-refined AH-isometry, and we obtained the same results when using either of the two methods by solving a set of examples in this paper. In addition to introduce the relationship between the identification method and the 3-refined ah-isometry.

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A Study on Blended Teaching Model Evaluation for English Major Courses in Higher Education: An Uncertainty-Based Approach

Chunmei Xue*
School of Foreign Languages, Sias University, Xinzheng, 451100, Henan, China
*Corresponding author, E-mail: 2004megan@sohu.com

Abstract: As businesses adopt remote work, online learning, and the digital workplace, the idea of blended learning keeps changing. Traditional trainer-led lectures and coaching sessions are still provided to learners, but they are combined with interactive, self-guided experiences that give employees practical training and allow them to operate as a team. The creation of a decision support system that facilitates the evaluation of blended learning of English courses in higher education for aiding students. In addition to qualitative metrics that show evaluation of blended learning, several quantitative metrics also influence blended learning performance. The purpose of this study is to evaluate blended learning performance using both qualitative and quantitative metrics. This study is performed to overcome the research gapes which ranking the alternatives without overcoming uncertainty in the decision making problem. So, the plithogenic sets and a multi-criteria decision-making (MCDM) approach are used within this context. Expert significance levels are determined using plithogenic sets. The performance rating of blended learning is determined using the plithogenic-simple weighted sum product (WISP) approach. The plithogenic-WISP hybrid model is thus created in this study, and an algorithm for this innovative decision-analytic model is provided. To illustrate the usefulness of the suggested hybrid paradigm, a case study concentrating on blending learning evaluation is created. The findings of the study support the plithogenic-WISP hybrid method's applicability and validate its robustness.

Keywords: Multi-criteria Decision-Making Approach; Decision Making; Uncertainty Model; Blended Teaching; English Major Courses; Higher Education.

1. Introduction

A training method called blended learning, sometimes referred to as flipped classroom, hybrid learning, or mixed learning, blends traditional in-person instruction and face-to-face learning with digital, online learning resources. It offers a flexible and all-encompassing learning experience by combining in-person encounters, like workshops or classroom sessions, with self-paced eLearning modules, virtual training, or digital resources. This hybrid technique is very effective at accommodating varying learning styles and improving knowledge retention since it enables students to interact with the content both alone and collectively[1], [2].

There are different benefits in blended learning such as:

I. Integrates theory and practice: Through online modules, blended learning enables staff members to study theoretical ideas before putting their understanding to use in real-world, hands-on situations like workshops or simulations. This combination strengthens comprehension and enhances practical application.
II. Reduces expenses and boosts return on investment: By eliminating the need for physical space, travel, and printed materials, blended learning reduces training expenses. The return on investment (ROI) for corporate training programs can be increased by using digital modules, which can be updated or reused for less money[3], [4].
III. Facilitates corporate training feedback: Real-time feedback and evaluations are made possible by digital platforms in blended learning, which helps trainers monitor progress, spot knowledge gaps, and modify the training as necessary to increase efficacy.
IV. Employees can study at their own pace thanks to blended learning, which offers flexibility by letting them finish online assignments at their own speed and arranging training around their personal and professional obligations. This guarantees that students won't feel hurried and can fully grasp the training content.
V. Cooperation and peer learning: By incorporating interactive components like discussion boards, group projects, or live sessions, blended learning promotes peer-to-peer learning and cooperation, which improves employee knowledge sharing and problem-solving skills[5], [6].
VI. Improved knowledge retention is a result of combining interactive, hands-on activities with self-paced learning. Digital content may be reviewed at any time, and experiential learning helps learners retain information better over the long term[7], [8].

Evaluation of the blended learning is a MCDM problem. The goal of research is to create a decision support system that can evaluate the blended learning in English courses in higher education. At the same time, the algorithm makes it possible to implement decision-making procedures that are dependent on blended learning performance[9], [10]. The MCDM approach, which is based on decision models, is the best method for creating a decision support system. To guarantee the precision and resilience of optimal assessments, the MCDM technique is applied in the literature and is advised. With this method, options are ranked based on the relative importance of the decision-making criteria[11], [12].

In this study, a group decision-making technique is combined with the MCDM approach to create a decision support system, considering the experts' assessment of criteria using plithogenic sets. Furthermore, a decision model that considers both qualitative and quantitative criteria simultaneously is created in the criterion determination process, as opposed to building a framework based just on qualitative or quantitative criteria[13], [14].

The suggested framework for decision support is set up as a hybrid approach. Expert assessments are carried out using plithogenic sets. Using plithogenic sets in conjunction with the basic weighted sum product (WISP) method, blending learning performance evaluation and ranking are carried out. The terms plithogenic set and WISP are used to describe this new hybrid paradigm[15], [16]. WISP plays a significant part in the performance ranking of blending learning English courses in higher education by helping to build a solution based on benefit scores. The plithogenic aggregation operator is also used in this hybrid approach to apply aggregation operations on plithogenic sets[17], [18].

The study ends with advice for managers in blending learning, students, scholarly insights, and implications in addressing blending learning in English courses in higher education selection difficulties.

Finally, by creating a decision support system for blending learning evaluation, the study advances the area.

1.1 Research Gaps

There are different research gaps in this study such as:

The research can use MCDM methods to compute the criteria weights and rank the alternatives without using the uncertainty framework to overcome the uncertainty and vague data in the decision-making process.

The previous research did not conduct a study to compute the criteria weights and rank the alternatives of the blended teaching model.

The previous research did not use the plithogenic sets to dela with uncertainty data in the evaluation MCDM problem.

1.2 Motivation of this study

There are different motivations of this study such as:

Create the decision-making model to evaluate the blended teaching model to evaluate the criteria and rank different alternatives.

Create uncertainty framework to deal with the vague and uncertainty information in the decision-making process.

Create the MCDM model to aid managers and students to evaluate English courses based on a set of criteria and alternatives.

1.3 Contribution of this study

More specifically, this research makes the following contributions:

1. Innovative methodological approach: Using plithogenic sets-based reasoning to evaluate blending learning, a novel technique based on an MCDM approach is presented.
2. Extensive criterion framework: After a thorough literature review that considers both qualitative and quantitative factors, a comprehensive set of criteria is developed for assessing blending learning.
3. Algorithm creation and case study application − The study helps put the suggested technique into practice by creating a methodical algorithm for assessing blending learning.
4. Additionally, it uses the constructed algorithm in a case study setting, proving that the suggested evaluation method is applicable in the actual world.
   By using the case study evaluation to determine which alternative is the best, the study aids in practical decision-making.

2. Methodological Framework

In this study it focused on computing the criteria weights and ranking the alternatives based on the plithogenic sets. The WISP steps are presented to rank the alternatives. Figure 1 shows the methodological flow.

Step 1. Each alternative and criterion are evaluated by each expert to design the decision matrix.

Step 2. Aggregate the decision matrix.

We used a plithogenic operator to combine the decision matrix.

Step 3. Compute the crisp initial decision matrix.

Step 4. Normalize the decision matrix.

$$r_{ij} = \frac{x_{ij}}{\max x_{ij}} \tag{1}$$

Step 5. Compute the four distinct benefits scores.

$$a_i = \sum_{g\in g^+} r_{ij}w_j - \sum_{g\in g^-} r_{ij}w_j \tag{2}$$

$$b_i = \prod_{g\in g^+} r_{ij}w_j - \prod_{g\in g^-} r_{ij}w_j \tag{3}$$

$$c_i = \sum_{d\in g^+} r_{ij}w_j \,/\, \sum_{d\in g^-} r_{ij}w_j \tag{4}$$

$$d_i = \prod_{g\in g^+} r_{ij}w_j \,/\, \prod_{g\in g^-} r_{ij}w_j \tag{5}$$

Step 6. Normalize the benefit scores

$$a_i^- = \frac{1+a_i}{1+\max a_i} \tag{6}$$

$$b_i^- = \frac{1+b_i}{1+\max b_i} \tag{7}$$

$$c_i^- = \frac{1+c_i}{1+\max c_i} \tag{8}$$

$$d_i^- = \frac{1+d_i}{1+\max d_i} \tag{9}$$

Step 7. Compute the overall alternatives.

$$y_i = \frac{a_i^- + b_i^- + c_i^- + d_i^-}{4} \tag{10}$$

Then the alternatives are ranked based on the largest values in overall score.

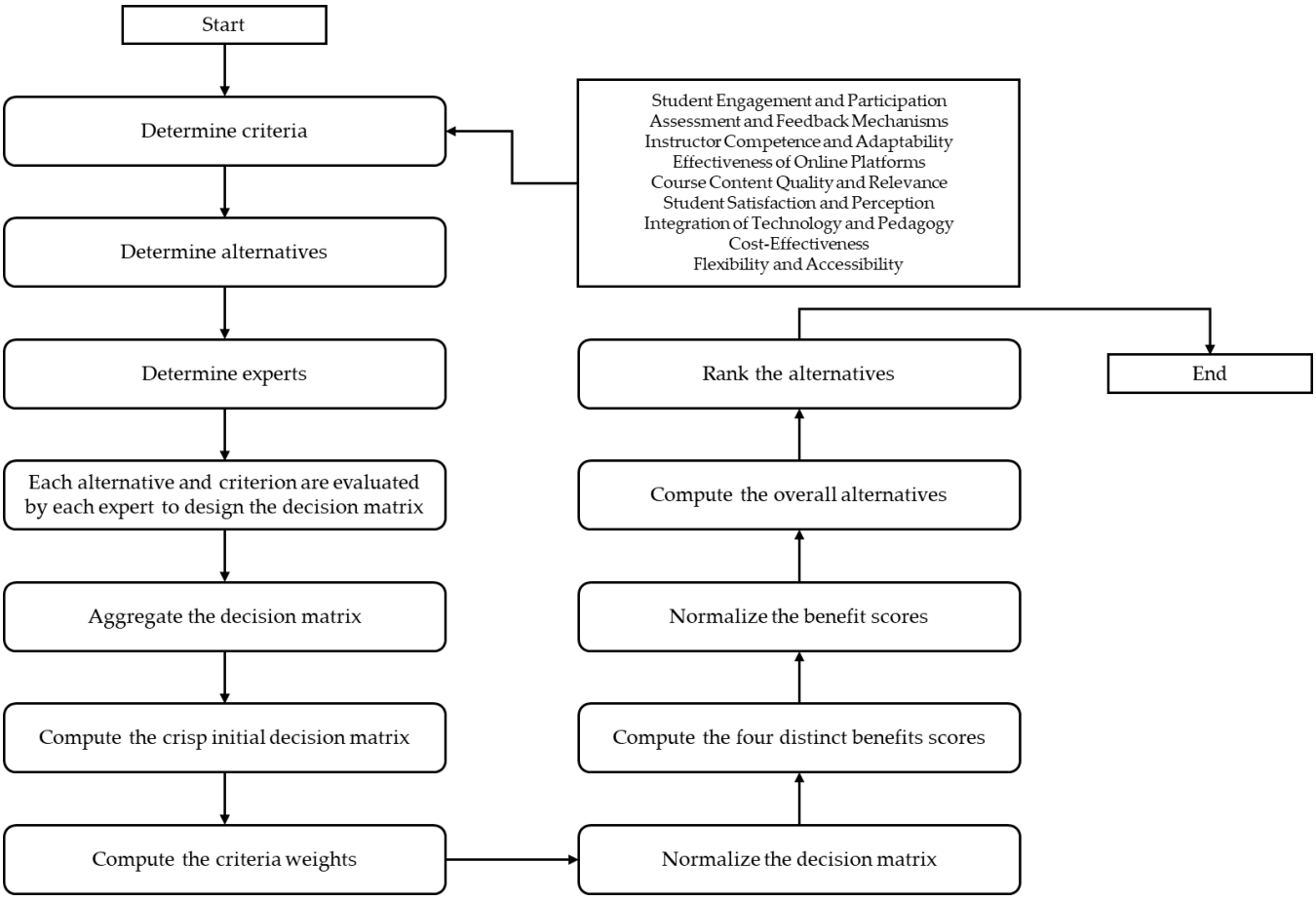
Start
Determine criteria
Determine alternatives
Determine experts
Each alternative and criterion are evaluated by each expert to design the decision matrix
Aggregate the decision matrix
Compute the crisp initial decision matrix
Compute the criteria weights
Normalize the decision matrix
Compute the four distinct benefits scores
Normalize the benefit scores
Compute the overall alternatives
Rank the alternatives
End
Student Engagement and Participation
Assessment and Feedback Mechanisms
Instructor Competence and Adaptability
Effectiveness of Online Platforms
Course Content Quality and Relevance
Student Satisfaction and Perception
Integration of Technology and Pedagogy
Cost-Effectiveness
Flexibility and Accessibility

Figure 1. The flowchart of this stud.

3. Case Study

This study proposes the MCDM approach for assessment English Major Courses in Higher Education. This study evaluated the MCDM approach under the WISP and plithogenic sets.

3.1 Definition of experts sets

The multi-criteria, expert-based method was created in this study with an emphasis on the English Major Courses evaluation. The applicability of this technique was tested by a real-world case analysis. An expert group was established for this case study to evaluate the English Major Courses with precision. Four experts are invited to this study to evaluate the criteria and alternatives.

3.2 Definition of criteria and alternatives

This study collects nine criteria and five alternatives to be evaluated by four experts to build the decision matrix.

Step 1. Tables 1-4 show the decision matrix between the criteria and alternatives. Four experts have evaluated the criteria and alternatives.

Table 1. The plithogenic numbers.

	A_1	A_2	A_3	A_4	A_5
C_1	(0.10, 0.70, 0.80)	(0.70, 0.30, 0.10)	(0.90, 0.10, 0.10)	(0.70, 0.30, 0.10)	(0.90, 0.10, 0.10)
C_2	(0.10, 0.70, 0.80)	(0.10, 0.70, 0.80)	(0.70, 0.30, 0.10)	(0.90, 0.10, 0.10)	(0.30, 0.40, 0.80)
C_3	(0.50, 0.40, 0.60)	(0.10, 0.70, 0.80)	(0.10, 0.70, 0.80)	(0.70, 0.30, 0.10)	(0.90, 0.10, 0.10)
C_4	(0.10, 0.70, 0.80)	(0.70, 0.30, 0.10)	(0.10, 0.70, 0.80)	(0.10, 0.70, 0.80)	(0.70, 0.30, 0.10)
C_5	(0.90, 0.10, 0.10)	(0.10, 0.70, 0.80)	(0.10, 0.70, 0.80)	(0.70, 0.30, 0.10)	(0.70, 0.30, 0.10)
C_6	(0.30, 0.40, 0.80)	(0.10, 0.70, 0.80)	(0.10, 0.70, 0.80)	(0.10, 0.70, 0.80)	(0.70, 0.30, 0.10)
C_7	(0.10, 0.70, 0.80)	(0.10, 0.70, 0.80)	(0.70, 0.30, 0.10)	(0.90, 0.10, 0.10)	(0.10, 0.70, 0.80)
C_8	(0.10, 0.70, 0.80)	(0.10, 0.70, 0.80)	(0.10, 0.70, 0.80)	(0.70, 0.30, 0.10)	(0.10, 0.70, 0.80)
C_9	(0.10, 0.70, 0.80)	(0.10, 0.70, 0.80)	(0.10, 0.70, 0.80)	(0.70, 0.30, 0.10)	(0.70, 0.30, 0.10)

Table 2. The plithogenic numbers.

	A_1	A_2	A_3	A_4	A_5
C_1	(0.30, 0.40, 0.80)	(0.10, 0.70, 0.80)	(0.50, 0.40, 0.60)	(0.50, 0.40, 0.60)	(0.70, 0.30, 0.10)
C_2	(0.30, 0.40, 0.80)	(0.30, 0.40, 0.80)	(0.10, 0.70, 0.80)	(0.50, 0.40, 0.60)	(0.50, 0.40, 0.60)
C_3	(0.50, 0.40, 0.60)	(0.30, 0.40, 0.80)	(0.30, 0.40, 0.80)	(0.10, 0.70, 0.80)	(0.50, 0.40, 0.60)
C_4	(0.70, 0.30, 0.10)	(0.30, 0.40, 0.80)	(0.10, 0.70, 0.80)	(0.50, 0.40, 0.60)	(0.70, 0.30, 0.10)
C_5	(0.90, 0.10, 0.10)	(0.30, 0.40, 0.80)	(0.10, 0.70, 0.80)	(0.50, 0.40, 0.60)	(0.50, 0.40, 0.60)
C_6	(0.30, 0.40, 0.80)	(0.10, 0.70, 0.80)	(0.30, 0.40, 0.80)	(0.10, 0.70, 0.80)	(0.50, 0.40, 0.60)
C_7	(0.30, 0.40, 0.80)	(0.10, 0.70, 0.80)	(0.50, 0.40, 0.60)	(0.50, 0.40, 0.60)	(0.10, 0.70, 0.80)
C_8	(0.30, 0.40, 0.80)	(0.10, 0.70, 0.80)	(0.50, 0.40, 0.60)	(0.50, 0.40, 0.60)	(0.10, 0.70, 0.80)
C_9	(0.10, 0.70, 0.80)	(0.30, 0.40, 0.80)	(0.10, 0.70, 0.80)	(0.50, 0.40, 0.60)	(0.50, 0.40, 0.60)

Table 3. The plithogenic numbers.

	A_1	A_2	A_3	A_4	A_5
C_1	(0.30, 0.40, 0.80)	(0.30, 0.40, 0.80)	(0.10, 0.70, 0.80)	(0.70, 0.30, 0.10)	(0.10, 0.70, 0.80)
C_2	(0.70, 0.30, 0.10)	(0.30, 0.40, 0.80)	(0.10, 0.70, 0.80)	(0.70, 0.30, 0.10)	(0.30, 0.40, 0.80)
C_3	(0.30, 0.40, 0.80)	(0.10, 0.70, 0.80)	(0.70, 0.30, 0.10)	(0.10, 0.70, 0.80)	(0.70, 0.30, 0.10)
C_4	(0.70, 0.30, 0.10)	(0.30, 0.40, 0.80)	(0.10, 0.70, 0.80)	(0.70, 0.30, 0.10)	(0.10, 0.70, 0.80)
C_5	(0.90, 0.10, 0.10)	(0.70, 0.30, 0.10)	(0.30, 0.40, 0.80)	(0.10, 0.70, 0.80)	(0.50, 0.40, 0.60)
C_6	(0.30, 0.40, 0.80)	(0.70, 0.30, 0.10)	(0.30, 0.40, 0.80)	(0.10, 0.70, 0.80)	(0.70, 0.30, 0.10)
C_7	(0.70, 0.30, 0.10)	(0.30, 0.40, 0.80)	(0.10, 0.70, 0.80)	(0.70, 0.30, 0.10)	(0.10, 0.70, 0.80)
C_8	(0.30, 0.40, 0.80)	(0.10, 0.70, 0.80)	(0.70, 0.30, 0.10)	(0.10, 0.70, 0.80)	(0.70, 0.30, 0.10)
C_9	(0.70, 0.30, 0.10)	(0.30, 0.40, 0.80)	(0.10, 0.70, 0.80)	(0.70, 0.30, 0.10)	(0.50, 0.40, 0.60)

Table 4. The plithogenic numbers.

	A_1	A_2	A_3	A_4	A_5
C_1	(0.90, 0.10, 0.10)	(0.10, 0.70, 0.80)	(0.10, 0.70, 0.80)	(0.90, 0.10, 0.10)	(0.10, 0.70, 0.80)
C_2	(0.30, 0.40, 0.80)	(0.90, 0.10, 0.10)	(0.10, 0.70, 0.80)	(0.10, 0.70, 0.80)	(0.30, 0.40, 0.80)
C_3	(0.10, 0.70, 0.80)	(0.70, 0.30, 0.10)	(0.90, 0.10, 0.10)	(0.10, 0.70, 0.80)	(0.10, 0.70, 0.80)
C_4	(0.70, 0.30, 0.10)	(0.90, 0.10, 0.10)	(0.10, 0.70, 0.80)	(0.10, 0.70, 0.80)	(0.70, 0.30, 0.10)
C_5	(0.90, 0.10, 0.10)	(0.10, 0.70, 0.80)	(0.10, 0.70, 0.80)	(0.70, 0.30, 0.10)	(0.90, 0.10, 0.10)
C_6	(0.30, 0.40, 0.80)	(0.90, 0.10, 0.10)	(0.10, 0.70, 0.80)	(0.10, 0.70, 0.80)	(0.30, 0.40, 0.80)
C_7	(0.70, 0.30, 0.10)	(0.30, 0.40, 0.80)	(0.90, 0.10, 0.10)	(0.10, 0.70, 0.80)	(0.10, 0.70, 0.80)
C_8	(0.10, 0.70, 0.80)	(0.90, 0.10, 0.10)	(0.10, 0.70, 0.80)	(0.10, 0.70, 0.80)	(0.10, 0.70, 0.80)
C_9	(0.90, 0.10, 0.10)	(0.10, 0.70, 0.80)	(0.10, 0.70, 0.80)	(0.30, 0.40, 0.80)	(0.70, 0.30, 0.10)

Step 2. We used plithogenic operators to evaluate the criteria and alternatives to obtain one matrix instead of different metrices.

Step 3. We compute the crisp values. Then we normalize the crisp values to obtain the criteria weights as shown in Figure 2.

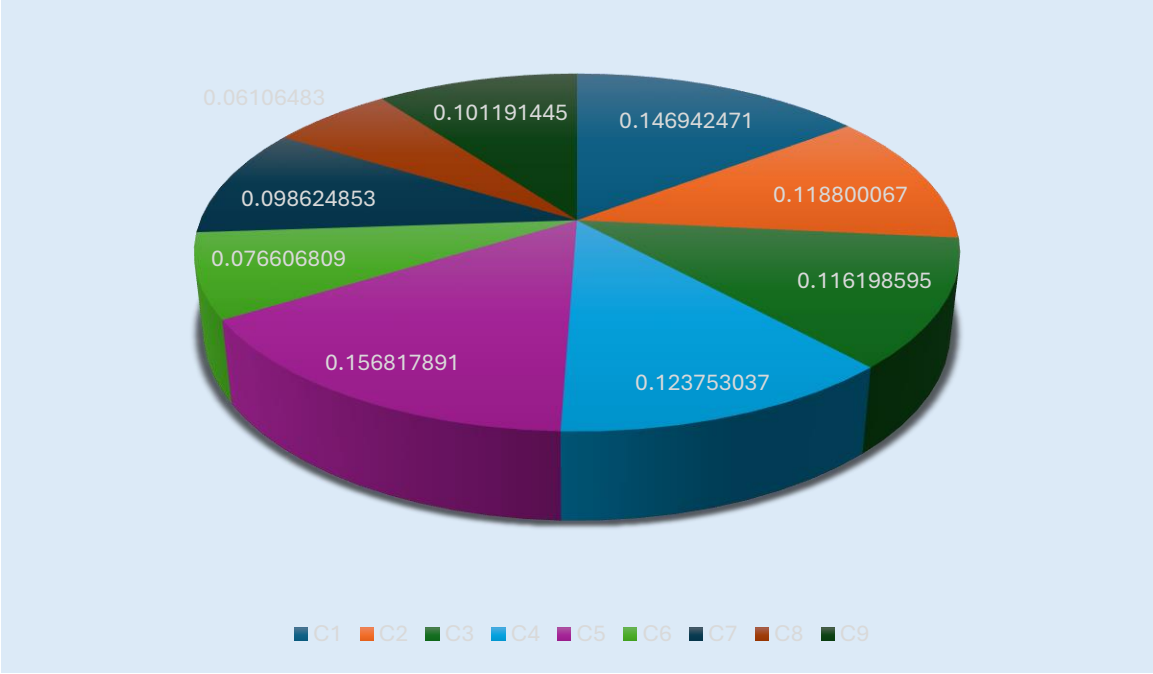

Figure 2. The criteria weights.

Step 4. We normalize the decision matrix between the criteria and alternatives as shown in Table 5.

Table 5. The normalized plithogenic numbers.

	A_1	A_2	A_3	A_4	A_5
C_1	0.68026	0.520415	0.6471	1	0.716647
C_2	0.715499	0.802853	0.512084	1	0.667692
C_3	0.648007	0.614202	0.944635	0.512084	1
C_4	0.95137	1	0.166898	0.652577	0.95137
C_5	1	0.441987	0.198174	0.657848	0.809246
C_6	0.579332	0.84382	0.376859	0.172517	1
C_7	0.857984	0.377743	1	1	0.172921
C_8	0.558685	0.887267	1	1	0.757376
C_9	0.807885	0.36081	0.16517	0.957413	1

Step 5. Then we compute the four distinct benefits scores.

Step 6. Then we normalize the benefits scores.

Step 7. Then we compute the overall alternatives. Table 6 shows the rank of alternatives.

Table 6. The rank of alternatives.

	a_i^-	b_i^-	c_i^-	d_i^-	Rank
A1	1	1	1	1	5
A2	0.888437	0.979226	0.50788	1	2
A3	0.818351	0.972099	0.37454	1	1
A4	0.968085	0.972099	0.558167	1	3
A5	0.989847	0.987438	0.744177	1	4

4. Results and Discussion

This study used the MCDM approach to compute the criteria weights and rank the alternatives. Four experts have evaluated the criteria and alternatives. They used the plithogenic terms in their assessment. Then we used the plithogenic numbers to build the decision matrix. We used the plithogenic operators to combine the decision numbers. Then we obtained the crips values. We used nine criteria and five alternatives in this study. The results showed that the Course Content Quality and Relevance criterion has the highest weights and the Cost criterion has the lowest weights.

In the rank of alternatives， we proposed the WISP approach to evaluate the alternatives and select the best one. The normalization decision matrix is performed between the criteria and alternatives. The results show that alternative 1 is the best and alternative 3 is the worst.

4.1 Comparison analysis

This part shows the comparison between the proposed approach and other MCDM methods. We compared the proposed approach with four different MCDM methods. Figure 3 shows the comparison analysis. We showed that all methods have identical rank except the EDAS method. We showed that there is a high correlation between the proposed method and other MCDM approaches. We showed that the proposed method is effective compared to other MCDM approaches.

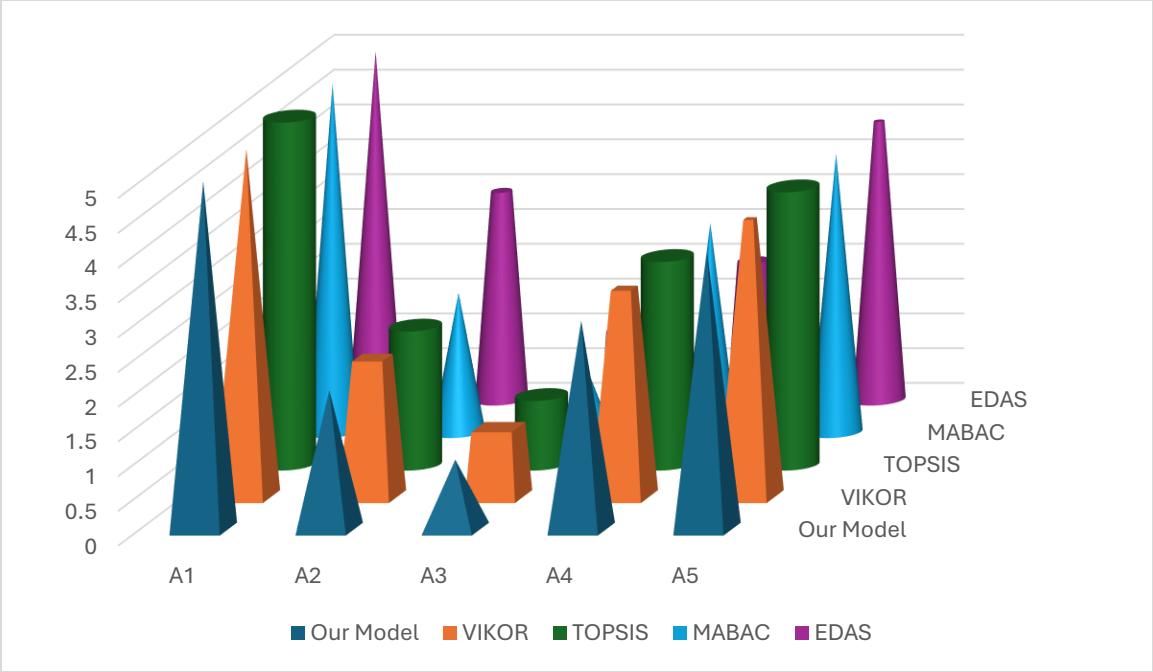

Figure 3. The comparison results.

4.2 Sensitivity Analysis

This section shows the sensitivity analysis to rank the alternatives under different criteria weights to show different rank of alternatives. Figure 4 shows the criteria weights under different cases. We proposed ten cases in criteria weights to show the rank of alternatives. We increased the criteria weights by 20% and other criteria have the same weights.

Then we applied the proposed approach under different cases to show the final score of each alternative and final rank of alternatives. The results show all cases have the same rank the alternative 1 is the best and alternative 3 is the worst. Figure 5 shows the final score of each alternative. Figure 6 shows the final rank of alternatives. All cases show the alternative 1 is the best, followed by alternative 5, alternative 4, alternative 2, and alternative 3 is the worst.

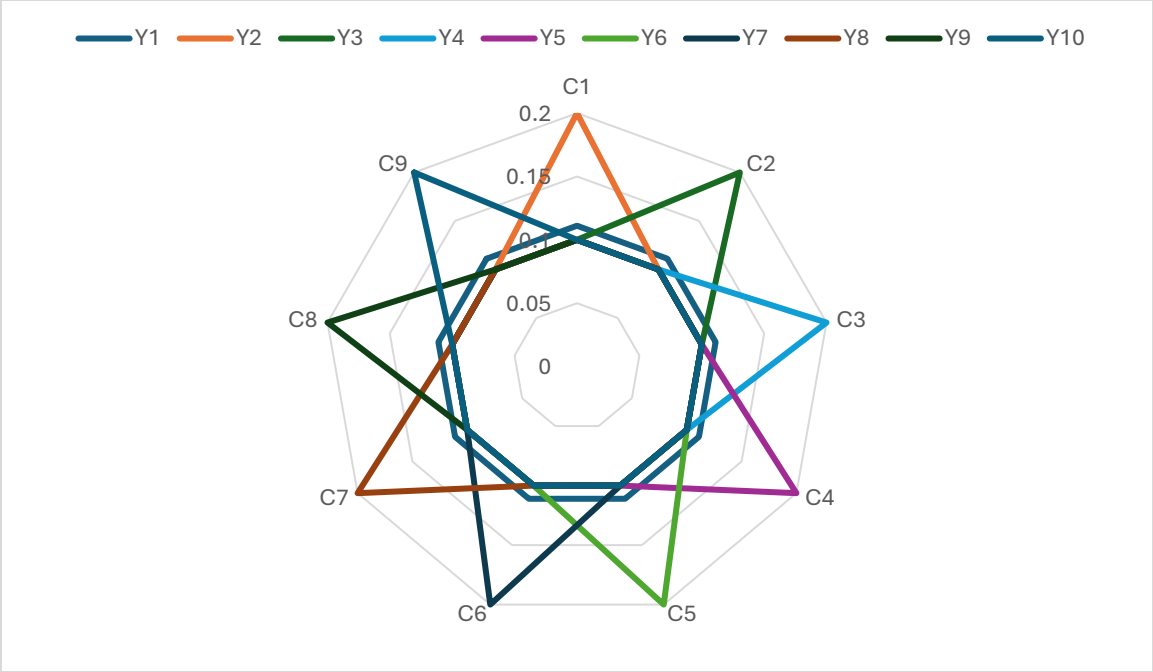

Figure 4. The criteria changes.

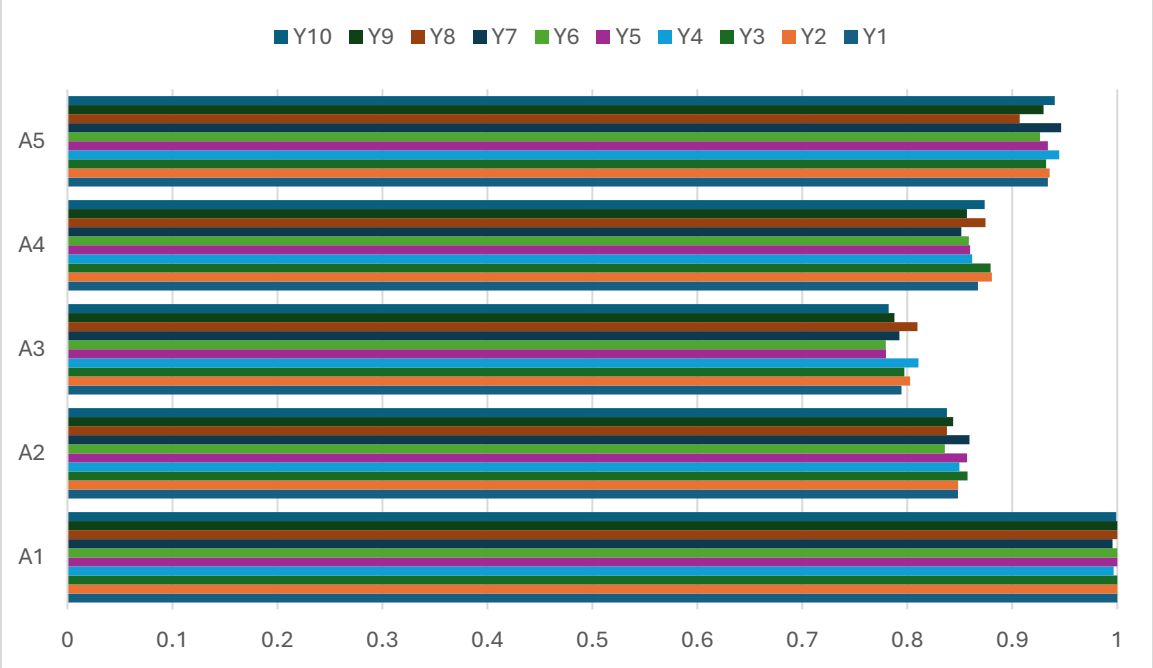

Figure 5. Final score of each alternative under different criteria weights.

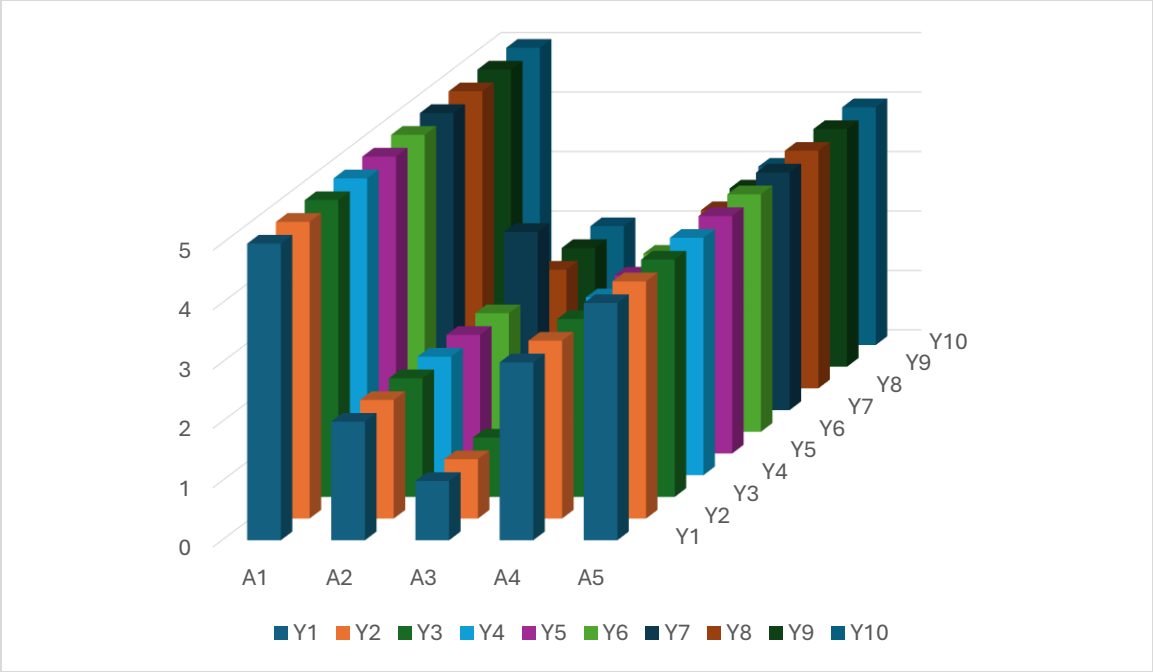

Figure 6. Final rank of each alternative under different criteria weights.

5. Conclusions, Limitations and Future works

The importance of blending learning structure and performance was emphasized by this study. The study presented a hybrid methodology to establish expert relevance levels, criterion weights, and alternative ranks by utilizing plithogenic sets and a group decision-based MCDM approach. Both qualitative and quantitative elements of a thorough literature review were used to develop the evaluation criteria for blending learning performance. The numerical values of blending learning were used to calculate the quantitative criteria values, and expert opinions were used to evaluate the qualitative criteria. The plithogenic -WISP hybrid model was suggested by the study to calculate blending learning performance. Alternatives were ranked using the plithogenic-WISP method that was presented. The results showed that alternative 1 is the best and alternative 3 is the worst.

The limitations of this study that can be solved in the future work are, limited number of criteria that can be extended to include all different criteria, limited number of the alternatives that can be extended in the future work, and the future work can use the proposed model in the different MCDM problem to compute the criteria weights and rank the alternatives.

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Implication-Based Neutrosophic Finite State Machine over a Finite Group

S. Jafari[1,*], N. Rajesh[2], G. Nordo[3] and M. Gorgone[4]

[1]Mathematical and Physical Science Foundation, 4200 Slagelse, Denmark; jafaripersia@gmail.com

[2] Department of Mathematics, Rajah Serfoji Government College, Thanjavur-613005, Tamilnadu, India; nrajesh_topology@yahoo.co.in

[3]MIFT Department – Mathematical and Computer Science, Physical Sciences and Earth Sciences University of Messina, 98166 Sant'Agata, Messina, Italy; giorgio.nordo@unime.it

[4]MIFT Department – Mathematical and Computer Science, Physical Sciences and Earth Sciences University of Messina, 98166 Sant'Agata, Messina, Italy; matteo.gorgone@unime.it

[*]Correspondence: jafaripersia@gmail.com

Abstract. In this paper, we define and investigate the concept of the implication-based neutrosophic finite state machine (IB-IFSA) over a finite group. Building upon the framework of neutrosophic logic, we introduce the implication-based neutrosophic kernel and the implication-based neutrosophic subsemiautomaton, establishing their formal definitions and properties. Additionally, we explore the structural interplay between these concepts and their implications within algebraic systems.

Keywords: Implication-based neutrosophic finite state machine, implication-based neutrosophic subgroup, neutrosophic kernel, neutrosophic subsemiautomaton, finite group theory, neutrosophic logic.

1. Introduction

In 1965, Zadeh [13] introduced the concept of fuzzy sets, providing a mathematical framework to model uncertainty and imprecision. This idea catalyzed significant developments in various fields, including automata theory. In 1969, Wee [10] proposed the concept of fuzzy automata, extending Zadeh's work to computational models designed for learning and decision-making processes. Later, in 1971, Rosenfeld [5] applied fuzzy set theory to algebraic group structures, which opened new research directions in the study of fuzzy algebraic systems.

The study of fuzzy subgroups, especially fuzzy normal subgroups, further enriched the algebraic applications of fuzzy logic. Notable contributions in this domain include the works of Dib and Hassan [1], Malik et al. [2], and Mukherjee and Bhattacharya [3], which investigated the structural properties of fuzzy groups. Additionally, Asok Kumar [4] explored the products of fuzzy subgroups, providing deeper insights into their composition and interactions.

The introduction of neutrosophic logic by Smarandache [7,8] significantly expanded the theoretical foundation of uncertainty modeling. Neutrosophic sets generalize fuzzy sets by introducing three membership degrees: truth, indeterminacy, and falsity. This enriched framework allows for a more comprehensive representation of incomplete, inconsistent, or indeterminate information, which is especially useful in fields requiring nuanced reasoning under uncertainty.

In this paper, we extend the application of neutrosophic logic to automata theory by defining the implication-based neutrosophic finite state machine (IB-IFSA) over a finite group, also referred to as the implication-based neutrosophic semiautomaton. We introduce and analyze the implication-based neutrosophic kernel and the implication-based neutrosophic subsemiautomaton, exploring their algebraic properties within the context of finite groups. By integrating neutrosophic logic with computational and algebraic systems, this work provides a foundational framework for further research, with potential applications in decision-making models, automated reasoning, and beyond.

2. Preliminaries

This section introduces the foundational concepts and definitions of neutrosophic sets, neutrosophic groups, and their properties, forming the mathematical framework for integrating neutrosophic logic with algebraic structures.

Definition 2.1. [9] *Let $\mathbb{U}$ be an initial universe set and $A \subseteq \mathbb{U}$, a **neutrosophic set** (more precisely, a single valued neutrosophic set) over $\mathbb{U}$ (**SVN-set** for short), denoted by $\widetilde{A} = \langle \mathbb{U}, \mu_A, \sigma_A, \omega_A \rangle$, is a set of the form*

$$\widetilde{A} = \{(u, \mu_A(u), \sigma_A(u), \omega_A(u)) : u \in \mathbb{U}\}$$

*where $\mu_A : \mathbb{U} \to I$, $\sigma_A : \mathbb{U} \to I$ and $\omega_A : \mathbb{U} \to I$ are the **membership function**, the **indeterminacy function** and the **nonmembership function** of A, respectively. For every $u \in \mathbb{U}$, $\mu_A(u)$, $\sigma_A(u)$ and $\omega_A(u)$ are said the **degree of membership**, the **degree of indeterminacy** and the **degree of nonmembership** of u, respectively.*

Definition 2.2. *Let $(\Omega, \cdot)$ be a group, a neutrosophic set $\widetilde{A} = \langle \Omega, \mu_A, \sigma_A, \omega_A \rangle$ over Ω is called a **neutrosophic group** on Ω, if the following conditions hold:*

$$\text{(i)} \begin{cases} \mu_A(xy) \geq \min\{\mu_A(x), \mu_A(y)\} \\ \sigma_A(xy) \geq \min\{\sigma_A(x), \sigma_A(y)\} \\ \omega_A(xy) \leq \max\{\omega_A(x), \omega_A(y)\} \end{cases}, \quad \textit{for every } x, y \in \Omega$$

$$\text{(ii)} \begin{cases} \mu_A(x^{-1}) \geq \mu_A(x) \\ \sigma_A(x^{-1}) \geq \sigma_A(x) \\ \omega_A(x^{-1}) \leq \omega_A(x) \end{cases}, \quad \textit{for every } x \in \Omega.$$

Definition 2.3. *Let $(\Omega, \cdot)$ be a group, a neutrosophic set $\widetilde{A} = \langle \Omega, \mu_A, \sigma_A, \omega_A \rangle$ over Ω is called a* ***neutrosophic normal subgroup*** *on Ω if:*

$$\begin{cases} \mu_A(xyx^{-1}) \geq \mu_A(y) \\ \sigma_A(xyx^{-1}) \geq \sigma_A(y) \\ \omega_A(xyx^{-1}) \leq \omega_A(y) \end{cases}$$

for every $x, y \in \Omega$.

Notation 2.4. *Let $\mathbb{U}$ be a universe set and $(\Omega, \cdot)$ a group. In neutrosophic logic, the truth value of a neutrosophic proposition α is denoted by $[\alpha]$. The following notations are used to describe the neutrosophic logical operations and their corresponding set-theoretical interpretations in this paper:*

$$(\nu \in \widetilde{A}) \equiv \widetilde{A}(\nu),$$

$$(\alpha \wedge \beta) \equiv \min\{[\alpha], [\beta]\},$$

$$(\alpha \to \beta) \equiv \min\{1, 1 - [\alpha] + [\beta]\},$$

$$(\forall \nu\, \alpha(\nu)) \equiv \inf_{\nu \in \mathbb{U}} [\alpha(\nu)],$$

$$(\exists \nu\, \alpha(\nu)) \equiv \sup_{\nu \in \mathbb{U}} [\alpha(\nu)],$$

$$\vDash \alpha \quad \textit{if and only if} \quad [\alpha] = 1 \quad \textit{for all valuations.}$$

This framework establishes the basis for applying neutrosophic logic within the algebraic context analyzed in this study. The truth valuation rules adopted follow the Łukasiewicz system of continuous-valued logic.

Additionally, the concept of λ-tautology, as introduced by Ying [11], is defined as $\vDash_\lambda \alpha$ if and only if $[\alpha] \geq \lambda$ for all valuations.

The following notions in neutrosophic theory are inspired by the frameworks outlined in [12] and [6].

Definition 2.5. *Let $\widetilde{A} = \langle \Omega, \mu_A, \sigma_A, \omega_A \rangle$ over Ω be a neutrosophic set on a finite group Ω, and let $\lambda \in (0, 1]$ be a fixed number. If, for any $\nu_1, \nu_2 \in \Omega$, the following conditions hold:*

(i) $\vDash_\lambda \big((\nu_1 \in \widetilde{A}) \wedge (\nu_2 \in \widetilde{A}) \to (\nu_1 \nu_2 \in \widetilde{A})\big)$,

(ii) $\vDash_\lambda \big((\nu_1 \in \widetilde{A}) \to (\nu_1^{-1} \in \widetilde{A})\big)$

then $\widetilde{A}$ is called an ***implication-based neutrosophic subgroup*** *of Ω.*

Definition 2.6. *Let $\widetilde{A}$ be an implication-based neutrosophic subgroup of Ω, $\lambda \in (0,1]$ a fixed number, and $f : \Omega \to \Omega$ a function defined on Ω. Then the* ***implication-based neutrosophic subgroup*** *$\widetilde{B}$ of $f(\Omega)$ is defined as:*

$$\vDash_\lambda \left(\exists\nu\{(\nu \in \widetilde{A})\}; \nu \in f^{-1}(\psi)\right) \to (\psi \in \widetilde{B}), \quad \forall\psi \in f(\Omega).$$

Conversely, if $\widetilde{B}$ is an implication-based neutrosophic subgroup of $f(\Omega)$, then the implication-based neutrosophic subgroup $\widetilde{A} = f \circ \widetilde{B}$ in Ω is defined as:

$$\vDash_\lambda \left((f(\nu) \in \widetilde{B}) \to (\nu \in \widetilde{A})\right), \quad \forall\nu \in \Omega,$$

and is called the ***pre-image*** *of $\widetilde{A}$ under f.*

Definition 2.7. *An implication-based neutrosophic subgroup $\widetilde{A}$ of Ω is called an* ***implication-based neutrosophic normal subgroup*** *if:*

$$\vDash_\lambda \left((\nu\psi \in \widetilde{A}) \to (\psi\nu \in \widetilde{A})\right), \quad \forall\nu, \psi \in \Omega,$$

where $\lambda \in]0,1]$ is a fixed number.

Proposition 2.8. *Let $\widetilde{A} = \langle \Omega, \mu_A, \sigma_A, \omega_A \rangle$ be an implication-based neutrosophic subgroup of a finite group $(\Omega, \cdot)$. Then, for any $\nu \in \Omega$, the following holds:*

$$\vDash_\lambda (\nu \in \widetilde{A}) \to (\epsilon \in \widetilde{A}),$$

where ϵ is the identity element of the group Ω.

From this point onward, let Ω be a finite group with identity element ϵ and $\lambda \in]0,1]$ a fixed parameter.

3. Implication-Based Neutrosophic Semiautomaton over a Finite Group

The concept of an implication-based neutrosophic semiautomaton combines the algebraic structure of finite groups with the logical framework of neutrosophic logic. This section formalizes the definition of a semiautomaton in this context, highlighting the interactions between group elements, logical variables, and neutrosophic conditions. The proposed model provides a versatile framework for analyzing transitions and behaviors in systems characterized by uncertainty, indeterminacy, and truth variability.

Definition 3.1. *Let $\mathbb{N}_{\mathcal{A}} = \langle A, B, C \rangle$ be an implication-based neutrosophic subgroup of a finite group Ω. An* ***implication-based neutrosophic semiautomaton*** *over the finite group $(\Omega, \cdot)$ is a triple $\mathbb{NS} = (\Omega, \Delta, \mathbb{N}_{\mathcal{A}})$, where Δ represents the set of all logical variables.*

Remark 3.2. *The set Δ of the definition above consists of all logical variables used to describe transitions within the semiautomaton. Each variable in Δ encodes a specific logic operation or transition condition.*

Notation 3.3. *If Δ is a set of logical variables, Δ^* denote the set of all finite combinations of these logical variables, including the special element $\boldsymbol{0}$, which represents the null operation or an empty transition.*

Definition 3.4. *Let $\mathbb{NS} = (\Omega, \Delta, \mathbb{N}_{\mathcal{A}})$ be an implication-based neutrosophic semiautomaton over the finite group Ω. Define $\mathbb{N}_{\mathcal{A}^*} = \langle A^*, B^*, C^* \rangle$ in $\Omega \times \Delta^* \times \Omega$ such that, for every $\alpha, \beta \in \Omega$, $\xi \in \Delta^*$, and $\omega \in \Delta$, the following conditions hold:*

(i) $\vDash_\lambda ((\alpha, \boldsymbol{0}, \beta) \in A^*) \to 0$ *(with $\lambda = 0$)*

(ii) $\vDash_\lambda ((\alpha, \boldsymbol{0}, \beta) \in B^*) \to 0$ *(with $\lambda = 0$)*

(iii) $\vDash_\lambda ((\alpha, \boldsymbol{0}, \beta) \in C^*) \to 1$

(iv) $\vDash_\lambda \big(\exists\gamma\{((\beta, \xi, \gamma) \in A^*) \wedge ((\gamma, \omega, \alpha) \in A^*)\}; \gamma \in \Omega\big) \to ((\beta, \xi \odot \omega, \alpha) \in A^*)$

(v) $\vDash_\lambda \big(\exists\gamma\{((\beta, \xi, \gamma) \in B^*) \wedge ((\gamma, \omega, \alpha) \in B^*)\}; \gamma \in \Omega\big) \to ((\beta, \xi \odot \omega, \alpha) \in B^*)$

(vi) $\vDash_\lambda \big(\forall\gamma\{((\beta, \xi, \gamma) \in C^*) \vee ((\gamma, \omega, \alpha) \in C^*)\}; \gamma \in \Omega\big) \to ((\beta, \xi \odot \omega, \alpha) \in C^*)$

Theorem 3.5. *Let $\mathbb{NS} = (\Omega, \Delta, \mathbb{N}_{\mathcal{A}})$ be an implication-based neutrosophic semiautomaton over the finite group Ω. Then, for all $\alpha, \beta \in \Omega$ and $\xi, \psi \in \Delta^*$, the following hold:*

(1) $\vDash_\lambda \big(\exists\gamma\{((\beta, \xi, \gamma) \in A^*) \wedge ((\gamma, \psi, \alpha) \in A^*)\}; \gamma \in \Omega\big) \to ((\beta, \xi \odot \psi, \alpha) \in A^*)$

(2) $\vDash_\lambda \big(\exists\gamma\{((\beta, \xi, \gamma) \in B^*) \wedge ((\gamma, \psi, \alpha) \in B^*)\}; \gamma \in \Omega\big) \to ((\beta, \xi \odot \psi, \alpha) \in B^*)$

(3) $\vDash_\lambda \big(\forall\gamma\{((\beta, \xi, \gamma) \in C^*) \vee ((\gamma, \psi, \alpha) \in C^*)\}; \gamma \in \Omega\big) \to ((\beta, \xi \odot \psi, \alpha) \in C^*)$

Proof. Let $\alpha, \beta \in \Omega$ and $\xi, \psi \in \Delta^*$. The proof proceeds by induction on $ord(\psi) = n$.

Base Case ($n = 0$): If $n = 0$, then $\psi = \mathbf{0}$ and $\xi \odot \psi = \xi \odot \mathbf{0} = \xi$. Thus:

$$\begin{aligned}
&\vDash_\lambda (\exists\gamma\{((\beta, \xi, \gamma) \in A^*) \wedge ((\gamma, \psi, \alpha) \in A^*)\}; \gamma \in \Omega) \\
&\to (\exists\gamma\{((\beta, \xi, \gamma) \in A^*) \wedge ((\gamma, \mathbf{0}, \alpha) \in A^*)\}; \gamma \in \Omega) \\
&\to ((\beta, \xi, \alpha) \in A^*) \\
&\to ((\beta, \xi \odot \psi, \alpha) \in A^*).
\end{aligned}$$

Similarly, the result holds for B^* and C^* using analogous reasoning. Thus, the theorem is valid for $n = 0$.

Inductive Step: Assume that the result holds for any $\zeta \in \Delta^*$ such that $ord(\zeta) = n - 1$ and $n > 0$. Let $\psi \in \Delta^*$ such that $\psi = \zeta \odot \omega$ with $\zeta \in \Delta^*$, $\omega \in \Delta$, and $ord(\zeta) = n - 1$. We need to show the result for ψ.

For A^* w have that:

$$\begin{aligned}
&\vDash_\lambda (\exists\gamma\{((\beta,\xi,\gamma)\in A^*)\wedge((\gamma,\psi,\alpha)\in A^*)\};\gamma\in\Omega)\\
&\to (\exists\gamma\{((\beta,\xi,\gamma)\in A^*)\wedge((\gamma,\zeta\odot\omega,\alpha)\in A^*)\};\gamma\in\Omega)\\
&\to (\exists\gamma\{((\beta,\xi,\gamma)\in A^*)\wedge(\exists\delta\{((\gamma,\zeta,\delta)\in A^*)\wedge((\delta,\omega,\alpha)\in A^*)\};\delta\in\Omega)\};\gamma\in\Omega)\\
&\to (\exists\delta\{((\beta,\xi\odot\zeta,\delta)\in A^*)\wedge((\delta,\omega,\alpha)\in A^*)\};\delta\in\Omega)\\
&\to ((\beta,\xi\odot\zeta\odot\omega,\alpha)\in A^*)\\
&\to ((\beta,\xi\odot\psi,\alpha)\in A^*).
\end{aligned}$$

For B^* and C^* the proofs follow similar steps, substituting B^* and C^* for A^*, with appropriate adjustments for logical operations.
So, by using the Induction Principle, the result holds for all $\psi\in\Delta^*$ with $ord(\psi)=n$ and this completes the proof. □

Definition 3.6. *Let $\mathbb{N}_{\mathcal{A}}=\langle A,B,C\rangle$ be an implication-based neutrosophic subgroup over a finite group $(\Omega,\cdot)$, and let $\mathbb{NS}=(\Omega,\Delta,\mathbb{N}_{\mathcal{A}})$ be an implication-based neutrosophic semiautomaton over the same finite group Ω. A neutrosophic subset $\langle\iota,\overline{A},\overline{B},\overline{C}\rangle$ of the group Ω is called an **implication-based neutrosophic subsemiautomaton** of $\mathbb{NS}$ if the following conditions hold:*

(i) $\mathbb{N}_{\tilde{\mathcal{A}}}=\langle\overline{A},\overline{B},\overline{C}\rangle$ *is an implication-based neutrosophic subgroup of* Ω.
(ii) $\vDash_\lambda \big(((\alpha,\xi,\beta)\in A)\wedge(\alpha\in\overline{A})\big)\to(\beta\in\overline{A})$.
(iii) $\vDash_\lambda \big(((\alpha,\xi,\beta)\in B)\wedge(\alpha\in\overline{B})\big)\to(\beta\in\overline{B})$.
(iv) $\vDash_\lambda \big((\beta\in\overline{C})\big)\to\big(((\alpha,\xi,\beta)\in C)\vee(\alpha\in\overline{C})\big)$,

for all $\alpha,\beta\in\Omega$ and $\xi\in\Delta$.

Theorem 3.7. *Let $\mathbb{NS}=\big(\Omega,\Delta,\mathbb{N}_{\tilde{\mathcal{A}}}\big)$ be an implication-based neutrosophic semiautomaton over the finite group $(\Omega,\cdot)$. Let $\mathbb{N}_{\tilde{\mathcal{A}}}=\langle\overline{A},\overline{B},\overline{C}\rangle$ be an implication-based neutrosophic subgroup of Ω. Then $\mathbb{N}_{\tilde{\mathcal{A}}}$ is an implication-based neutrosophic subsemiautomaton of $\mathbb{NS}$ if and only if the following conditions hold:*

(1) $\vDash_\lambda \big(((\alpha,\xi,\beta)\in A^*)\wedge(\alpha\in\overline{A})\big)\to(\beta\in\overline{A})$
(2) $\vDash_\lambda \big(((\alpha,\xi,\beta)\in B^*)\wedge(\alpha\in\overline{B})\big)\to(\beta\in\overline{B})$
(3) $\vDash_\lambda \big((\beta\in\overline{C})\big)\to\big(((\alpha,\xi,\beta)\in C^*)\vee(\alpha\in\overline{C})\big)$

for all $\alpha,\beta\in\Omega$ and $\xi\in\Delta^*$.

Proof. Let $\mathbb{N}_{\tilde{\mathcal{A}}}=\langle\overline{A},\overline{B},\overline{C}\rangle$ be an implication-based neutrosophic subsemiautomaton of $\mathbb{NS}$. Let $\alpha,\beta\in\Omega$ and $\xi\in\Delta^*$. We prove the theorem by induction on $ord(\xi)=n$.

Base Case: If $n = 0$, then $\xi = \mathbf{0}$. For A^*:

$$\begin{aligned}\vDash_\lambda \big(((\alpha,\xi,\beta) \in A^*) \wedge (\alpha \in \overline{A})\big) &\to \big(((\alpha,\mathbf{0},\beta) \in A^*) \wedge (\alpha \in \overline{A})\big)\\ &\to \big(0 \in A^*\big) \wedge (\alpha \in \overline{A})\\ &\to (\beta \in \overline{A}).\end{aligned}$$

Analogously, the same logic applies to B^*, while for $\overline{C}$, we have:

$$\vDash_\lambda (\beta \in \overline{C}) \to ((\alpha,\mathbf{0},\beta) \in C^*) \vee (\alpha \in \overline{C}) \to ((\alpha,\xi,\beta) \in C^*) \vee (\alpha \in \overline{C}).$$

Thus, the base case holds.

Inductive Step: Assume the result is true for all $\psi \in \Delta^*$ such that $ord(\psi) = n-1$ and $n > 0$. Let $\xi = \psi \odot \omega$ where $\omega \in \Delta$. For A^*:

$$\begin{aligned}\vDash_\lambda \big(((\alpha,\xi,\beta) \in A^*) \wedge (\alpha \in \overline{A})\big) &\to \big(((\alpha,\psi\odot\omega,\beta) \in A^*) \wedge (\alpha \in \overline{A})\big)\\ &\to \big(\exists\gamma\{((\alpha,\psi,\gamma) \in A^*) \wedge ((\gamma,\omega,\beta) \in A)\};\gamma \in \Omega\big) \wedge (\alpha \in \overline{A})\\ &\to \big(\exists\gamma\{(\gamma \in \overline{A}) \wedge ((\gamma,\omega,\beta) \in A)\};\gamma \in \Omega\big)\\ &\to (\beta \in \overline{A}).\end{aligned}$$

The case for B^* is analogous, while for $\overline{C}$, we have that:

$$\begin{aligned}\vDash_\lambda (\beta \in \overline{C}) &\to \big(\forall\gamma\{((\gamma,\omega,\beta) \in C) \vee (\gamma \in \overline{C})\};\gamma \in \Omega\big)\\ &\to \big(\forall\gamma\{((\alpha,\psi,\gamma) \in C^*) \vee ((\gamma,\omega,\beta) \in C)\};\gamma \in \Omega\big) \vee (\alpha \in \overline{C})\\ &\to ((\alpha,\xi,\beta) \in C^*) \vee (\alpha \in \overline{C}).\end{aligned}$$

Thus, by induction, the forward implication has been proven for all $ord(\xi) = n$, showing that if $\mathbb{N}_{\tilde{A}}$ satisfies the conditions of an implication-based neutrosophic kernel, then the stated properties hold for A^*, B^*, and C^*.

Since the converse is trivial, the theorem is completely proved. □

Definition 3.8. *Let $\mathbb{N}_{\mathcal{A}} = \langle A, B, C\rangle$ be an implication-based neutrosophic subgroup over a finite group $(\Omega,\cdot)$, and let $\mathbb{NS} = (\Omega,\Delta,\mathbb{N}_{\mathcal{A}})$ be an implication-based neutrosophic semiautomaton over the finite group $(\Omega,\cdot)$. A neutrosophic subset $\langle \iota, \overline{A}, \overline{B}, \overline{C}\rangle$ of the group Ω is called an* ***implication-based neutrosophic kernel*** *of $\mathbb{NS}$ if the following conditions are satisfied:*

(i) $\mathbb{N}_{\tilde{A}} = \langle \overline{A}, \overline{B}, \overline{C}\rangle$ *is an implication-based neutrosophic normal subgroup of Ω.*

(ii) $\vDash_\lambda \big(((\beta\kappa,\xi,\alpha) \in A) \wedge ((\beta,\xi,\gamma) \in A) \wedge (\kappa \in \overline{A})\big) \to (\alpha\gamma^{-1} \in \overline{A})$.

(iii) $\vDash_\lambda \big(((\beta\kappa,\xi,\alpha) \in B) \wedge ((\beta,\xi,\gamma) \in B) \wedge (\kappa \in \overline{B})\big) \to (\alpha\gamma^{-1} \in \overline{B})$.

(iv) $\vDash_\lambda \big((\alpha\gamma^{-1} \in \overline{C})\big) \to \big(((\beta\kappa,\xi,\alpha) \in C) \vee ((\beta,\xi,\gamma) \in C) \vee (\kappa \in \overline{C})\big)$,

for all $\alpha,\beta,\gamma,\kappa \in \Omega$ and $\xi \in \Delta$.

Theorem 3.9. *Let $\mathbb{NS} = (\Omega, \Delta, \mathbb{N}_{\mathcal{A}})$ be an implication-based neutrosophic semiautomaton over the finite group $(\Omega, \cdot)$ and $\mathbb{N}_{\tilde{\mathcal{A}}} = \langle \overline{A}, \overline{B}, \overline{C} \rangle$ be an implication-based neutrosophic subgroup of Ω. Then $\mathbb{N}_{\tilde{\mathcal{A}}}$ is an implication-based neutrosophic kernel of $\mathbb{NS}$ if and only if the following conditions hold:*

(1) $\vDash_{\lambda} \big(((\beta\kappa, \xi, \alpha) \in A^*) \wedge ((\beta, \xi, \gamma) \in A^*) \wedge (\kappa \in \overline{A})\big) \rightarrow (\alpha\gamma^{-1} \in \overline{A})$

(2) $\vDash_{\lambda} \big(((\beta\kappa, \xi, \alpha) \in B^*) \wedge ((\beta, \xi, \gamma) \in B^*) \wedge (\kappa \in \overline{B})\big) \rightarrow (\alpha\gamma^{-1} \in \overline{B})$

(3) $\vDash_{\lambda} \big((\alpha\gamma^{-1} \in \overline{C})\big) \rightarrow \big(((\beta\kappa, \xi, \alpha) \in C^*) \vee ((\beta, \xi, \gamma) \in C^*) \vee (\kappa \in \overline{C})\big)$

for all $\alpha, \beta, \gamma, \kappa \in \Omega$ and $\xi \in \Delta^*$.

Proof. Let $\mathbb{N}_{\tilde{\mathcal{A}}} = \langle \overline{A}, \overline{B}, \overline{C} \rangle$ be an implication-based neutrosophic kernel of $\mathbb{NS}$. We prove the result by induction on $ord(\xi) = n$.

Base Case: If $n = 0$, then $\xi = \mathbf{0}$. For A^* we have:

$$\begin{aligned}
\vDash_{\lambda} &\big(((\beta\kappa, \xi, \alpha) \in A^*) \wedge ((\beta, \xi, \gamma) \in A^*) \wedge (\kappa \in \overline{A})\big) \\
&\rightarrow \big(((\beta\kappa, \mathbf{0}, \alpha) \in A^*) \wedge ((\beta, \mathbf{0}, \gamma) \in A^*) \wedge (\kappa \in \overline{A})\big) \\
&\rightarrow \big(0 \wedge 0 \wedge (\kappa \in \overline{A})\big) \\
&\rightarrow (\alpha\gamma^{-1} \in \overline{A}).
\end{aligned}$$

Similarly, for B^* the argument follows the same reasoning, while for $\overline{C}$ we have that:

$$\begin{aligned}
\vDash_{\lambda} \big((\alpha\gamma^{-1} \in \overline{C})\big) &\rightarrow 1 \vee 1 \vee (\kappa \in \overline{C}) \\
&\rightarrow ((\beta\kappa, \mathbf{0}, \alpha) \in C^*) \vee ((\beta, \mathbf{0}, \gamma) \in C^*) \vee (\kappa \in \overline{C}) \\
&\rightarrow ((\beta\kappa, \xi, \alpha) \in C^*) \vee ((\beta, \xi, \gamma) \in C^*) \vee (\kappa \in \overline{C}).
\end{aligned}$$

Thus, the base case holds.

Inductive Step: Assume the result holds for all $\psi \in \Delta^*$ such that $ord(\psi) = n - 1$. Let $\xi = \psi \odot \omega$ where $\omega \in \Delta$. For A^* we have:

$$\begin{aligned}
&\vDash_{\lambda} ((\beta\kappa, \xi, \alpha) \in A^*) \wedge ((\beta, \xi, \gamma) \in A^*) \wedge (\kappa \in \overline{A}) \\
&\rightarrow ((\beta\kappa, \psi \odot \omega, \alpha) \in A^*) \wedge ((\beta, \psi \odot \omega, \gamma) \in A^*) \wedge (\kappa \in \overline{A}) \\
&\rightarrow (\exists\zeta\{((\beta\kappa, \psi, \zeta) \in A^*) \wedge ((\zeta, \omega, \alpha) \in A)\}; \zeta \in \Omega) \wedge (\exists\eta\{((\beta, \psi, \eta) \in A^*) \wedge \\
&\qquad ((\eta, \omega, \gamma) \in A)\}; \gamma \in \Omega) \wedge (\kappa \in \overline{A})
\end{aligned}$$

$$\to \big(\exists\eta\{\exists\zeta\{((\beta\kappa,\psi,\alpha)\in A^*)\wedge((\alpha,\omega,\alpha)\in A)\wedge(\kappa\in\overline{A})\wedge((\beta,\psi,\eta)\in A^*)\wedge$$
$$((\eta,\omega,\gamma)\in A)\};\zeta\in\Omega\};\eta\in\Omega)$$
$$\to \big(\exists\eta\{\exists\zeta\{((\beta\kappa,\psi,\alpha)\in A^*)\wedge((\beta,\psi,\eta)\in A^*)\wedge(\kappa\in\overline{A})\wedge((\zeta,\omega,\alpha)\in A)\wedge$$
$$((\eta,\omega,\gamma)\in A)\};\zeta\in\Omega\};\eta\in\Omega)$$
$$\to \big(\exists\eta\{\exists\zeta\{(\zeta\eta^{-1}\in\overline{A})\wedge((\zeta,\omega,\alpha)\in A)\wedge((\eta,\omega,\gamma)\in A)\};\zeta\in\Omega\};\eta\in\Omega\big)$$
$$\to \big(\exists\eta\{\exists\zeta\{((\eta,\omega,\gamma)\in A)\wedge(((\eta.\eta^{-1})\alpha,\omega,\alpha)\in A)\wedge(\alpha\eta^{-1}\in\overline{A})\};\zeta\in\Omega\};\eta\in\Omega\big)$$
$$\to \big(\exists\eta\{\exists\zeta\{((\eta,\omega,\gamma)\in A)\wedge((\eta.(\alpha\eta^{-1}),\omega,\alpha)\in A)\wedge(\zeta\eta^{-1}\in\overline{A})\};\zeta\in\Omega\};\eta\in\Omega\big)$$
$$\to \big(\exists\eta\{\exists\zeta\{((\eta.(\zeta\eta^{-1}),\omega,\alpha)\in A)\wedge((\eta,\omega,\gamma)\in A)\wedge(\zeta\eta^{-1}\in\overline{A})\};\zeta\in\Omega\};\eta\in\Omega\big)$$
$$\to (\alpha\gamma^{-1}\in\overline{A})$$

Analogously for B^*, while for $\overline{C}$ we have that:

$$\vDash_\lambda (\alpha\gamma^{-1}\in\overline{C})$$
$$\to \big(\forall\eta\{\forall\zeta\{((\eta.(\zeta\eta^{-1}),\omega,\alpha)\in C)\vee((\eta,\omega,\gamma)\in C)\vee(\zeta\eta^{-1}\in\overline{C})\};\zeta\in\Omega\};\eta\in\Omega\big)$$
$$\to \big(\forall\eta\{\forall\zeta\{((\eta,\omega,\gamma)\in C)\vee((\eta.(\zeta\eta^{-1}),\omega,\alpha)\in C)\vee(\zeta\eta^{-1}\in\overline{C})\};\zeta\in\Omega\};\eta\in\Omega\big)$$
$$\to \big(\forall\eta\{\forall\zeta\{((\eta,\omega,\gamma)\in C)\vee(((\eta.\eta^{-1})\zeta,\omega,\alpha)\in C)\vee(\zeta\eta^{-1}\in\overline{C})\};\zeta\in\Omega\};\eta\in\Omega\big)$$
$$\to \big(\forall\eta\{\forall\zeta\{(\zeta\eta^{-1}\in\overline{C})\vee((\zeta,\omega,\alpha)\in C)\vee((\eta,\omega,\gamma)\in C)\};\zeta\in\Omega\};\eta\in\Omega\big)$$
$$\to \big(\forall\eta\{\forall\zeta\{((\beta\kappa,\psi,\alpha)\in C^*)\vee((\beta,\psi,\eta)\in C^*)\vee(\kappa\in\overline{C})\vee((\zeta,\omega,\alpha)\in C)\vee$$
$$((\eta,\omega,\gamma)\in C)\};\zeta\in\Omega\};\eta\in\Omega)$$
$$\to \big(\forall\eta\{\forall\zeta\{((\beta\kappa,\psi,\alpha)\in C^*)\vee((\zeta,\omega,\alpha)\in C)\vee(\kappa\in\overline{C})\vee((\beta,\psi,\eta)\in C^*)\vee$$
$$((\eta,\omega,\gamma)\in C)\};\zeta\in\Omega\};\eta\in\Omega)$$
$$\to (\forall\zeta\{((\beta\kappa,\psi,\alpha)\in C^*)\vee((\zeta,\omega,\alpha)\in C)\};\zeta\in\Omega)\vee(\forall\eta\{((\beta,\psi,\eta)\in C^*)\vee$$
$$((\eta,\omega,\gamma)\in C)\};\eta\in\Omega)\vee(\kappa\in\overline{C})$$
$$\to ((\beta\kappa,\psi\odot\omega,\alpha)\in C^*)\vee((\beta,\psi\odot\omega,\gamma)\in C^*)\vee(\kappa\in\overline{C})$$
$$\to ((\beta\kappa,\xi,\alpha)\in C^*)\vee((\beta,\xi,\gamma)\in C^*)\vee(\kappa\in\overline{C})$$

Thus, the result holds for $ord(\xi)=n$ and so, by applying the Principle of Induction, the forward implication has been established.

Since the reverse implication is straightforward, the proof is complete. □

Theorem 3.10. *Let $\mathbb{N}_{\tilde{\mathcal{A}}}=\langle\overline{A},\overline{B},\overline{C}\rangle$ be an implication-based neutrosophic kernel of the implication-based neutrosophic semiautomaton $\mathbb{NS}=(\Omega,\Delta,\mathbb{N}_{\mathcal{A}})$ over the finite group $(\Omega,\cdot)$. Then $\mathbb{N}_{\tilde{\mathcal{A}}}$ is an implication-based neutrosophic subsemiautomaton if and only if the following conditions are satisfied:*

(1) $\vDash_\lambda \big(((\epsilon,\xi,\alpha)\in A)\wedge(\epsilon\in\overline{A})\big)\to(\alpha\in\overline{A})$

(2) $\vDash_\lambda \big(((\epsilon,\xi,\alpha)\in B)\wedge(\epsilon\in\overline{B})\big)\to(\alpha\in\overline{B})$

(3) $\vDash_\lambda \big(\alpha\in\overline{C}\big)\to\big(((\epsilon,\xi,\alpha)\in C)\vee(\epsilon\in\overline{C})\big)$

for all $\alpha\in\Omega$ and $\xi\in\Delta$, where ϵ is the identity element of the finite group $(\Omega,\cdot)$.

Proof. Let $\mathbb{N}_{\tilde{A}}=\langle\overline{A},\overline{B},\overline{C}\rangle$ be an implication-based neutrosophic kernel of $\mathbb{NS}$.
Assume $\mathbb{N}_{\tilde{A}}$ satisfies:

(i) $\vDash_\lambda \big(((\epsilon,\xi,\alpha)\in A)\wedge(\epsilon\in\overline{A})\big)\to(\alpha\in\overline{A})$,

(ii) $\vDash_\lambda \big(((\epsilon,\xi,\alpha)\in B)\wedge(\epsilon\in\overline{B})\big)\to(\alpha\in\overline{B})$,

(iii) $\vDash_\lambda \big((\alpha\in\overline{C})\big)\to\big(((\epsilon,\xi,\alpha)\in C)\vee(\epsilon\in\overline{C})\big)$,

for all $\alpha\in\Omega$ and $\xi\in\Delta$.

Using these conditions, for A^*, we have:

$$\vDash_\lambda \big(((\alpha,\xi,\beta)\in A)\wedge(\alpha\in\overline{A})\big)\to\big((\beta\in\overline{A})\big).$$

Similarly, for B^*, we deduce:

$$\vDash_\lambda \big(((\alpha,\xi,\beta)\in B)\wedge(\alpha\in\overline{B})\big)\to(\beta\in\overline{B}).$$

Finally for $\overline{C}$, we conclude:

$$\vDash_\lambda \big((\beta\in\overline{C})\big)\to\big(((\alpha,\xi,\beta)\in C)\vee(\alpha\in\overline{C})\big).$$

Thus, $\mathbb{N}_{\tilde{A}}$ satisfies the definition of an implication-based neutrosophic subsemiautomaton.

Conversely, suppose $\mathbb{N}_{\tilde{A}}$ is an implication-based neutrosophic subsemiautomaton. Then:

$$\vDash_\lambda \big(((\epsilon,\xi,\alpha)\in A)\wedge(\epsilon\in\overline{A})\big)\to(\alpha\in\overline{A}),$$

$$\vDash_\lambda \big(((\epsilon,\xi,\alpha)\in B)\wedge(\epsilon\in\overline{B})\big)\to(\alpha\in\overline{B}),$$

$$\vDash_\lambda \big((\alpha\in\overline{C})\big)\to\big(((\epsilon,\xi,\alpha)\in C)\vee(\epsilon\in\overline{C})\big).$$

Thus, $\mathbb{N}_{\tilde{A}}$ satisfies the necessary conditions and the theorem is completely proved. □

4. **Conclusions**

This study has explored the extension of neutrosophic logic to the algebraic framework of finite groups through the introduction of implication-based neutrosophic subgroups. The analysis has focused on key properties of these subgroups, including their connection to the identity element and their behavior under group operations. Additionally, the concept of implication-based neutrosophic normal subgroups has been introduced, providing a systematic approach to studying these structures within group theory.

The results obtained generalize existing theories of fuzzy subgroups and demonstrate the capacity of neutrosophic logic to incorporate varying degrees of truth, indeterminacy, and

falsity into algebraic reasoning. By parametrizing these notions with λ, the proposed framework allows for a refined and adjustable characterization of subgroup properties, enriching the theoretical understanding of algebraic systems under uncertainty.

Potential directions for future research include the extension of these methods to more complex algebraic structures, such as rings and fields, as well as the development of computational techniques for their practical application. Furthermore, exploring the interaction between neutrosophic logic and other branches of mathematics may reveal additional insights and broaden the scope of this approach in both theoretical and applied contexts.

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Non-Destructive Detection of Fillet Fish Quality Using MQ135 Gas Sensor and Neutrosophic Logic-Enhanced System

M. Y. Shams[1], M. R. Darwesh[2,3], Roheet Bhatnagar[4], N. S. A. Al-Sattary[2], A. A. Salama[5], M. S. Ghoname[2]

[1] Deptartment of Machine Learning, Faculty of Artificial Intelligence, Kafrelsheikh University, Kafrelsheihk 33516, Egypt; mahmoud.yasin@ai.kfs.edu.eg,
[2] Department of Agriculture Engineering, Faculty of Agriculture, Tanta University, Tanta 31527, Egypt; mohamed.darwish@agr.tanta.edu.eg, salahnora94@gmail.com, mohamed.ghonaim@agr.tanta.edu.eg
[3] Department of Food Processing, Faculty of Industry and Energy, Samannoud Technological University, Egypt.
[4] Department of Computer Science and Engineering Manipal University Jaipur, Rajasthan, India, roheet.bhatnagar@jaipur.manipal.edu
[5] Department of Mathematics and Computer Science, Faculty of Science, Port Said University, Egypt, ahmed_salama_2000@sci.psu.edu.eg

***** Correspondence: Roheet Bhatnagar; Email: roheet.bhatnagar@jaipur.manipal.edu

Abstract: This paper demonstrates the feasibility of using an electronic nose to assess fish quality by analyzing air quality and examining volatile organic compounds (VOCs) alongside physical variables, with pH, protein content, and VOCs serving as chemical reference points. Artificial intelligence algorithms were employed to predict quality and calculate regression coefficients. Using the reference neural network algorithm based on chemical and physical compounds, regression coefficients (R values) achieved were 0.99, 0.98, and 0.97, respectively. Additionally, ANFIS (Adaptive Neuro-Fuzzy Inference System) produced R values of 0.99, 0.85, and 0.98. Both fuzzy logic and ANFIS proved effective for predicting fish quality. Image processing techniques, including histogram analysis, color mapping, and edge detection, were also applied to assess fish quality. To enhance the inference process, Neutrosophic Logic-Enhanced Fuzzy Logic Systems were utilized, addressing uncertainty and imprecision in fish quality assessment. Neutrosophic logic combines fuzzy logic's partial truth with indeterminacy, represented by three membership functions: truth, indeterminacy, and falsity. Neutrosophic fuzzy inference integrates steps like neutrosophication, rule evaluation, aggregation, and defuzzification, ensuring improved expressiveness and fidelity. For instance, neutrosophic fuzzy rules evaluated fish freshness and appearance to determine quality ratings such as poor, good, or excellent. This integration enhances decision-making by accurately modeling complex real-world uncertainties. These methods, combining electronic nose technology, artificial intelligence, and neutrosophic inference, provide a robust, non-destructive, and cost-effective approach to detecting spoilage in fillet fish.

Keywords: Adaptive Neuro-Fuzzy Inference System, Neural Networks, Fuzzy Logic, Neutrosophic Logic.

1. Introduction

Fish is one of the most traded foods in the world because it contains nutrients beneficial to human health. The chemical composition of fish consists of the following: Fish contains 18% to 24% lipoprotein, 2% calcium and contains 22 kg / 100 grams of phosphorous, it also contains soluble vitamins **[1]**. There are many factors that affect the proportions of the chemical composition of fish, including type of food, sex, seasonal differences, differences in the composition of parts of the same fish, such as the difference between white muscle and red muscle, and the difference on the right side from the left side **[2–4]**. There are several methods for assessing the freshness of fish, firstly, sensory methods, which include change in smell, gill color, texture, buoyancy test, flesh and skin consistency, eye color, and texture **[5]**. Most of the current organic gas sensors are considered as individual

machine learning methodologies and are considered as the best modern non-destructive methods for assessing freshness during fish storage period. A gas sensor was used to assess TVB-N of salmon. The correlation coefficient between freshness values and TVB-N gas sensor readings was determined to determine the sensitive gas sensor indicators. Linear models, ensemble machine learning, which includes the following models SVR, MLP, KNN, Gaussian model and decision tree were used to determine the freshness indicators by improving the analysis models in detecting the quality and freshness of salmon, which proves the effectiveness of sensors in detecting fish freshness **[6]**. There are also analytical methods for detecting the freshness of fish, such as estimation of VOCs, estimation of ammonia, estimation of sedimentation, selection of thiobarbutic acid, study of the pH concentration of the refractive index of eye fluids, total number of bacteria, and the test for total volatile nitrogen **[7,8]**. There are several factors that cause fish spoilage, including: Bacterial spoilage that occurs due to the presence of spoilage bacteria in the intestines of fish, and the degree of bacterial spoilage varies according to the temperature and the type of bacteria **[9]**. Bacteriological spoilage begins with the appearance of unpleasant odors, as a result of which a trimethylamine complex is formed, and then the formation of indole compounds, amines and hydrogen sulfide increases. The type of bacterial spoilage varies depending on several factors, the most important of which is the type of fish **[10,11]**. Red muscle fish are more prone to spoilage than white muscle fish because they contain a high percentage of hemoglobin and the condition of the fish when fishing in terms of the fullness of the stomach and the pressure on the fish, the temperature of transporting the fish after fishing, the type of pollution to which the fish are exposed. In terms of the percentage of pollution and mud in the water in it, as well as pollution, manual work in fishing, transporting and trading fish during and after fishing. There are several methods for preserving and processing fish, including fish cryopreservation, by reducing the temperature of fish to a point close to freezing, which leads to reducing biological and chemical reactions, which delays bacterial reactions **[12,13]**. Spoilage of fillet fish causes a change in color as a result of the change of myoglobin and hemoglobin responsible for the red-pink color to yellow and brown and a change in odor as a result of the release of volatile organic compounds, therefore, researchers tried to find new non-destructive detection methods for detection **[14]**. On the freshness of fish using artificial intelligence **[15]**, spectroscopic computer vision, electronic nose and NIR. Through the applications of artificial intelligence and machine learning algorithms, it is possible to classify foods and predict food quality **[16–18]**. Machine learning algorithms were also used to predict the quality of fish built on real time platform [19]. Artificial intelligence contain predictable algorithms such as The adaptive neuro-fuzzy inference system (ANFIS) and fuzy inference system that used to predict the quality of fish [20]. As for analytical detection methods, they are not suitable in fish factories because they are considered destructive detection methods that take time, effort and costs. Recently, the fish fillet industry has spread and the consumer's demand for eating fillet fish has increased, as it contains advantages represented in providing the process of washing and cleaning to the consumer, fish fillets can be cut as desired, longitudinal or transverse slices, which facilitates the processing process for the consumer. Certain weights can also be obtained, which facilitates the marketing work [21]. Spoilage of fish causes food poisoning to humans when eaten. In this study, three models were used, Back Propagation Neural Network (BP-ANN) [22], Adaptive Neuro Fuzzy Inference System (ANFIS) to determine the extent of correlation between the chemical variables represented in pH, protein values and reference TVB-N and the physical variables represented in image features and TVB-N reading from MQ135 gas sensors. Fuzzy Inference System (FIS) was used to predict the quality of fish fillets. Despite these advancements, the inherent uncertainty and imprecision in fish spoilage assessment present significant challenges. This study aims to address these limitations by integrating neutrosophic logic—a framework that models uncertainty using three membership functions: truth (T), indeterminacy (I), and falsity (F)—with predictive algorithms to enhance the reliability and robustness of fish quality evaluation systems.

The main contribution of this study is to use the gas sensor as a non-destructive way to detect the extent of freshness by measuring volatile organic compounds through:

1. Creating a suitable system for obtaining computer data and computer vision,
2. The use of predictive algorithms such as neural networks and ANFIS for the chemical variables represented in the acidity number, protein values, measurement of volatile organic compounds in the laboratory, and the physical variables represented in the variables in the characteristics of the images and the reading of the output from the gas sensor of the volatile organic compounds
3. Building a fuzzy logic model to predict fish quality, depending on the gas sensor reading of VOCs, reference VOCs, protein values, and pH number.
4. Neutrosophic logic enhances fish quality assessment by effectively modeling uncertainty and imprecision through its truth, indeterminacy, and falsity dimensions, improving decision-making accuracy.

The study involved several key steps, including sample preparation, chemical analysis of fish fillets, image acquisition using an image conception system, and VOC measurement via a gas sensor. Machine learning models, such as a Back Propagation Neural Network and an Adaptive Neural Fuzzy Inference System (ANFIS), were developed, evaluated, and refined using the ANFIS algorithm and alongside a fuzzy logic, and neutrosophic-logic approaches with an inference system to enhance accuracy. The related work are illustrated in Section 3, the methodologies are detailed in Section 3. Section 34 presents the results and discussion, while Section 5 concludes the study and highlights directions for future research.

2. Related work

The assessment of fish quality has witnessed significant advancements over the years, transitioning from traditional destructive techniques to modern non-invasive approaches **[23]**. Traditional methods rely on measuring chemical and physical attributes such as appearance, color, texture, odor, and taste **[19,24]**. While effective, these methods have notable drawbacks, including their destructive nature, high costs, time consumption, and the need for highly skilled operators. Consequently, researchers have focused on developing innovative non-destructive techniques such as biosensors, electronic sensors, and spectroscopic methods. These approaches minimize sample preparation, avoid destruction, and provide comprehensive data from a single test, making them ideal for online and at-line process controls in food quality assessment **[25]**. Machine learning algorithms have also played a pivotal role in advancing fish quality evaluation. Models like Back Propagation Artificial Neural Networks (BP-ANN) and Adaptive Neuro-Fuzzy Inference Systems (ANFIS) have been widely used to predict correlations between chemical variables (e.g., pH and protein content) and physical variables (e.g., VOC sensor readings and image features). These predictive models improve the reliability of quality assessments by analyzing large datasets and identifying intricate patterns in the data **[26]**.

However, the integration of neutrosophic logic into these frameworks has introduced a transformative approach to handling uncertainty, imprecision, and indeterminacy inherent in real-world conditions. Neutrosophic logic enhances traditional fuzzy systems by incorporating three membership functions: truth, indeterminacy, and falsity. This allows for a more comprehensive modeling of ambiguous scenarios, such as variations in fish spoilage conditions. For example, neutrosophic fuzzy rules can classify fish quality into categories like "poor," "good," or "excellent" with greater expressiveness and accuracy. By addressing data uncertainty, neutrosophic logic improves decision-making capabilities and ensures higher fidelity in fish freshness evaluation systems.

In one study **[27]**, a hydrogel-pH-electrode-based near-field passive volatile sensor was developed for real-time monitoring of fish spoilage. The sensor utilized a varactor-based LC resonator, which was interrogated via inductive coupling to detect changes in resonant frequency caused by total volatile basic nitrogen (TVB-N) in the fish headspace. This sensor exhibited a linear response to ammonia gas concentrations, with a detection limit of 0.001 mg/L (1.5 ppm). Trials on tilapia stored at 24 °C and 4 °C demonstrated that the sensor's readings correlated with bacterial growth patterns and accurately identified the spoilage threshold (107 cfu/g) for both storage conditions. This wireless sensor, designed for embedding in packaging materials, offers a cost-effective and scalable solution for real-time spoilage detection, leveraging advancements in printed electronics for mass production. Another notable study **[28]** employed Arduino-compatible MQ-series sensors (MQ3, MQ4, MQ5, MQ8, MQ9, and MQ135) to measure odor changes in trout, sea bream, and sea bass during storage. The odor intensity readings from the sensors were compared with microbiological and sensory data to establish spoilage thresholds. An electronic nose box, controlled by an Arduino microprocessor, was developed to rapidly assess the quality of 10 g fish samples. The system accurately identified "Fresh Fish" samples with total viable counts below 3 log CFU/g, highlighting the potential of Arduino-controlled odor sensors as a fast, simple, and low-cost solution for evaluating food quality.

3. Material and methods

3.1. Sample preparation

A 94 slices of fish fillet were purchased from the local market and divided into four sets: 24 slices for protein analysis, 24 slices for pH analysis, 24 slices for TVBN reference value analysis, and 24 slices for air quality analysis using the MQ135 gas sensor. The mass of each slice was 2.4 ± 0.48 g. The fish fillet sets were placed in sterile plastic bags and stored in a refrigerator at 4°C for 10 days. Six samples were analyzed on four different days (days 1, 4, 7, and 10) for each set.

3.2. Chemical analysis

The volatile organic compounds, protein, and pH number in fish fillets were estimated in the laboratories of the Food Technology Department at the Faculty of Agriculture, Kafr El Sheikh University, Egypt in march 2024. The pH was measured by a PH meter of fish samples during the storage period. The protein of fillet fish was estimated by the Kildahl method through the following steps: 2.2 grams of fish fillet was weighed after it is ground and placed in a beaker Kildahl capacity from 500 to 800 ml, then 15gm of anhydrous potassium sulfate powder are added, 25ml of concentrated sulfuric acid 93-98% free of nitrogen and 0.7g of mercury oxide free of nitrogen. The Kildahl beaker is heated in the Kildahl apparatus for 30 minutes until it boils, then it is left to cool down, then 200ml of water is added, then a quantity of mercury granules is added and then the beaker is placed on the distillation device, then 80ml of sodium hydroxide solution are added and then 5 to 7 drops of sodium are added. The methyl red guide and green bromocresol are dissolved in 0.16g of methyl red, then the distillation flask is shaken and heated well until the ammonia is volatilized and condensed in the receiving flasks, where at least 150ml of nitrogen is extracted and is equivalent to adding hydrochloric acid (LU et al., 2010). The Planck test is done, i.e. repeat the same previous steps without samples, then calculate the protein percentage through the following equations (1) and (2):

$$\%\ \text{of nitrogen} = \frac{(\text{volume of hydrochloric} \times \text{standard hydrochloric standard} \times 14.007 \times 100)}{(1000 \times \text{weight samples})} \quad (1)$$

$$\%\ \text{of protein in fillet fish} = \%\ \text{of nitrogen} \times (16/100) \quad (2)$$

The content of the volatile organic compounds in the fillet was measured during the storage period using the steam distillation method through the following steps: The samples were ground with a grinder 10 ± 0.1g of the ground fish samples were taken and placed in a beaker and 100 ml of distilled water were added for 30 minutes with shaking The beaker every 10 minutes, then the solution was filtered using filter paper, then 10ml of magnesium oxide was added to produce 5ml of the alkaline filter, then the resulting solution was distilled by steam distillation for 5 minutes, then 10ml of boric acid was added to absorb the resulting solution from the distillation flask with steam **[29]**. Titration was performed by adding 0.01 hydrochloric acid. TVB-N for fish was calculated by the following equation (3):

$$\text{TVB-N} = \frac{(v_1 - v_2) \times c \times 14}{m \times 5/100} \times 100 \quad (3)$$

where v_1 is size of the sample that was calibrated, v_2 is size without samples (Blank), c is actual concentration of hydrochloric acid, Sample weight of ground fish (g).

3.3. Image conception system

Images of fillet fish were taken during the four analysis days with a charge coupled device (CCD) camera. The images were calibrated by taking ten white reference images with a reflection rate of 99% and ten black reference images with a reflection rate of 0% using the following equation (4) **[30]**.

$$\text{M} = \frac{I_S - I_D}{I_W - I_D} \times 100\% \quad (4)$$

where M is the calibrated image, I_S is the raw image, I_D is the mean dark reference image, and I_W is the mean white reference image. The size of the captured images was (2534 ×2786). The median filter was used to remove noise before determine the Region of Interest (ROI). Region of Interest in images is defined as the portion of the image being processed where the pixel value is 1 and the remaining pixels are 0 (Cheng et al., 2015 and Taheri-Garavand et al., 2019). The size of ROI in the images was (364 ×246) using a binary mask. Edge deduction of image processing was used to determine the dashed curves as pixels of image to extract the features from the images through using five types of filters they are: Sobel, log, canny, Roberts, and Prewitt filter.

3.4. Measuring VOC from fillet fishes via gas sensor

Volatile organic compounds are among the most important indicators of the quality of fish meat, which result from the decomposition of basic chemical compounds due to the activity of bacteria and the conditions of keeping fish, which is the decomposition of protein into ammonia. Hydrogen sulfide, ethyl mercaptan, etc. Carbohydrates in hydrocarbons, alcohols, ketones, aldehydes and carboxylic acid gases hydrolyze lipids into aldehydes and aldehydes **[27,31–36]**. In this research, MQ135 gas sensor was used as a non-destructive method to detect the freshness of fish by determining the percentages of volatile organic compounds produced by the fish during the storage period Figure 1. The gas sensor is a metal (MOS) because the resistance changes according to the amount of gas produced of the article. The gas sensor specification is based on a DC voltage of 5V, and can identify the gas concentration for the MQ135 gas sensor VOCs in the range between 10 to 10,000 ppm. The MQ135 gas sensor measures air quality gases, which are represented in (ammonia, nitrogen alcohols, sulifide, and oxygen), which are considered among the most important volatile organic compounds that express the quality of fish.

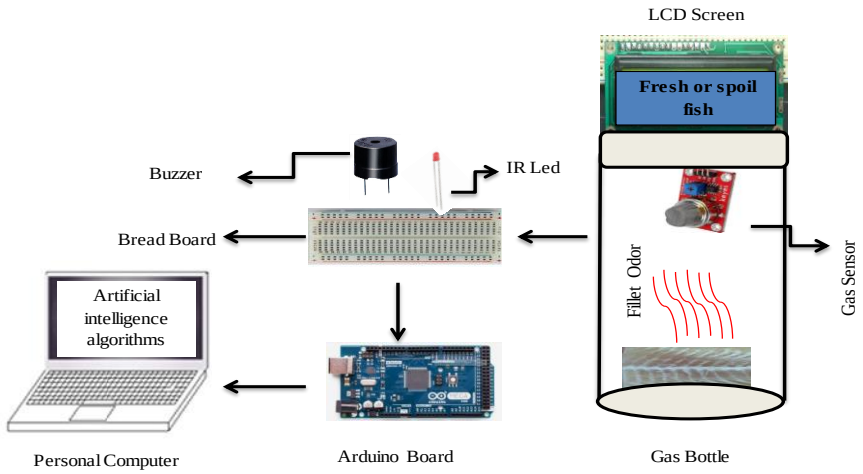

Figure 1. Scheme of gas sensor system for measuring VOCs of fish

The MQ135 gas sensor was placed in the gas bottle and readings were taken through three stages: The first stage is to calibrate the gases inside the bottle to ensure that there are no volatile organic components inside the bottle, and the second stage is to place 40 ± 0.5 grams of fish samples for 15 minutes until we obtain the gas values. VOC (Headspace Phase), by connecting the gas sensors to the massive Arduino 2560 board through four pins, A0 represents the voltage proportional to the gas concentration, VCC represents the voltage supplied to the sensor, GND is the ground pin of the sensor, and D0 represents the values of the gases produced. The gas sensor is set by the gas output voltage as an increase in gas concentration results in an increase in the output voltage which indicates an increase in VOCs and fish damage. The third stage: After taking a reading of each sample, the gas bottle is disinfected with water and air for ten minutes, and the previous two stages are repeated for each sample of fish at a temperature of 25 degrees Celsius. Humidity was kept at a level of 30±1% during the experiment. The gas sensor was set to determine the quality of fish by turning on a red infrared LED light, emitting a siren and reading the quality of the fish (fresh or spoiled) on the LCD screen when a concentration of volatile gases greater than 9000 ppm is reached, which are the air quality values after the second day of analysis. Designated as an indicator of fish spoilage.

3.5. Back Propagation Neural Network

The artificial neural network is one of the most important artificial intelligence algorithms that are used in prediction and classification, which simulate the way of human thinking. An artificial neural network consists of nodes, input layers and output layers. There are several types of neural networks, including Back Propagation - Artificial Neural Network (BP-ANN), Radial Basis Function - Neural Network (RBF-NN), and Self-Organizing - Neural Network (SO-NN) **[37]**.

Back Propagation Neural Network (BP-ANN) is a type of neural network that transmits information in the opposite direction to the original direction, and it is a type of machine learning under supervision where the network needs inputs **[38,39]**. The network is trained using forward propagation to obtain the predicted output value **[40]**. then the error value between the predicted output and the desired output is calculated, then the back propagation stage begins, in which the error value is calculated for each neuron in the network and at the end of the weights are updated and compensated with the new calculated values **[41]** as shown in Figure 2. One of the most important uses of BP-ANN in artificial intelligence are prediction and classification processes of the data **[42]**. Furthermore, it used in simulating human thinking, as it trains the network by adjusting

and updating the weights that express the interconnections between the layer and the layers that follow it, the network contains only one input layer, one hidden layer, and one of output layer, Figure 3.

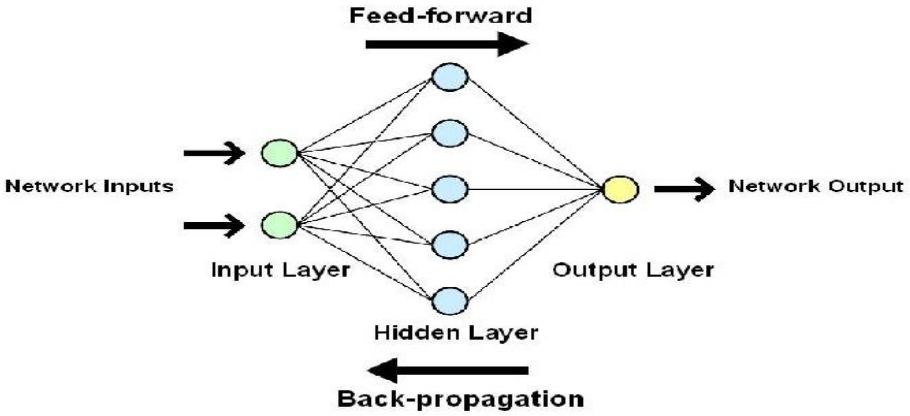

Figure 2. Architecture of BP-ANN

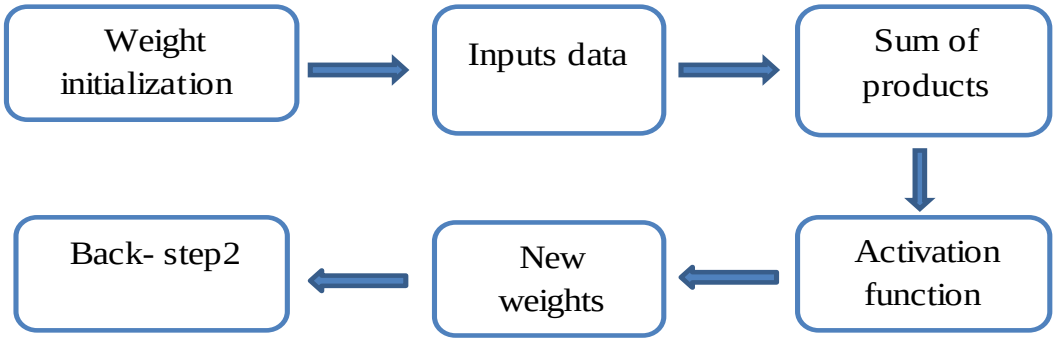

Figure 3. Flowchart of training steps of neural network

A back propagation neural network consists of a set of nodes, input layer, hidden layers, activation functions, and a set of weights that indicate the extent of the connection between communications and some of them **[43,44]**. The learning rate from 0 to 1. In this study, 2 hidden layers were used, and the number of neurons in each layer was 10 and 6 Figure 4, respectively. Number of epochs were of 1000. The learning rate of the network was set to 0.1, and the type of activation function were purline and tansiq in the two hidden layers.

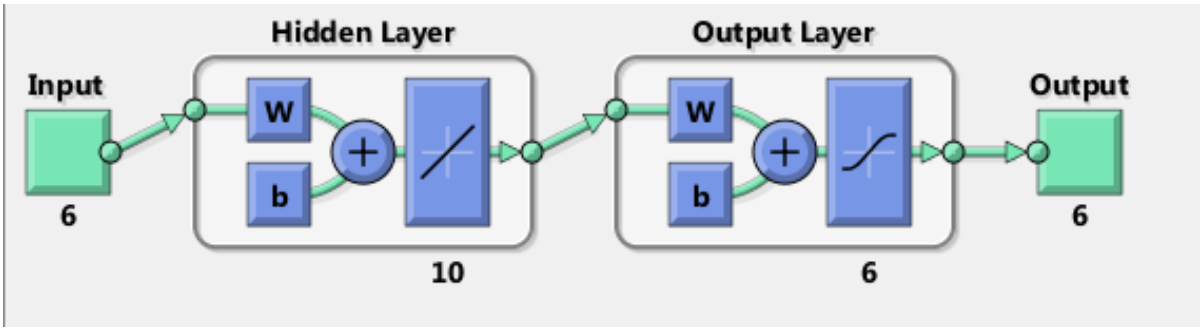

Figure 4. Type of activation functions of purline and tansiq that used in the hidden layer and output layer, respectively

3.6. The Adaptive Neural Inference System

The Adaptive Neural Inference System (ANFIS) is a system that combines the characteristics of artificial neural networks and fuzzy logic, where through this system it is possible to modify network parameters such as adjusting the number of neurons for the learning parameter and improving methods of fuzzy modeling through neural networks **[45,46]**. Figure 5 shows the components of the nervous system, which contains five layers, as illustrated by the following equations from (5) to (11). The first layer is called the "fuzzification layer", which includes the inputs and determines the type of membership function **[47,48]**. However the type of membership function were trimf (triangular function). ANFIS also determines the participation rates for the inputs.

$$\boldsymbol{\mu_{A_j}(x) = \max\left(\min\left(\frac{x-a_j}{b_j-a_j}, 1, \frac{c_j-x}{c_j-b_j}\right), 0\right)} \tag{5}$$

$$\mathbf{O_j^1} = \boldsymbol{\mu_{Aj}(x)}, \quad \text{for j = 1, 2\&3 for linguistic labels number of x} \tag{6}$$

where, $\{\boldsymbol{a_j, b_j, c_j}\}$, and $\{\boldsymbol{c_{mn}, \sigma_n^2}\}$ are the parameter sets of $\boldsymbol{\mu_{Ai}(x)}$, and $\boldsymbol{\mu_{f_{mn}}(z_n)}$, respectively. These parameters are called premise parameters. x and z are the inputs to the nodes, and $\boldsymbol{A_j, and\ f_{mn}}$ are the linguistic labels. The second layer is called "rule layer" determines the extent of the relationship between the rule and other rules, equation 12.

$$\boldsymbol{O_i^2} = \boldsymbol{w_i} = \boldsymbol{\mu_{A_j}(x).\mu_{\beta_k}(y)}, \quad \text{for} \quad \text{j = 1, 2, 3 and k = 1, 2, 3} \tag{7}$$

where

$\boldsymbol{i = j \times k = 9}$ rules = 1, 2, 3, ……., 9.

$\boldsymbol{i}$ = number of rules which is the product of number of states of inputs x and y.
$\boldsymbol{w_i}$ is the node output which represents the firing strength of a rule.
Sugeno fuzzy system was used to express rules in ANFIS model, equation 19.

$$\text{Rule } \boldsymbol{i}\text{: If x is } \boldsymbol{A_j} \text{ and y is } \boldsymbol{\beta_k}\text{, then } \boldsymbol{f_i} = \boldsymbol{p_i}\text{x} + \boldsymbol{q_i}\text{y} + \boldsymbol{r_i}, \tag{8}$$

where $\boldsymbol{A_j}$ and $\boldsymbol{\beta_k}$ are the fuzzy sets, $\boldsymbol{f_i}$ is the output, and $\boldsymbol{p_i}$, $\boldsymbol{q_i}$, and $\boldsymbol{r_i}$ are parameters sets that are used in the training process.

Third layer is called "defuzzification layer" This layer determines the ratio of the rule and other rules.

$$\boldsymbol{O_i^3 = \overline{w_i} = \frac{w_i}{w_1 + w_2 + \cdots + w_9}} \quad \text{for} \quad \boldsymbol{i} = 1, 2, \ldots..,9 \tag{9}$$

Four layer is called "normalization layer" this layer can add some additional data.

$$\boldsymbol{O_i^4 = \overline{w_i} * f_i = \overline{w_i}\ (\ p_i x + q_i\ y + r_i)} \tag{10}$$

Five layer is called "output layer" this layer can produce the desired final product.

$$\boldsymbol{O_i^5 = \sum_i \overline{w_i} * f_i = \frac{\sum_i \overline{w_i} * f_i}{\sum_i w_i}} \tag{11}$$

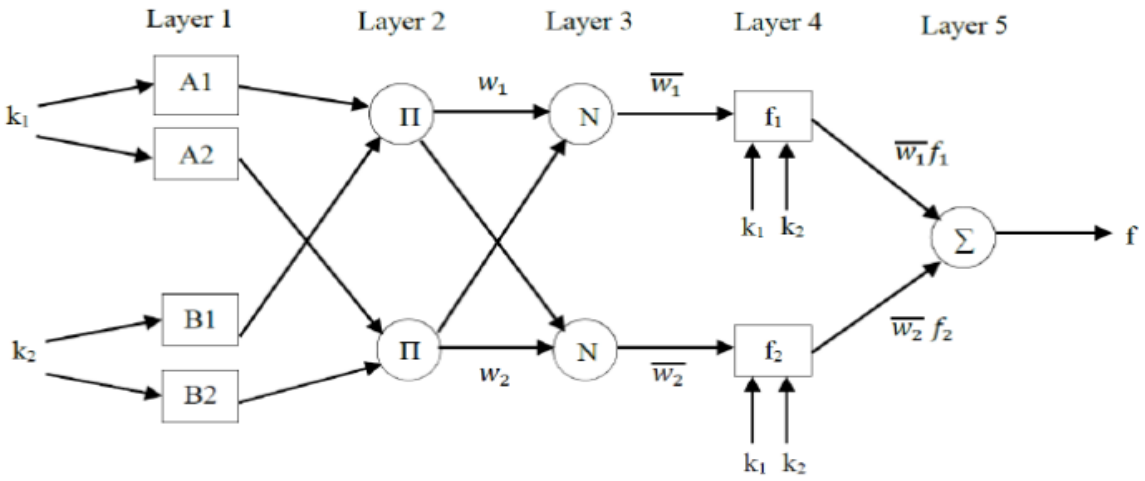

Figure 5. The architecture of ANFIS

3.7. Performance of ANFIS model

In this study, the hybrid nervous system was used with Takagi-Sugeno inference model, then the system parameters were adjusted. Members acceptance and rejection rate that determine the number of shares rejected **[49]**. Therefore, in order to calculate the regression coefficient in the ANFIS system, the previous transactions were determined and were as follows: influence radius was 0.3, maximum number of epochs was 200, error goal was 0, initial step size was 0.01, step size decrease rate was 0.9, and step size increase rate was 1.1.To reduce the Mean Square Error (MSE), neural fuzzy system parameters were set as follows, error tolerance was 0.005. Epochs was 100, range of influence was 0.5, squash factor was 1.25, accept ratio 0.5, and reject ratio was 0.15, the equation (12) of MSE as follows:

$$MSE = \frac{1}{n}\sum_{s=1}^{n}(y_j - \hat{y}_j)^2 \tag{12}$$

where n is number of value, s the absolute value of the residual, $\boldsymbol{y_j}$ is the original value, $\hat{y}_j$ is predicted value of y.

3.8. Building of machine learning model by ANFIS algorithm

The ANFIS system relies on three main steps: logical operations, membership functions, and IF-THEN rules **[32,50,51]**. Figure 6 illustrates the structure of the ANFIS model for the three different models as follows: contains one input representing in chemical analysis (protein values, pH number and TVB-N reference values) and one output that representing in physical analysis (air quality values produced from the MQ135 Gas sensor), number of structure was 3 mf . ANFIS algorithm was also used to predict quality of fish the inputs were (protein values, pH number, TVB-N reference values and air quality values produced from the MQ135 Gas sensor) and the output was quality of fish. The number of IF then-rules as the number of nodes equals 3, and 81 node, respectively, which is the product of number of states of inputs x, y and z etc., as shown in Figure 6.

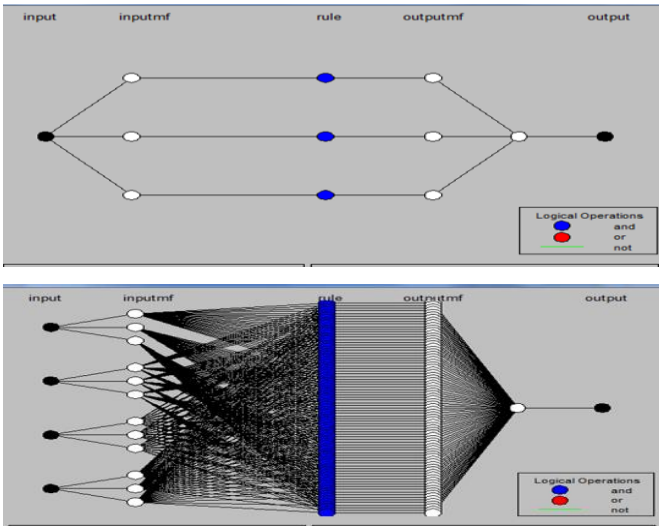

Figure 6. Structures of ANFIS to predict chemical values from physical values and quality of fillet fishes

3.9. Neutrosophic Fuzzy logic inference system

The traditional logic or Aristotle logic assumes it is impossible for the element to belong to the class and to contradict it at the same time, but fuzzy logic assumes the element belongs to a certain class, but to a certain degree **[52–54]**. Membership function, that determines degree the element that belongs to a certain category, such as (high, medium, and low), and the inference step, through which the ambiguous outputs resulting from the application of certain rules are known, Figure 7. In this study, fuzzy logic was applied as an application of artificial intelligence algorithms in determining the quality of fillet fish. The type of membership function was used, trimf as shown by the following equation:

$$f(x,a,b,c) = \max\left(\min\left(\frac{x-a}{b_j-a}, 1, \frac{c-x}{c-b}\right), 0\right) \tag{13}$$

where a, b, c, and d are the parameter set of the trapezoidal and triangle of membership function.

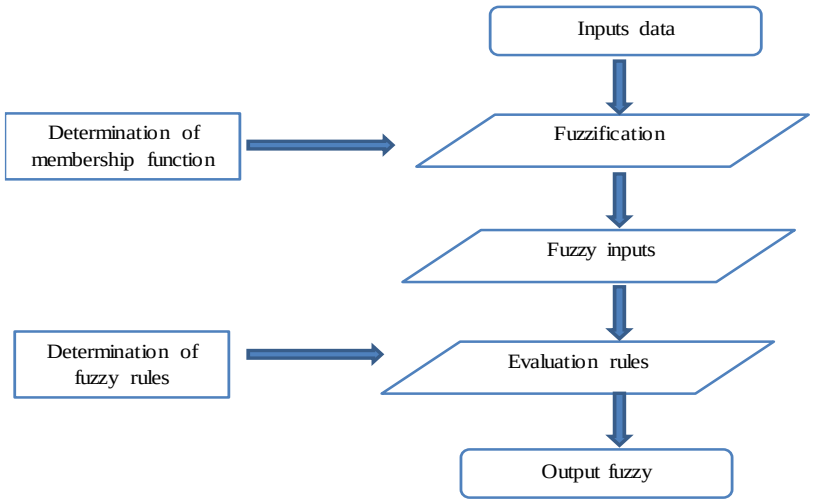

Figure 7. Steps of Fuzzy-logic inference system

Figure 8 illustrates a structured process for handling fish data using neutrosophic logic, which is designed to deal with uncertainty, imprecision, and incomplete information. The steps are as follows:

1. Input Dataset (x): The process begins with an input dataset, denoted as x which contains raw data to be analyzed.

2. Neutrosophication: This step involves transforming the input dataset into a neutrosophic representation. This process converts the data into a format compatible with neutrosophic logic by considering three aspects: truth (T), indeterminacy (I), and falsity (F).

3. Neutrosophic Inputs: The transformed dataset is represented as neutrosophic inputs, which consist of three components: T (truth degree), F (falsity degree), and I (indeterminacy degree).

4. Truth (T), Falsity (F), and Indeterminacy (I): The neutrosophic inputs are split into their respective components. These components represent the degrees of truth, falsity, and indeterminacy for each element in the dataset.

5. Inference Engine: Using the T, F, and I components, the inference engine applies rules to derive meaningful conclusions. These rules are defined as T(x), F(x), and I(x).

6. Deneutrosophication: After the inference process, the neutrosophic representation is converted back into a standard format. This step, known as deneutrosophication, produces actionable results by consolidating the T, F, and I components.

7. Output (Y(T,F,I)): Finally, the process yields an output, Y(T,F,I), Y(T,F,I), which represents the result of the analysis based on the neutrosophic framework.

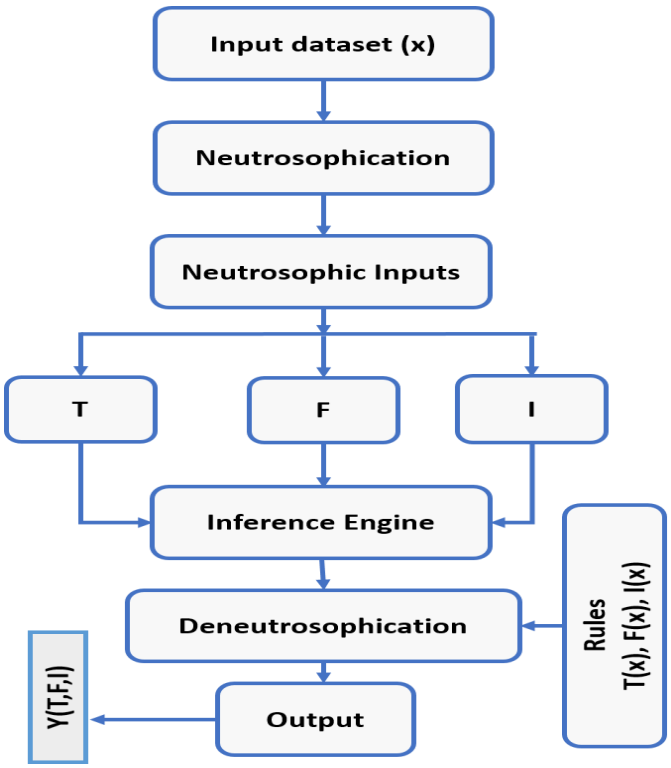

Figure 8. Steps of Neutrosophic-logic inference system

4. Results and dissuasions

3.1. Chemical and physical analysis variance

The values of the chemical parameters of the fish were measured experimentally, that represent in the protein values, pH number, TVB-N, using the Kildahl method, PH meter, and steam distillation method as a reference measure to predict the quality and viability of the fish, respectively. 72 fish slices were used to perform three different chemical analyzes and 24 fish slices were also used to do physical analysis during 10 days of storage. Table 1 shows the change in fish protein content, pH number TVB-N and air quality appeared from the second analysis day. When calculating the lowest significant difference (LSD) for differences between the averages at the level of significance is 0.05, it was found the value of ($p < 0.05$), that indicates there are significant differences between the mean values of the different variables. As a result of the proliferation of bacteria that cause spoilage of cooled fish, such as Pseudomonas, Achromobacter, Flavobacterium, they feed on the protein found in fish, which grows in neutral pH level, which results in volatilization of volatile organic compounds, which is also consider the important indicator of fish spoilage. The mean ± standard deviation of the calculated values of protein and pH count TVB-N for fillet fish fillets, at 0.05 significance.

Table 1. Change variance between chemical and physical sample during analysis days

fish samples	Analysis days			
	1-day	4-day	7-day	10 -day
Protein	0.615377 ±0.784	25.30667 ±. 626^{a}	23.95333 ±0.3126^{b}	23.3687 ±0.2^{c}
pH	6.45±0.297	6.456±0.053^{a}	6.238±0.089^{b}	5.5998 ±0.324^{c}
TVB-N	3.89±.4	6.44±1.04^{a}	23.92±1.18^{b}	30.2754 ±1.60^{c}
Air quality	1736.333± 275.5196	5405.833 ±886.03^{a}	12929 ±2689.12^{b}	26755.83 ±4270.39^{c}

3.2. Extracting image features of fillet fish by image processing

Figure 9 shows the jet color map of fish fillets, which shows how much fish change by color, as fish spoilage increases, from Reddish-pink color to pale pink yellow-gray and finally to brown.

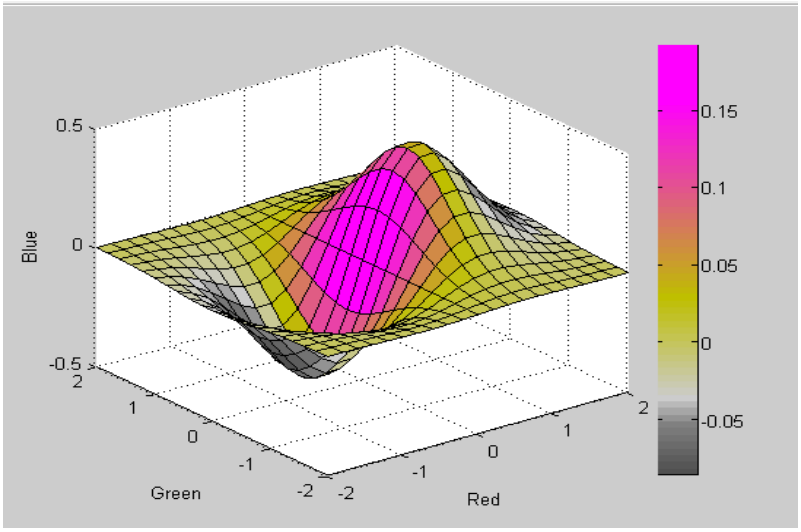

Figure 9. RGB of color bar of fillet fish during analysis

Figure 10 also shows a grayscale histogram, which shows the color gradients of images of fish fillets during the spoilage stages. The histogram consists of two axes, the first is the horizontal axis of three regions as follows: the first region represents the light color, the second region represents the medium color in the color, and the third region represents the dark color. As a result, by increasing spoil in fillet fish the yellow color increase in color map, and the number of frequencies in the right part that represents the light area decreases after the second day of analysis, the second axis is the vertical axis, which represents the size that expresses the number of pixels in each of the previous three parts. Figure 11 shows the relation between chemical and physical analysis during the four analysis days for fillet fish.

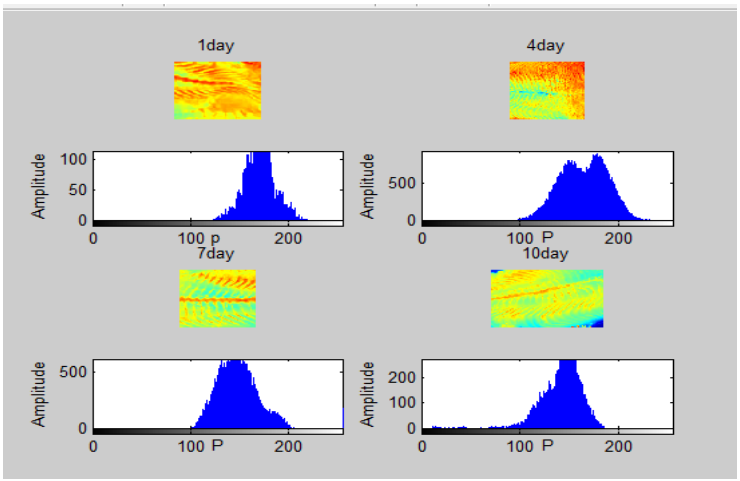

Figure 10. Color map and gray histogram of collective fillet image during analysis days

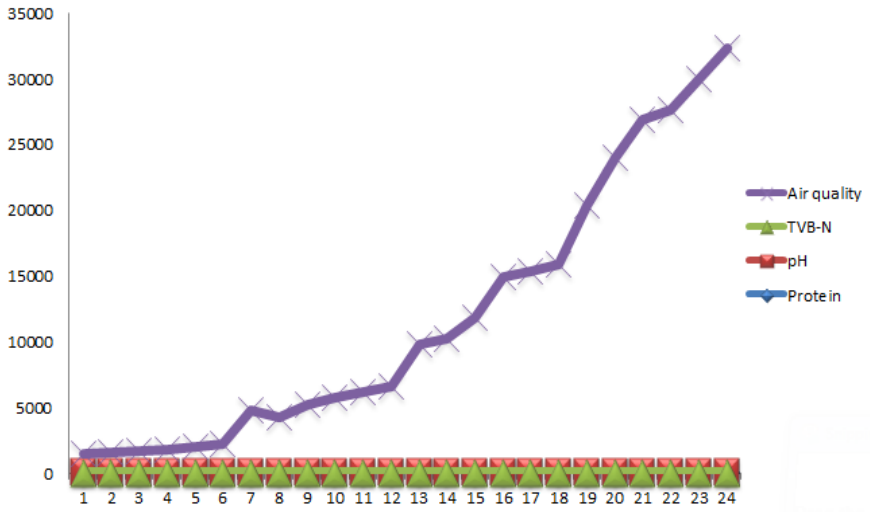

Figure 11. Charts between chemical and physical analysis during storage days

Figure 12 shows the four fillet images were used during the four analysis days to determine features from edge deduction, when the spoil of fillet increases the noise or pixels decreased in all filters in edge deduction that used.

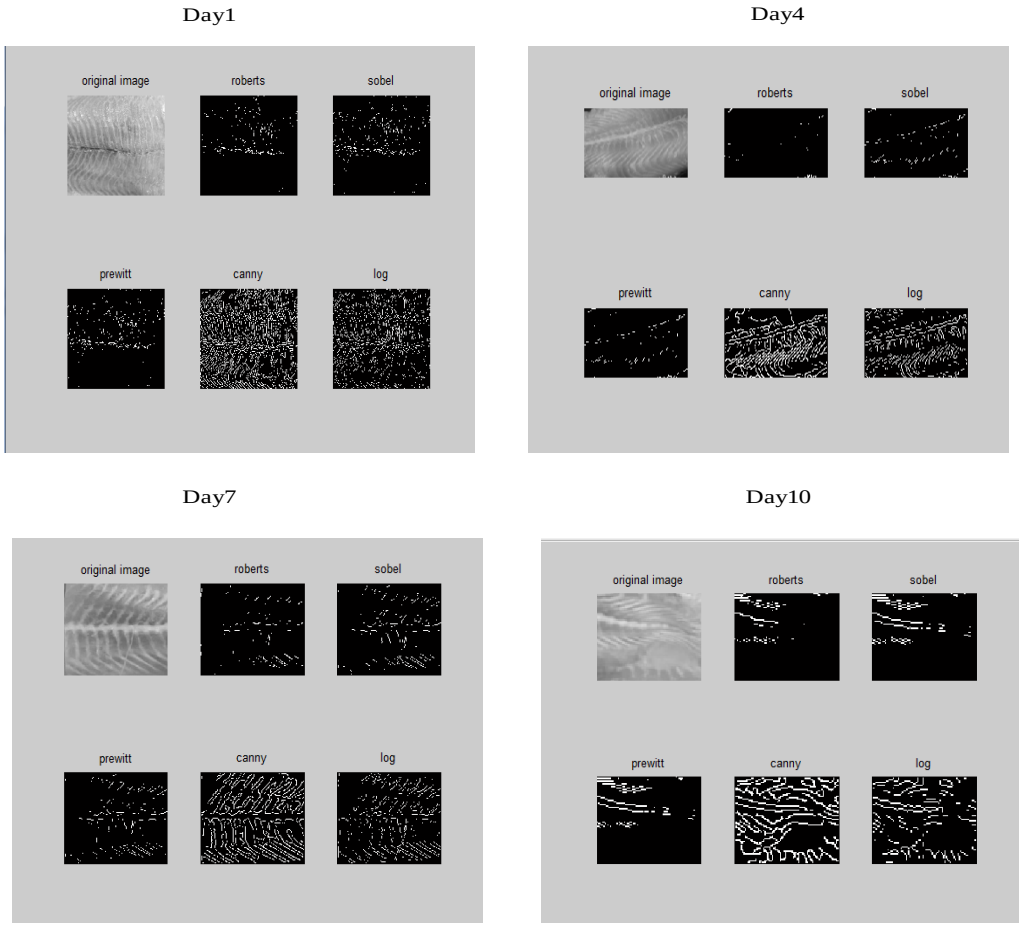

Figure 12. Digital image histograms for fillet fishes slice during different analysis days

3.3. Machine learning model BP-ANN-based for fillet fish inspection

The BP-ANN network was used for the three different prediction models to predict protein values, pH number, and the reference TVB-N from the inputs values that represent in air quality from MQ135 gas sensor for the fillet images. It was found that high regression coefficient between chemical analysis (protein, pH, and the reference TVB-N) and physical components (air quality value) in fillet through spoilage stages as shown in Table 2.

Table 2. Performance coefficients of prediction models (BP-NN)

Model No.	**Variables**		**Number of samples**	**Samples set**	
	Inputs	**Outputs**		**MSE**	**R**
1	Air quality from gas sensor	pH	48	4.15	0.999
2	Air quality from gas sensor	Protein	48	1.96	0.973
3	Air quality from gas sensor	TVB-N	48	2.02	0.987

3.4. Machine learning model ANFIS-based for fillet fish inspection

The ANFIS model was built in order to calculate values of regression coefficient and the Root Mean Square Error (MSE) to predict the protein values, the acidity number, and the reference TVB-N values for the three models that represent and the TVB-N values produced from the Gas sensor, as shown in Table 3.

Table 3. Performance coefficients of prediction models (ANFIS)

Model No.	**Variables**		**Number of samples**	**Samples set**	
	Inputs	**Outputs**		**MSE**	**R**
1	Air quality from gas sensor	pH	48	1.94	0.998
2	Air quality from gas sensor	Protein	48	3.06	0.859
3	Air quality from gas sensor	TVB-N	48	.59	0.999

Each of the inputs in the chemical analysis (protein values, pH and TVB-N reference values) has 3 different grades (high, medium and low), and the TVB-N generated by the gas sensor (air quality) also has 3 grades of linguistic nomenclature (high, medium and low). It was organic type trim. Figure 13 shows the fish quality value represented by the ANFIS model and which depends on the inputs (protein values, pH, TVB-N reference values and air quality values produced from the MQ135 gas sensor). IF rules were also generated in ANFIS as shown in Table 4 and Figure 14. Each value of the quality predictor was multiplied by % in order to obtain the output as a percentage.

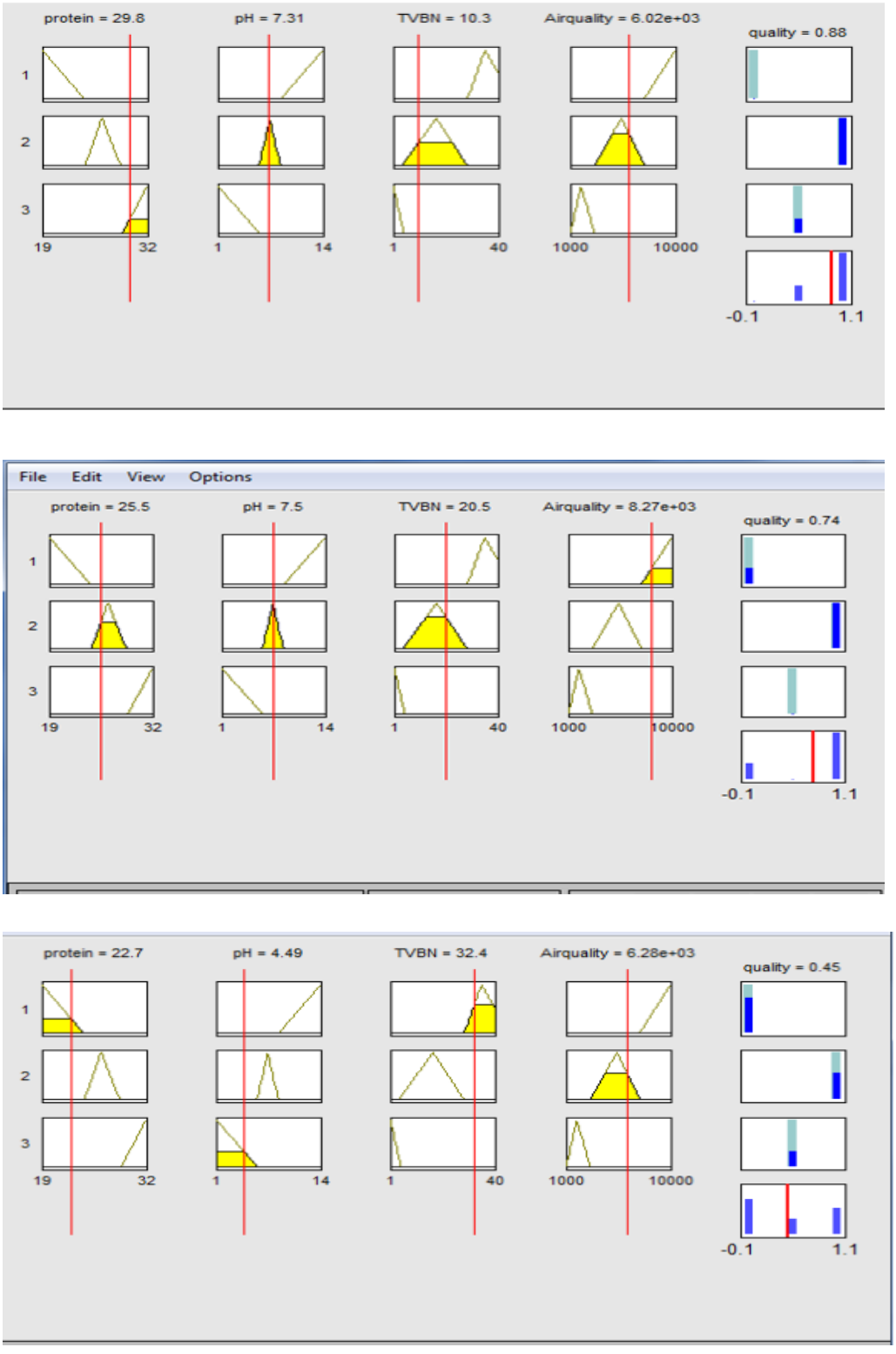

Figure 13. Quality of fillet fish in three different cases (fresh, acceptable and spoil)

3.5. Estimating quality of fillet fish via fuzzy logic system

Previous two models: BP-ANN, and ANFIS were used to predict protein values, PH, and reference TVB-N, but there is still a need to use a model to predict fish quality. The fuzzy logic model was built

based on Mamdani system to predict fish quality. The inputs were chemicals analysis (pH, protein and TVBN) and physical analysis (air quality from MQ135 gas sensor). Type of membership function for input and outputs were trimf,. Three categories of inputs were (high, acceptable low, acid, neutral and alkaline) and the categories of outputs were (fresh, acceptable and spoiled). Figure 15 and Table 4 illustrate if then- rules were used to determine the quality of fillet fish.

Table 4. If-Then rules for the proposed scenario.

No.	Rules
1	IF 'protein' IS 'high' AND 'pH' IS ' acid' AND 'TVBN' IS 'low' AND 'air quality' IS ' low' AND THEN 'quality' IS 'fresh'.
2	IF 'protein' IS 'acceptable' AND 'pH' IS ' neutral' AND 'TVBN' IS 'acceptable' AND 'air quality' IS 'acceptable' AND THEN 'quality' IS 'acceptable'.
3	IF 'protein' IS 'high' AND 'pH' IS 'acid' AND 'TVBN' IS 'low' AND 'air quality' IS 'low' AND THEN 'quality' IS 'spoil'.

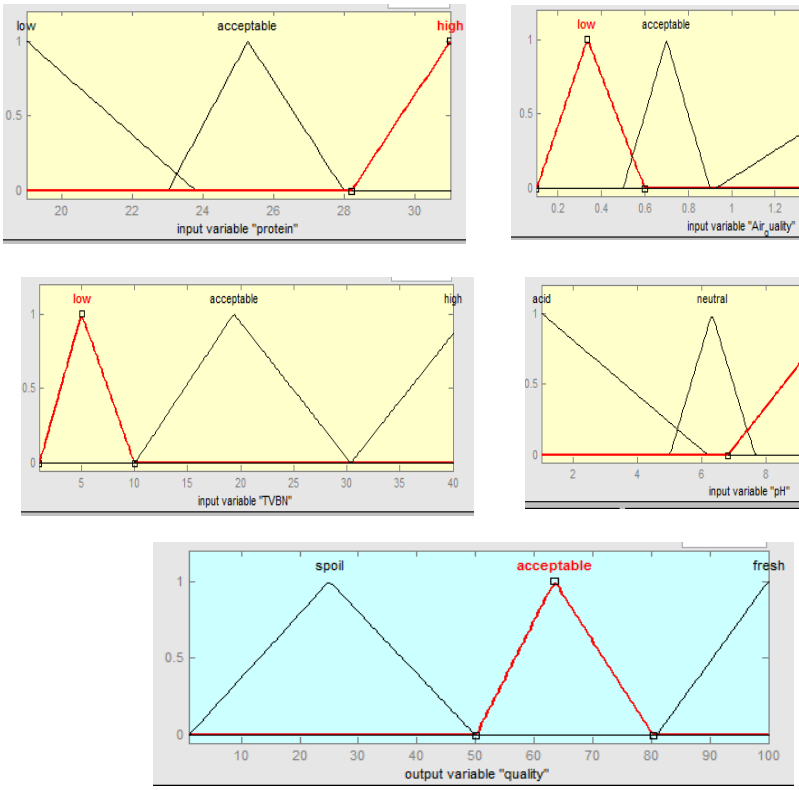

Figure 14. Membership function of the inputs values and output: pH, TVB-N, protein and air quality from gas sensor, and quality

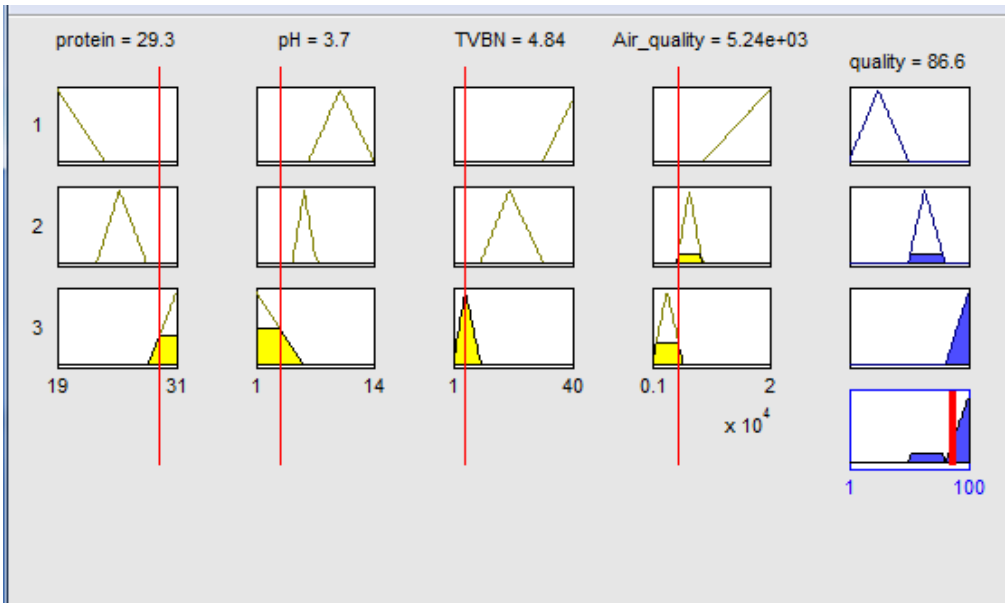

Figure 15. Predicted fish quality from if then- rules viewer

4.6. Neutrosophic Logic-Enhanced Fuzzy Logic Inference Systems

Neutrosophic Logic-Enhanced Fuzzy Logic Inference Systems combine the strengths of fuzzy sets and neutrosophic sets to address uncertainty and imprecision from different perspectives [47]. Fuzzy logic focuses on partial truth by assigning membership values between 0 and 1, while neutrosophic logic incorporates truth, indeterminacy, and falsity through three distinct membership functions [55]. This integration allows for a more comprehensive representation of complex real-world issues. Neutrosophic fuzzy sets extend traditional fuzzy sets by including indeterminacy, enabling richer behavior in uncertain environments [56]. The inference process involves neutrosophication (converting crisp inputs into neutrosophic fuzzy sets), rule evaluation using neutrosophic fuzzy rules, aggregation of rule outcomes, and defuzzification to produce crisp outputs [57]. For instance, assessing fish quality based on freshness and appearance utilizes neutrosophic fuzzy rules to determine quality ratings (e.g., poor, good, excellent). Neutrosophic values, expressed as triplets (truth, indeterminacy, falsity), capture uncertainty effectively, such as representing medium freshness as (0.7, 0.2, 0.1), indicating high truth, some indeterminacy, and low falsity as shown in Table 5. This approach improves expressiveness, fidelity, and decision-making in uncertain scenarios, despite challenges like computational complexity, data collection, and standardization.

These illustrative values would need to be adjusted based on specific data and technical expertise, as some definitions of neutrosophic sets do not require that T + I + F equals 1. This structure serves as the foundation for encoding neutrosophic concepts within the fish quality assessment as shown in Figure 16. The chart suggests that the fish is likely to have medium freshness and a good appearance, with some level of uncertainty regarding the appearance.

In practical implementation, these values would be integrated into the neutrosophic fuzzy rule-based system. Neutrosophic logic, as a generalization of fuzzy logic, addresses typical limitations such as representing incomplete or contradictory information, while fuzzy logic inference systems provide a robust framework for reasoning under uncertainty. By combining these approaches, this strategy can deliver stronger and more reliable solutions.

Table 5. Neutrosophic Values for Input Variables in Fish Quality Assessment

Input Variable	Linguistic Term	Truth (T)	Indeterminacy (I)	Falsity (F)
Freshness	Low	0.1	0.2	0.7
Freshness	Medium	0.7	0.2	0.1
Freshness	High	0.9	0.05	0.05
Appearance	Good	0.8	0.1	0.1
Appearance	Fair	0.5	0.3	0.2
Appearance	Poor	0.1	0.1	0.8
Quality	Very Poor	0.1	0.1	0.8
Quality	Poor	0.2	0.2	0.6
Quality	Okay	0.5	0.3	0.2
Quality	Good	0.7	0.2	0.1
Quality	Excellent	0.9	0.05	0.05

Further research should focus on exploring alternative neutrosophic aggregation operators and defuzzification methods, designing efficient algorithms for inference with neutrosophic fuzzy information, and applying neutrosophic fuzzy systems to real-world problems across various domains. Neutrosophic fuzzy logic offers powerful tools for developing new rule classes and gaining insights into the uncertain, gray, and aleatory aspects of complex systems, helping to characterize their behaviors, events, and outcomes when integrated into decision-making frameworks.

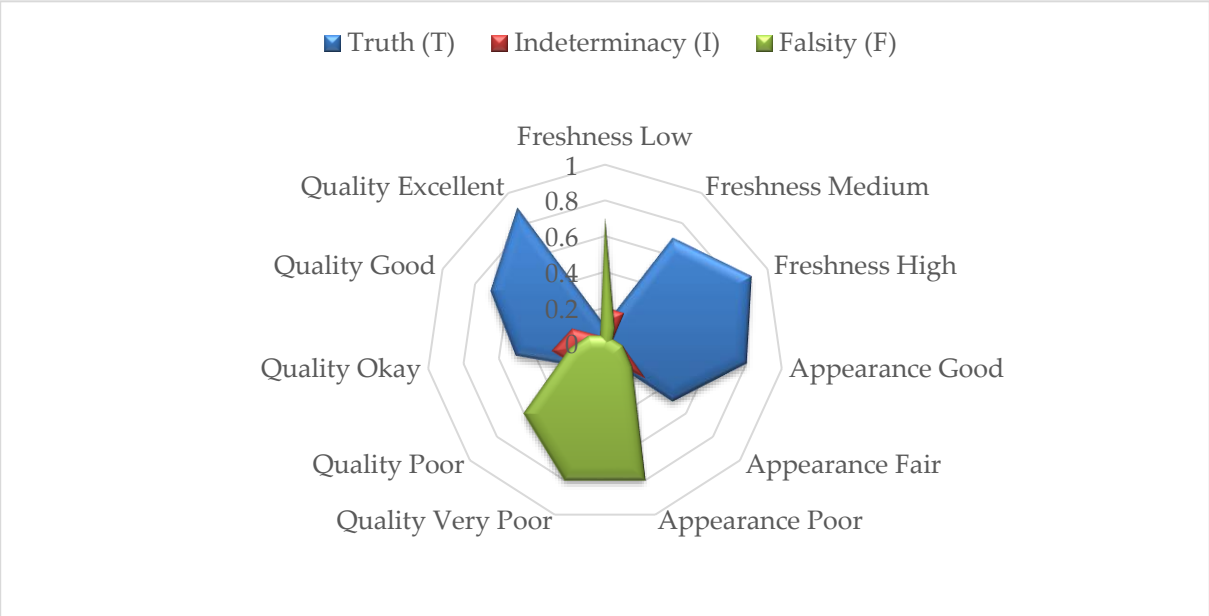

Figure 16. Neutrosophic Logic-Enhanced Fuzzy Evaluation of Fish Quality

5. Conclusions

This study demonstrates the effectiveness of combining electronic nose technology, artificial intelligence algorithms, and neutrosophic logic to assess fish quality. Two predictive models were used: a neural network model and the Adaptive Neuro-Fuzzy Inference System (ANFIS), based on chemical and physical analyses, including protein levels, pH, and TVB-N values, as well as data from an MQ135 gas sensor. The neural network model achieved the highest regression coefficients, indicating its superiority for predicting fish quality. Both fuzzy logic and ANFIS models proved to be reliable, non-destructive methods for assessing fish quality. Image processing techniques, including color mapping and edge detection, were also applied to visualize spoilage stages. The introduction of neutrosophic values for input variables such as freshness and appearance added further depth to the analysis by incorporating dimensions of truth, indeterminacy, and falsity, improving the accuracy of the quality assessment. Neutrosophic fuzzy inference processes, including neutrosophication, rule evaluation, and defuzzification, enhanced expressiveness and fidelity, allowing for a more precise representation of real-world uncertainties in the evaluation of fish quality. Ultimately, the integration of these technologies provided a robust, cost-effective, and non-destructive solution for detecting spoilage in fillet fish, demonstrating the potential for advanced decision-making in food quality assessment.

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Quality Evaluation in Hospital Operations Management Using Multi-Criteria Decision-Making with Plithogenic Approach to Address Uncertainty and Enhance Decision-Making Processes

Meng Zhang [1], Yafang Li[2]*
[1]Jining Yucai High School, Jining, 272100, Shandong, China
[2]Jining Medical University, Jining, 272100, Shandong, China
*Corresponding author, E-mail: liyafang@mail.jnmc.edu.cn

Abstract: Coordination of medical facilities' operations depends on operations management. Hospitals must function effectively to meet the medical demands of their patients, who get critical and emergency treatment. The decision support system for Quality Evaluation in Hospital Operations Management is created in this study. Plithogenic sets and multicriteria decision-making (MCDM) form the foundation of this decision assistance system. The study presents a novel approach to evaluating and rating Hospital Operations Management. Alternatives are ranked using the plithogenic reference-based normalization alternative ranking (RBNAR) technique. Thus, the algorithm for plithogenic-RBNAR is constructed and presented. An analysis of Hospital Operations Management is conducted using the new hybrid decision support model. The plithogenic-RBNAR hybrid method's applicability is supported by the research findings. The case study and sensitivity analysis results support the plithogenic-RBNAR hybrid model's robustness and applicability.

Keywords: Uncertainty; Multi-Criteria Decision Making; Hospital Operations Management; Decision making.

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1. Introduction

The term "hospital operations management" describes the variety of administrative tasks required to maintain hospital operations. Budgeting, staffing, employee issues, services offered, care quality, and hospital policies are all included[1], [2]. To maintain a hospital's operations and guarantee that its patients receive the treatment they require, operations management specialists are crucial. They frequently collaborate to complete the various administrative duties necessary to maintain the hospital's operations[3], [4]. A hospital's operations management department handles a variety of duties pertaining to managing the facility's daily operations. Among their duties are the following: Budget optimization, grant and other funding applications, financial goal-setting, and strategy implementation[5], [6]. Developing regulations that facilitate operations and the delivery of high-quality care-making sure the hospital complies with all legal requirements, including making sure all healthcare professionals participate in professional development and update their licenses on a regular basis Establish collaborations with other nearby medical establishments to offer community care. Managing the hospital's workforce requirements and carrying out duties like hiring, educating, and acclimating new hires Examining processes and revising

tactics to increase service effectiveness Quality[7], [8]. Evaluation in Hospital Operations Management is a MCDM issue[9], [10].

1.1 Motivation of this study

There are different primary components to the system for Quality Evaluation in Hospital Operations Management that was created in this study. The first section uses plithogenic sets to determine weighting for the efficiency level of experts in the decision-making process. plithogenic sets, which express the membership, non-membership, and indeterminacy degrees of set constituents, offer a useful method for quantifying qualitative evaluations. This makes it possible to lower the level of uncertainty in expert assessments. Additionally, plithogenic sets are excellent for capturing qualitative uncertainties. A novel reference-based normalization alternative ranking (RBNAR) technique is created to rate alternatives[11], [12]. The RBNAR method enables the computation of values based on their separations from the optimal point[13], [14].

1.2 Aims of this study

Linguistic expressions used by experts to assess criteria are converted into plithogenic sets for computing the criteria weights. For reference values. The RBNAR approach for ranking the options is introduced in the study. Using both Z-score normalization and Aytekin's reference-based normalization, this approach integrates reference-based normalization procedures. All these techniques are then used in the study to create the plithogenic-RBNAR hybrid methodology.

1.3 Contributions of this study

The study adds a new approach to Quality Evaluation in Hospital Operations Management. Based on the research findings, the conclusions for hospital managers are also examined. In the end, this study offers insights into plithogenic sets, introduces the RBNAR method for ranking Hospital Operations Management. All these elements are combined into a hybrid decision support system for evaluating Quality Evaluation in Hospital Operations Management.

This study offers a comprehensive and cutting-edge decision support system that may be used by experts and decision makers.

RBNAR method for ranking alternatives. The creation of the RBNAR system for evaluating and ranking alternatives is a notable contribution. This approach offers a thorough ranking solution by combining two reference-based normalizing techniques: Z-score normalization and Aytekin's reference-based normalization.

Robustness testing: By conducting thorough sensitivity studies on the case study findings and the plithogenic-RBNAR hybrid approach, this study goes beyond theoretical validation. The results of these investigations support the hybrid model's resilience and dependability.

1.4 Organization of this study

There are five sections in this study. The approach is described in Section 2, which also includes details on plithogenic sets and the steps and algorithm of the plithogenic-RBNAR hybrid model. In Section 3, the MCDM issue is evaluated through a case study. The sensitivity analysis results are presented in Section 4. Section 5 shows the conclusions of this study.

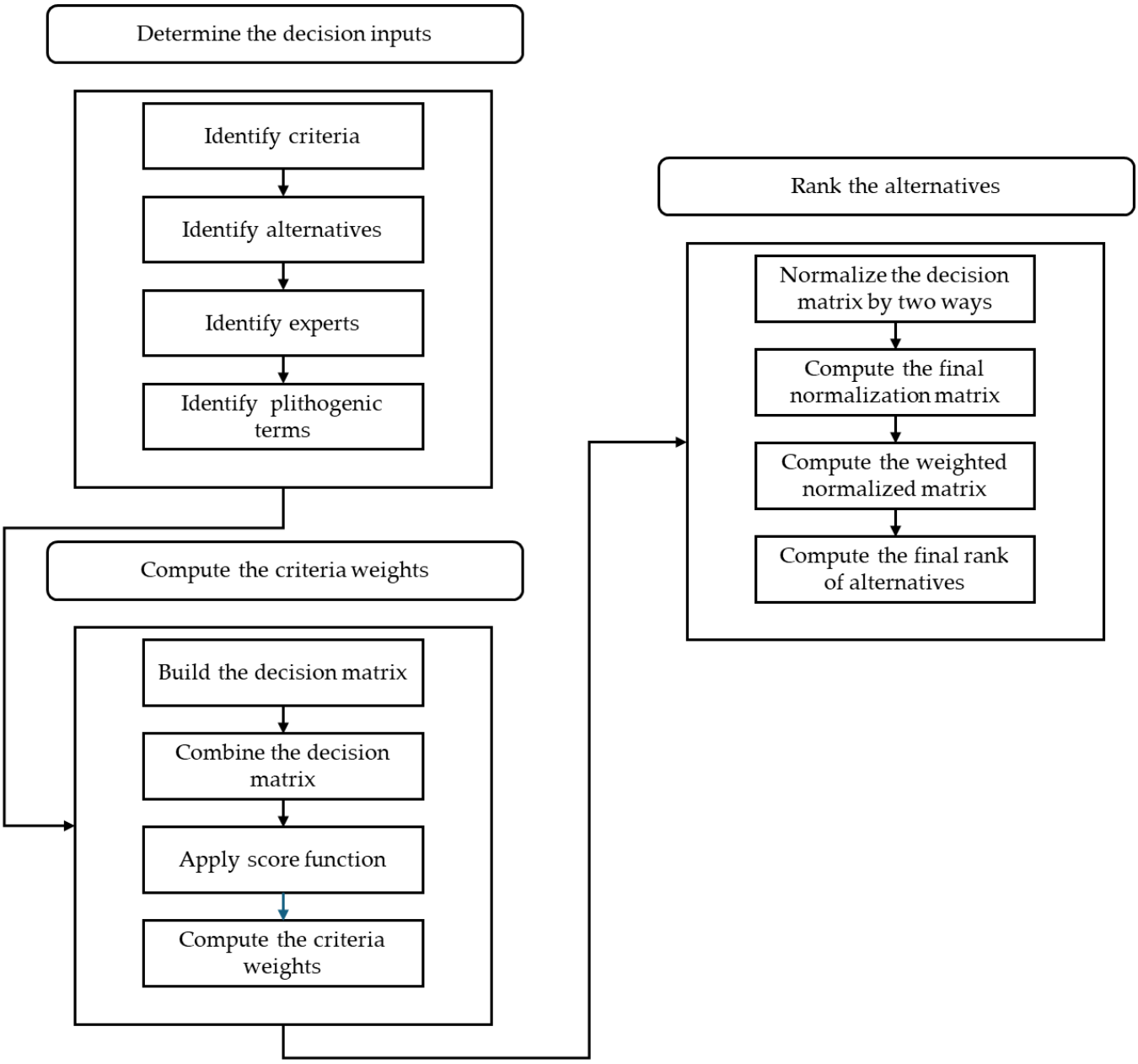

Figure 1. The steps of the RBNAR approach.

2. Methodology

This section shows the steps of the proposed approach. The RBNAR approach is integrated with the plithogenic sets to deal with uncertainty data[15], [16]. Figure 1 shows the steps of the proposed approach.

Step 1. Build the decision matrix.

Experts use plithogenic terms to evaluate the criteria and alternatives.

Step 2. Combine the decision matrix

The decision matrix is built using the plithogenic numbers between the criteria and the alternatives.

Step 3. Obtain crisp values and compute the criteria weights.

We obtained crisp values from the combined decision matrix. Then we normalize crisp values to obtain the criteria weights.

Step 4. The two distinct normalization methods are used to compute the normalized decision matrix. Among these is the non-monotonic Z-score normalization method. The other is the reference-based normalizing method developed by Aytekin. Normalizing the data using reference values and standard deviations enables a normalization procedure that is based on the presumption of a normal distribution, which is the main reason for using the Z-score normalization technique. However, the ability of Aytekin's reference-based normalization technique to standardize data scales makes it possible to compare datasets with disparate scales. The Heron mean is then used to aggregate two normalized decision matrices.

$$r_{ij} = e^{\left(\frac{\left(x_{ij}-F_j\right)}{-2\left(std_j\right)^2}\right)} \tag{1}$$

Where F_j refers to the reference value and std_j refers to the standard deviation.

$$q_{ij} = 1 - \left(\frac{x_{ij}-F_j}{|F_j|+10^t}\right) \tag{2}$$

Where t>0

Step 5. Compute the final normalization matrix

$$d_{ij} = u\sqrt{r_{ij}q_{ij}} + (1-u)\frac{r_{ij}+q_{ij}}{2} \tag{3}$$

$u \in [0,1]$

Step 6. Compute the weighted normalized matrix.

$$y_{ij} = w_j d_{ij} \tag{4}$$

Step 7. Compute the final rank of alternatives.

$$L_{ij} = \sum_{j=1}^{m} y_{ij} \tag{5}$$

Figure 2. The criteria list.

3. Case Study

The evaluation of hospital operational management quality focuses on assessing how effectively a hospital delivers healthcare services while ensuring efficiency, patient satisfaction, and sustainability. It examines the overall performance of hospital systems, including resource allocation, staff coordination, service delivery, and infrastructure utilization. High-quality operational management ensures that hospitals provide timely, safe, and patient-centered care while optimizing costs and maintaining a supportive work environment for healthcare professionals. Effective management also addresses challenges such as patient flow, staff workload, and emergency preparedness to ensure smooth daily operations. Additionally, it emphasizes compliance with healthcare regulations, ethical standards, and the adoption of innovative practices to enhance service quality. By continuously improving operational processes, hospitals can achieve better outcomes for patients, reduce waiting times, and improve resource efficiency. Hospital operational management quality evaluation is essential for identifying gaps, driving improvements, and aligning services with the evolving needs of patients and communities, ultimately contributing to the overall effectiveness and sustainability of healthcare systems. This section evaluates the Hospital Operations Management using the MCDM approach under the uncertainty environment. We collected eight criteria and five alternatives in this study as shown in Figure 2. Four experts have evaluated the criteria and alternatives. Then we used the plithogenic numbers to assess the criteria and alternatives to build the decision matrix as shown in Table 1. Then we used the plithogenic operators to combine the decision matrix. Then we obtained crips values. Then we normalize the decision matrix to obtain the criteria weights as shown in table 2.

Table 1. The decision matrix.

	A_1	A_2	A_3	A_4	A_5
C_1	(0.90, 0.10, 0.10)	(0.10, 0.70, 0.80)	(0.10, 0.70, 0.80)	(0.90, 0.10, 0.10)	(0.10, 0.70, 0.80)
C_2	(0.30, 0.40, 0.80)	(0.90, 0.10, 0.10)	(0.10, 0.70, 0.80)	(0.10, 0.70, 0.80)	(0.30, 0.40, 0.80)
C_3	(0.10, 0.70, 0.80)	(0.70, 0.30, 0.10)	(0.90, 0.10, 0.10)	(0.10, 0.70, 0.80)	(0.10, 0.70, 0.80)
C_4	(0.70, 0.30, 0.10)	(0.90, 0.10, 0.10)	(0.10, 0.70, 0.80)	(0.10, 0.70, 0.80)	(0.70, 0.30, 0.10)

C_5	(0.90, 0.10, 0.10)	(0.10, 0.70, 0.80)	(0.10, 0.70, 0.80)	(0.70, 0.30, 0.10)	(0.90, 0.10, 0.10)
C_6	(0.90, 0.10, 0.10)	(0.10, 0.70, 0.80)	(0.10, 0.70, 0.80)	(0.90, 0.10, 0.10)	(0.30, 0.40, 0.80)
C_7	(0.70, 0.30, 0.10)	(0.90, 0.10, 0.10)	(0.10, 0.70, 0.80)	(0.10, 0.70, 0.80)	(0.90, 0.10, 0.10)
C_8	(0.10, 0.70, 0.80)	(0.10, 0.70, 0.80)	(0.90, 0.10, 0.10)	(0.10, 0.70, 0.80)	(0.10, 0.70, 0.80)
	A_1	A_2	A_3	A_4	A_5
C_1	(0.70, 0.30, 0.10)	(0.30, 0.40, 0.80)	(0.10, 0.70, 0.80)	(0.30, 0.40, 0.80)	(0.10, 0.70, 0.80)
C_2	(0.70, 0.30, 0.10)	(0.30, 0.40, 0.80)	(0.10, 0.70, 0.80)	(0.70, 0.30, 0.10)	(0.30, 0.40, 0.80)
C_3	(0.50, 0.40, 0.60)	(0.70, 0.30, 0.10)	(0.30, 0.40, 0.80)	(0.10, 0.70, 0.80)	(0.70, 0.30, 0.10)
C_4	(0.70, 0.30, 0.10)	(0.70, 0.30, 0.10)	(0.70, 0.30, 0.10)	(0.30, 0.40, 0.80)	(0.10, 0.70, 0.80)
C_5	(0.90, 0.10, 0.10)	(0.70, 0.30, 0.10)	(0.30, 0.40, 0.80)	(0.10, 0.70, 0.80)	(0.50, 0.40, 0.60)
C_6	(0.70, 0.30, 0.10)	(0.30, 0.40, 0.80)	(0.10, 0.70, 0.80)	(0.10, 0.70, 0.80)	(0.70, 0.30, 0.10)
C_7	(0.70, 0.30, 0.10)	(0.70, 0.30, 0.10)	(0.30, 0.40, 0.80)	(0.10, 0.70, 0.80)	(0.10, 0.70, 0.80)
C_8	(0.10, 0.70, 0.80)	(0.70, 0.30, 0.10)	(0.70, 0.30, 0.10)	(0.30, 0.40, 0.80)	(0.10, 0.70, 0.80)
	A_1	A_2	A_3	A_4	A_5
C_1	(0.50, 0.40, 0.60)	(0.30, 0.40, 0.80)	(0.30, 0.40, 0.80)	(0.50, 0.40, 0.60)	(0.70, 0.30, 0.10)
C_2	(0.30, 0.40, 0.80)	(0.50, 0.40, 0.60)	(0.30, 0.40, 0.80)	(0.30, 0.40, 0.80)	(0.50, 0.40, 0.60)
C_3	(0.50, 0.40, 0.60)	(0.30, 0.40, 0.80)	(0.30, 0.40, 0.80)	(0.10, 0.70, 0.80)	(0.50, 0.40, 0.60)
C_4	(0.70, 0.30, 0.10)	(0.50, 0.40, 0.60)	(0.30, 0.40, 0.80)	(0.30, 0.40, 0.80)	(0.70, 0.30, 0.10)
C_5	(0.90, 0.10, 0.10)	(0.10, 0.70, 0.80)	(0.50, 0.40, 0.60)	(0.30, 0.40, 0.80)	(0.30, 0.40, 0.80)
C_6	(0.30, 0.40, 0.80)	(0.50, 0.40, 0.60)	(0.30, 0.40, 0.80)	(0.30, 0.40, 0.80)	(0.30, 0.40, 0.80)
C_7	(0.50, 0.40, 0.60)	(0.30, 0.40, 0.80)	(0.30, 0.40, 0.80)	(0.50, 0.40, 0.60)	(0.10, 0.70, 0.80)
C_8	(0.10, 0.70, 0.80)	(0.50, 0.40, 0.60)	(0.30, 0.40, 0.80)	(0.30, 0.40, 0.80)	(0.10, 0.70, 0.80)
	A_1	A_2	A_3	A_4	A_5
C_1	(0.10, 0.70, 0.80)	(0.70, 0.30, 0.10)	(0.90, 0.10, 0.10)	(0.70, 0.30, 0.10)	(0.90, 0.10, 0.10)
C_2	(0.10, 0.70, 0.80)	(0.10, 0.70, 0.80)	(0.70, 0.30, 0.10)	(0.90, 0.10, 0.10)	(0.30, 0.40, 0.80)
C_3	(0.50, 0.40, 0.60)	(0.10, 0.70, 0.80)	(0.70, 0.30, 0.10)	(0.90, 0.10, 0.10)	(0.90, 0.10, 0.10)
C_4	(0.10, 0.70, 0.80)	(0.70, 0.30, 0.10)	(0.90, 0.10, 0.10)	(0.70, 0.30, 0.10)	(0.90, 0.10, 0.10)
C_5	(0.90, 0.10, 0.10)	(0.10, 0.70, 0.80)	(0.70, 0.30, 0.10)	(0.90, 0.10, 0.10)	(0.70, 0.30, 0.10)
C_6	(0.30, 0.40, 0.80)	(0.10, 0.70, 0.80)	(0.70, 0.30, 0.10)	(0.90, 0.10, 0.10)	(0.90, 0.10, 0.10)
C_7	(0.10, 0.70, 0.80)	(0.70, 0.30, 0.10)	(0.90, 0.10, 0.10)	(0.90, 0.10, 0.10)	(0.10, 0.70, 0.80)
C_8	(0.10, 0.70, 0.80)	(0.70, 0.30, 0.10)	(0.90, 0.10, 0.10)	(0.90, 0.10, 0.10)	(0.10, 0.70, 0.80)

Table 2. The criteria weights.

	Weights	Rank
C_1	0.129109	6
C_2	0.113023	2
C_3	0.12521	4
C_4	0.145429	8
C_5	0.139794	7
C_6	0.124244	3
C_7	0.125568	5
C_8	0.097623	1

Eq. (1) is used to first normalization decision matrix between the criteria and alternatives.
Then we used Eq. (2) is used to second normalization of decision matrix as shown in Table 3. Then we obtained the final normalized decision matrix by using Eq. (3). We put value of u with 0.5

Table 3. The normalized decision matrix.

	A_1	A_2	A_3	A_4	A_5
C_1	1.760065	1.778128	1.778991	1.754727	1.769856
C_2	1.778128	1.768725	1.78456	1.76358	1.781164
C_3	1.777716	1.769082	1.757774	1.785468	1.760065
C_4	1.760973	1.748623	1.76358	1.778128	1.755575
C_5	1.735327	1.791043	1.774249	1.76358	1.754727
C_6	1.757774	1.79396	1.78456	1.758101	1.757774
C_7	1.765515	1.750706	1.772582	1.775067	1.785468
C_8	1.812577	1.765515	1.74583	1.772582	1.812577

Then we obtain weighted decision matrix using Eq. (4). Then we compute the final rank of alternatives using Eq. (5) as shown in Figure 3.

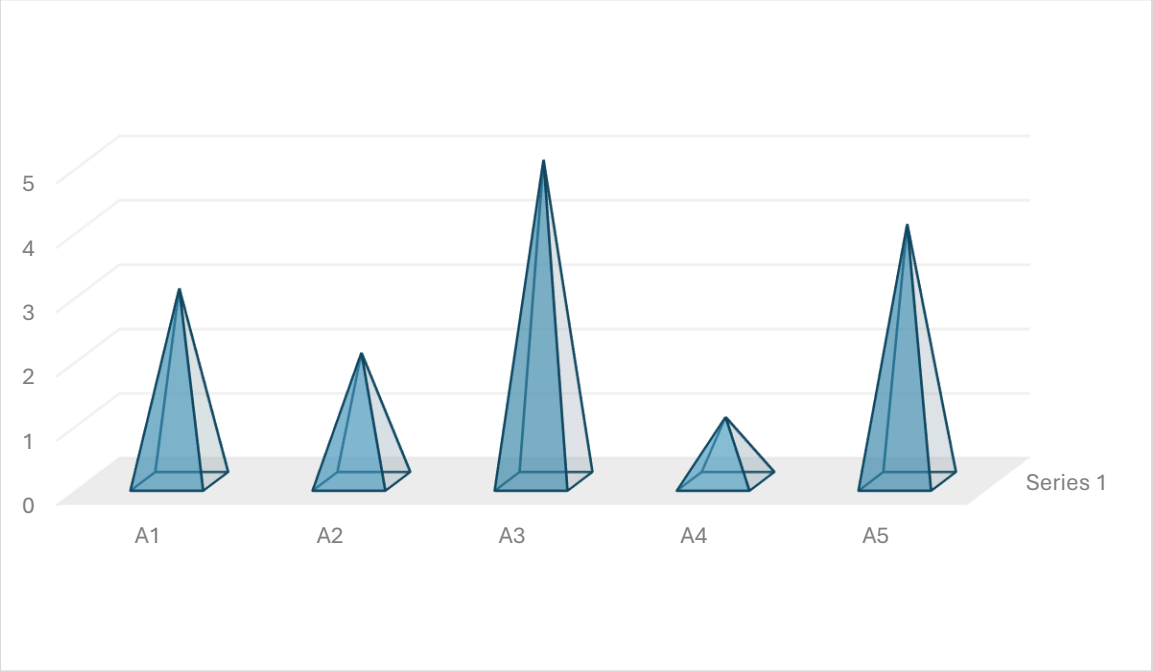

Figure 3. The final rank of alternatives.

4. Sensitivity Analysis

This section shows the sensitivity analysis under different cases to show the different ranks of alternatives. We change the value of u between 0 to 1 and then we rank the alternatives under different cases. Table 4 shows different ranks of alternatives.

We show alternative 3 is the best and alternative 1 is the worst. So, our model is effective and stable in the ranks under different ranks.

Table 4. The rank of alternatives under different cases.

	D_1	D_2	D_3	D_4	D_5	D_6	D_7	D_8	D_9	D_{10}	D_{11}
A_1	3	3	3	3	3	3	3	3	3	3	3
A_2	2	2	2	2	2	2	2	2	2	2	2

A_3	5	5	5	5	5	5	5	5	5	5	5
A_4	1	1	1	1	1	1	1	1	1	1	1
A_5	4	4	4	4	4	4	4	4	4	4	4

In the second case in the sensitivity analysis, we change the criteria weights under different cases, then we rank the alternatives to show the different rank of alternatives. This study proposed nine cases in criteria weights. These cases show the criteria weights can increase with 23% and then decrease with 23% to show different criteria weights. Figure shows the criteria weights under nine cases.

Then we rank the alternatives under these cases. We applied the proposed approach under the new criteria weights to show the different ranks of alternatives. The results show the rank of alternatives is stable under different cases in criteria weights. We show alternative 3 is the highest rank in all cases and alternative 4 is the worst rank in all cases. Figure shows the rank of alternatives. Figure shows the histogram of rank of alternatives.

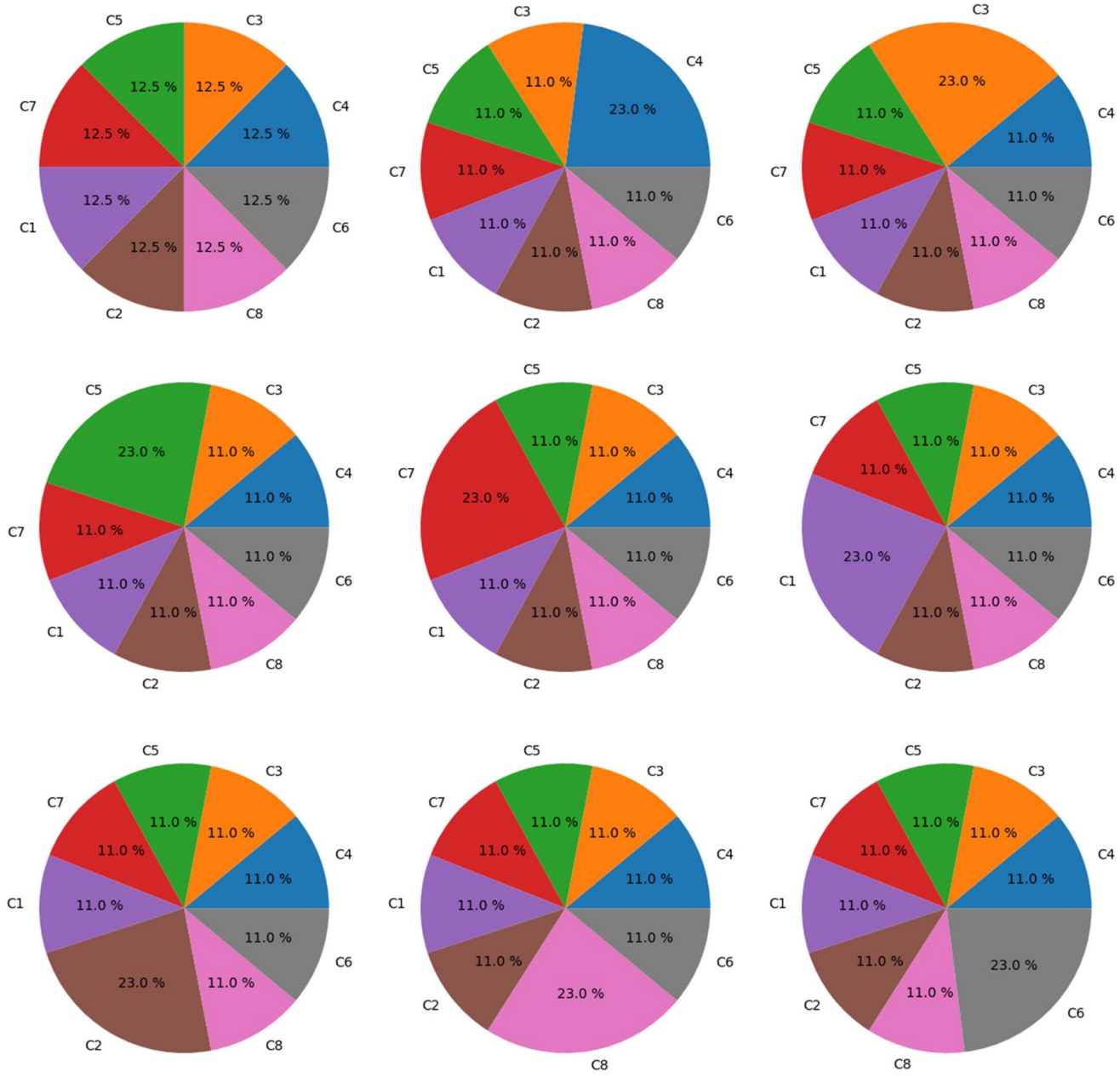

Figure . The different criteria weights.

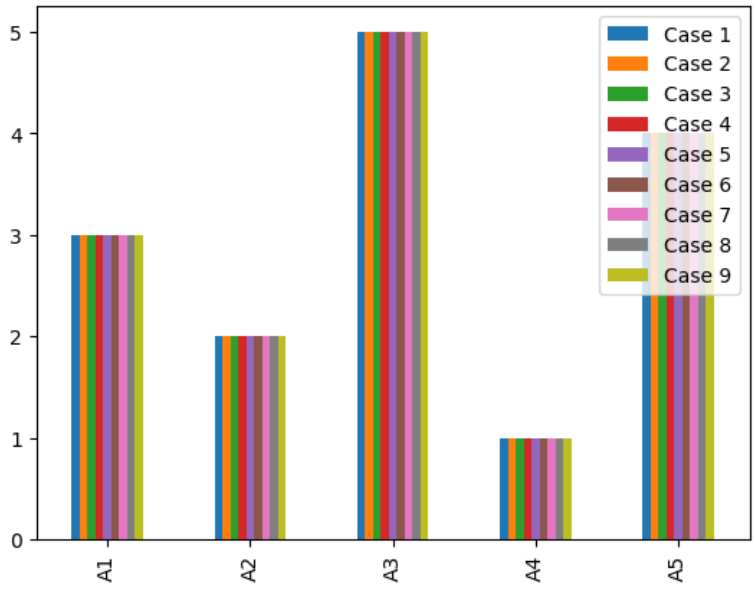

Figure . The rank of alternatives under different criteria weights.

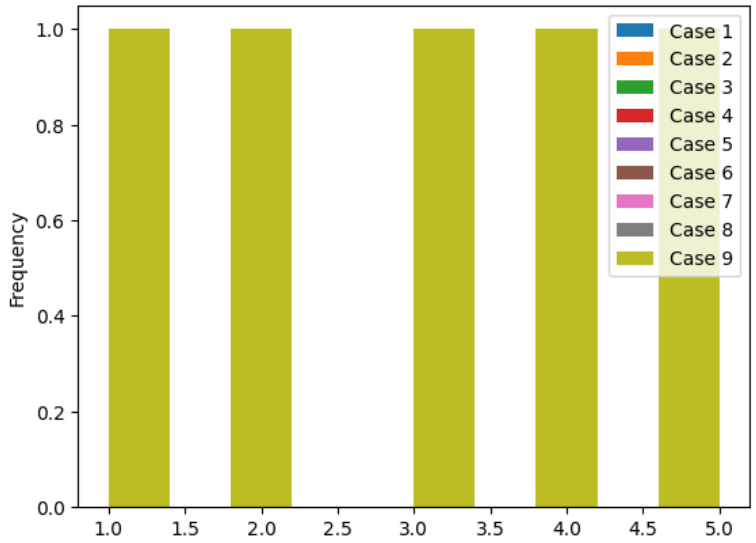

Figure . The histogram of rank of alternatives under different criteria weights.

Research Implications

With its novel insights and approaches that add to the body of existing literature, this study has important implications for the field of hospital operations management assessment and decision-making methodologies.

A. Hybrid decision support system − The creation of the plithogenic-RBNAR hybrid method offers a comprehensive decision support system that simultaneously weighs criteria and matrix-based criteria, fostering a more sophisticated and flexible approach to operations management analysis and opening the door for further research and applications.

B. Criterion weighing techniques − This study offers a sophisticated comprehension of criterion weighting. A deeper comprehension of the significance of criteria is provided by the normalization technique, which is under plithogenic environment.
C. Validation of real case studies: The research's practical significance is increased when the created method is applied to a genuine case study involving hospital operations management. When applying comparable approaches, researchers can find inspiration in the case study's design and implementation.

5. Conclusions and Future Directions

By creating a unique decision support system for identifying and ranking the Hospital Operations Management, this research made a contribution. Hospital Operations Management was evaluated using MCDM approach. Hospital Operations Management was evaluated and ranked as alternatives using the created RBNAR approach, which enables the ranking of alternatives using reference value-based normalization procedures. By combining these approaches, the hybrid plithogenic-RBNAR method was created, and the paper included a description of its algorithm. The results showed that the alternative one is the best and alternative 4 is the worst.

In the future directions, the proposed algorithm can be applied in different decision-making issues to compute the criteria weights and rank the alternatives. The criteria can be increased to show different directions of the decision making issue. Different decision-making models can be applied to compute the criteria weights and rank the alternatives in this study.

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The improper neutrosophic integrals

Yaser Ahmad Alhasan[1,*], Issam Mohammed Abdallah Abdalaziz[2], Raja Abdullah Abdulfatah[3] and Florentin Smarandache[4]

[1]Deanship of the Preparatory Year, Prince Sattam bin Abdulaziz University, Alkharj, Saudi Arabia.; y.alhasan@psau.edu.sa
[2]Deanship the Preparatory Year, Prince Sattam bin Abdulaziz University, Alkharj Saudi Arabia.; I.abdalaziz@psau.edu.sa
[3]Deanship the Preparatory Year, Prince Sattam bin Abdulaziz University, Alkharj, Saudi Arabia.; r.abdulfatah@psau.edu.sa
[4]University of New Mexico, Gallup Campus, NM 87301, USA; fsmarandache@gmail.com

[*]Corresponding author: y.alhasan@psau.edu.sa

Abstract: In light of the expansion of research that dealt with the logic of the neutrosophic and its clear spread at the level of science, it was necessary for us to continue in the series of research that covers the calculus of neutrosophic integration, and this is what led us to study the improper neutrosophic integrals. This study dealt with the types of the improper neutrosophic integrals of the first and second kind. In addition, we discussed the concept of convergent and divergent neutrosophic integrals. As well as providing a set of examples related to these ideas.

Keywords: neutrosophic, integrals, improper, limit, divergent, converges.

1. Introduction and Preliminaries

In contrast to the current logics, Smarandache suggested the Neutrosophic Logic to describe a mathematical model of uncertainty, vagueness, ambiguity, imprecision, undefined, unknown, incompleteness, inconsistency, redundancy, and contradiction. Smarandache introduced the concept of neutrosophy as a new school of philosophy [4]. He presented the definition of the standard form of neutrosophic real number [3-5], the Neutrosophic statistics [5-6], and professor Smarandache entered the concept of preliminary calculus of the differential and integral calculus, where he introduced for the first time the notions of neutrosophic mereo-limit, mereo-continuity, mereoderivative, and mereo-integral [1]. A number of studies in the area of integration and differentiation were given by Y. Alhasan [8-11-14], also he presented the definition of the concept of neutrosophic complex numbers and its properties, in addition, he studied the general exponential form of a neutrosophic complex number [2-9]. Madeleine Al- Taha presented results on single valued neutrosophic (weak) polygroups [12]. The AH isometry was used to study many structures such as conic sections, real analysis concepts, and geometrical surfaces [11-15].

Definition [15]

Let $f: R(I) \to R(I)$; $f = f(X)$ and $X = x + yI \in R(I)$ then f is called a neutrosophic real function with one neutrosophic variable. a neutrosophic real function $f(X)$ written as follows:

$$f(X) = f(x + yI) = f(x) + I[f(x + y) - f(x)]$$

2. Main Discussion

2.1 Improper neutrosophic integrals of the first kind

Definition 1

We will consider the improper neutrosophic integral whose integrands are bound on an infinite interval. Let $f(x,I)$ be a function defined on the infinite interval $[a+bI,+\infty)$, and integrable over any bounded closed interval $[a+bI,k]\subset[a+bI,+\infty)$, then we call $f(x,I)$ over the interval $[a+bI,+\infty)$ the improper neutrosophic integral and write:

$$\int_{a+bI}^{+\infty} f(x,I)\,dx = \lim_{k\to+\infty}\int_{a+bI}^{k} f(x,I)\,dx$$

We call the improper neutrosophic integral convergent when the limit exists, and the neutrosophic integral is of first kind, where it has defined limit, it is the value of the neutrosophic integral. Otherwise the improper neutrosophic integral is divergent (when the improper integral doesn't exist or equal to $\pm\infty$).

Example 1

$$\int_{1+I}^{+\infty}\frac{dx}{x+3+5I} = \lim_{k\to+\infty}\int_{1+I}^{k}\frac{dx}{x+3+5I} = \lim_{k\to+\infty} ln(x+3+5I)|_{1+I}^{k}$$

$$= \lim_{k\to+\infty} ln(x+3+5I)|_{1+I}^{k} = \lim_{k\to+\infty}[ln(k+3+5I) - ln(4+6I)]$$

$$= \lim_{k\to+\infty}[ln(k+3+5I)] - ln(4+6I)$$

by applying:

$$f(X) = f(x+yI) = f(x) + I[f(x+y) - f(x)]$$

we get on:

$$\int_{1+I}^{+\infty}\frac{dx}{x+3+5I} = \lim_{k\to+\infty}\big[ln(k+3) + I[ln(k+3+5) - ln(k+3)]\big] - \big[ln\,4 + I[ln\,10 - ln\,4]\big]$$

$$= \lim_{k\to+\infty}\left[ln(k+3) + I\left[ln\left(\frac{k+3+5}{k+3}\right)\right]\right] - \left[ln\,4 + I\,ln\frac{5}{2}\right]$$

$= +\infty + I[ln(1)] - \left[ln\,4 + I\,ln\frac{5}{2}\right] = +\infty - I\,ln\frac{5}{2} = \infty_I$

(the improper neutrosophic integral diverges)

Example 2

$$\int_{0+0I}^{+\infty} e^{-(3+3I)x}\,dx = \lim_{k\to+\infty}\int_{0+0I}^{k} e^{-(3+3I)x}\,dx = \lim_{k\to+\infty}\frac{-1}{3+3I}e^{-(3+3I)x}\Big|_{0+0I}^{k}$$

$$= \frac{-1}{3+3I}\lim_{k\to+\infty}\big[e^{-(3+3I)k} - 1\big]$$

by applying:

$$f(X) = f(x+yI) = f(x) + I[f(x+y) - f(x)]$$

we get on:

$$\int_{0+0I}^{+\infty} e^{-(3+3I)x}\,dx = \frac{-1}{3+3I}\lim_{k\to+\infty}[e^{-3k}+I[e^{-6k}-e^{-3k}]-1]$$

$$=\frac{1}{3+3I}=\frac{1}{3}-\frac{1}{6}I$$

the improper neutrosophic integral converges.

Definition 2

We will consider the improper neutrosophic integral whose integrands are bound on an infinite interval. Let $f(x,I)$ be a function defined on the infinite interval $(-\infty, c+dI]$, and integrable over any bounded closed interval $[k, c+dI] \subset (-\infty, c+dI]$, then we call $f(x,I)$ over the interval $(-\infty, c+dI]$ the improper neutrosophic integral and write:

$$\int_{-\infty}^{c+dI} f(x,I)\,dx = \lim_{k\to-\infty}\int_{k}^{c+dI} f(x,I)\,dx$$

We call the improper neutrosophic integral convergent when the limit exists, and the integral is of first kind, where it has defined limit, it is the value of the integral. Otherwise the improper neutrosophic integral is divergent (when the improper integral doesn't exist or equal to $\pm\infty$).

Example 3

$$\int_{-\infty}^{0+0I} cos\left(x+\frac{\pi}{3}I\right)dx = \lim_{k\to-\infty}\int_{k}^{0+0I} cos\left(x+\frac{\pi}{3}I\right)dx$$

$$=\lim_{k\to-\infty}\left[sin\left(x+\frac{\pi}{3}I\right)\right]\Big|_{k}^{0+0I} = sin\left(\frac{\pi}{3}I\right)-\lim_{k\to-\infty} sin\left(k+\frac{\pi}{3}I\right)$$

$$=I\frac{\sqrt{3}}{2}-\lim_{k\to-\infty}\left(sin(k)+I\left[sin\left(k+\frac{\pi}{3}\right)-sin(k)\right]\right)$$

(Does not exist, so the improper neutrosophic integral is divergent)

Example 4

$$\int_{-\infty}^{0+0I}(x+2I)e^{x}dx = \lim_{k\to-\infty}\int_{k}^{0+0I}(x+2I)e^{x}dx$$

$$u=(x+2I) \implies du=dx$$
$$v\,dx=e^{x}dx \implies v=e^{x}$$

then:

$$\int_{-\infty}^{0+0I}(x+2I)e^{x}dx = \lim_{k\to-\infty}[(x+2I)e^{x}]|_{k}^{0+0I}-\lim_{k\to-\infty}\int_{k}^{0+0I} e^{x}dx$$

$$=\lim_{k\to-\infty}[(x+2I)e^{x}]|_{k}^{0+0I}-\lim_{k\to-\infty}\int_{k}^{0+0I} e^{x}dx = 2I-\lim_{k\to-\infty}\left((k+2I)e^{k}\right)-\lim_{k\to-\infty}e^{x}|_{k}^{0+0I}$$

$$= 2I - \lim_{k\to-\infty}(ke^k + 2Ie^k) - 1 + \lim_{k\to-\infty} e^k = 2I - 1$$

Theorem 1

$$\int_{1+I}^{+\infty} \frac{dx}{x^p} = \begin{cases} \dfrac{p}{(1+I)^{p-1}} & if \quad p > 1 \\ \text{Divergent} & if \quad p \le 1 \end{cases}$$

Example 5

$$\int_{1+I}^{+\infty} \frac{dx}{x^2} = \lim_{k\to+\infty} \int_{1+I}^{k} \frac{dx}{x^2} = \lim_{k\to+\infty} \frac{-2}{x}\Big|_{1+I}^{k}$$

$$= \lim_{k\to+\infty} \left[\frac{-2}{k} + \frac{2}{1+I}\right] = \frac{2}{1+I} = 2 - I$$

the improper neutrosophic integral converges.

Example 6

$$\int_{1+I}^{+\infty} \frac{dx}{x^3} = \lim_{k\to+\infty} \int_{1+I}^{k} \frac{dx}{x^3} = \lim_{k\to+\infty} \frac{-3}{x^2}\Big|_{1+I}^{k}$$

$$= \lim_{k\to+\infty} \left[\frac{-3}{k^2} + \frac{3}{(1+I)^2}\right] = \frac{3}{(1+I)^2} = \frac{3}{1+3I} = 3 - \frac{9}{4}I$$

the improper neutrosophic integral converges.

Example 7

$$\int_{1+I}^{+\infty} \frac{dx}{x^4} = \lim_{k\to+\infty} \int_{1+I}^{k} \frac{dx}{x^4} = \lim_{k\to+\infty} \frac{-4}{x^3}\Big|_{1+I}^{k}$$

$$= \lim_{k\to+\infty} \left[\frac{-4}{k^3} + \frac{4}{(1+I)^3}\right] = \frac{4}{(1+I)^3} = \frac{4}{1+7I} = 4 - \frac{7}{2}I$$

the improper neutrosophic integral converges.

Example 8

$$\int_{1+I}^{+\infty} \frac{dx}{x} = \lim_{k\to+\infty} \int_{1+I}^{k} \frac{dx}{x} = \lim_{k\to+\infty} \ln x|_{1+I}^{k}$$

$$= \lim_{k\to+\infty} [\ln k + \ln(1+I)] = \lim_{k\to+\infty} \ln k - (\ln 1 + I[\ln 2 - \ln 1])$$

$= \infty - I \ln 2 = \infty_I$ (the improper neutrosophic integral diverges)

Example 9

$$\int_{2+2I}^{+\infty} \frac{dx}{x(\ln x)^2} = \lim_{k\to+\infty} \int_{2+2I}^{k} \frac{dx}{x(\ln x)^2} == \lim_{k\to+\infty} \frac{-1}{\ln x}\Big|_{2+2I}^{k}$$

$$= \lim_{k\to+\infty} \frac{-1}{\ln x} + \frac{1}{\ln(2+2I)} = \frac{1}{\ln 2 + I[\ln 4 - \ln 2]} = \frac{1}{\ln 2 + I \ln 2}$$

$$= \frac{1}{\ln 2} - \frac{\ln 2}{\ln 2\,(2 \ln 2)} = \frac{1}{\ln 2} - \frac{1}{2 \ln 2} I$$

(the improper neutrosophic integral converges)

Remark 1

The improper neutrosophic integral of $f(x,I)$ over $(-\infty,\infty)$ is defined as:

$$\int_{-\infty}^{+\infty} f(x,I)\,dx = \int_{-\infty}^{c} f(x,I)\,dx + \int_{c}^{+\infty} f(x,I)\,dx$$

Example 10

$$\int_{-\infty}^{+\infty} \frac{dx}{1+3I+x^2} = \int_{-\infty}^{0+0I} \frac{dx}{1+3I+x^2} + \int_{0+0I}^{+\infty} \frac{dx}{1+3I+x^2}$$

$$= \lim_{k_1\to-\infty} \int_{k_1}^{0+0I} \frac{dx}{1+3I+x^2} + \lim_{k_2\to+\infty} \int_{0+0I}^{k_2} \frac{dx}{1+3I+x^2}$$

$$= \lim_{k_1\to-\infty} \frac{1}{1+I} tan^{-1}\frac{x}{1+I}\Big|_{k_1}^{0+0I} + \lim_{k_2\to+\infty} \frac{1}{1+I} tan^{-1}\frac{x}{1+I}\Big|_{0+0I}^{k_2}$$

$$= \lim_{k_1\to-\infty} \left[\frac{1}{1+I} tan^{-1}(0) - \frac{1}{1+I} tan^{-1}\frac{k_1}{1+I}\right] + \lim_{k_2\to+\infty} \left[\frac{1}{1+I} tan^{-1}\frac{k_2}{1+I} - \frac{1}{1+I} tan^{-1}(0)\right]$$

$$= \lim_{k_1\to-\infty} \left[-\frac{1}{1+I} tan^{-1}\left(1-\frac{1}{2}I\right)k_1\right] + \lim_{k_2\to+\infty} \left[\frac{1}{1+I} tan^{-1}\left(1-\frac{1}{2}I\right)k_2\right]$$

$$= \lim_{k_1\to-\infty} \left[-\frac{1}{1+I}\left(tan^{-1}(k_1) + I\left[tan^{-1}\left(\frac{1}{2}k_1\right) - tan^{-1}(k_1)\right]\right)\right] + \lim_{k_2\to+\infty} \left[\frac{1}{1+I}\left(tan^{-1}(k_2) + I\left[tan^{-1}\left(\frac{1}{2}k_1\right) - tan^{-1}(k_1)\right]\right)\right]$$

$$= -\frac{1}{1+I}\left(\frac{-\pi}{2} + I\left[\frac{-\pi}{2} + \frac{\pi}{2}\right]\right) + \frac{1}{1+I}\left(\frac{\pi}{2} - I\left[\frac{\pi}{2} - \frac{\pi}{2}\right]\right)$$

$$= \frac{\pi}{1+I} = \pi - \frac{\pi}{2}I$$

the improper neutrosophic integral converges.

2.2 Improper neutrosophic integrals of the second kind

Definition 3

If $f(x,I)$ is continuous on the interval $[a+bI, c+dI]$ and not defined at $c+dI$, then the improper neutrosophic integral of $f(x,I)$ over the interval $[a+bI, c+dI]$ is defined as:

$$\int_{a+bI}^{c+dI} f(x,I)\,dx = \lim_{k\to c+dI^-} \int_{a+bI}^{k} f(x,I)\,dx$$

The improper neutrosophic integral converges when the limit is existing, and the integral is of first kind, where it has defined limit, it is the value of the integral. Otherwise the improper neutrosophic integral diverges (when the improper neutrosophic integral doesn't exist or equal to $\pm\infty$).

Example 11

$$\int_{0+0I}^{1+I} \frac{1}{1+I-x}\,dx = \lim_{k\to 1+I^-} \int_{0+0I}^{k} \frac{1}{1+I-x}\,dx$$

$$= \lim_{k\to 1+I^-} ln(1+I-x)\big|_{0+0I}^{k} = \lim_{k\to 1+I^-} ln(1+I-k) - ln(1+I)$$

$$= \lim_{k\to 1+I^-} ln(1+I-k) - ln(1+I) = -\infty - I\,ln\,2 = -\infty_I$$

So, the improper neutrosophic integral diverges

Example 12

$$\int_{0+0I}^{2-I} \frac{dx}{\sqrt{4-3I-x^2}} = \lim_{k\to 2-I^-} \int_{0+0I}^{k} \frac{dx}{\sqrt{4-3I-x^2}}$$

$$= \lim_{k\to 2-I^-} sin^{-1}\left(\frac{x}{2-I}\right)\Big|_{0+0I}^{k} = \lim_{k\to 2-I^-} sin^{-1}\left(\frac{x}{2-I}\right) - sin^{-1}(0)$$

$$= sin^{-1}\left(\frac{2-I}{2-I}\right) = sin^{-1}(1) = \frac{\pi}{2}$$

The improper neutrosophic integral converges.

Definition 4

If $f(x,I)$ is continuous on the interval $[a+bI, c+dI]$, except for a singular point (discontinuity) $a+bI$, then the improper neutrosophic integral of $f(x,I)$ over the interval $[a+bI, c+dI]$ is defined as:

$$\int_{a+bI}^{c+dI} f(x,I)\,dx = \lim_{k\to a+bI^+} \int_{k}^{c+dI} f(x,I)\,dx$$

The improper neutrosophic integral converges when the limit is existing, and the integral is of first kind, where it has defined limit, it is the value of the integral. Otherwise the improper neutrosophic integral diverges (when the improper neutrosophic integral doesn't exist or equal to $\pm\infty$).

Example 13

the neutrosophic integral is improper and the integrand neutrosophic function is not defined at $2+2I$:

$$\int_{1+I}^{4+4I} \frac{1}{\sqrt[3]{(x-2-2I)^2}}\,dx = \int_{1+I}^{2+2I} \frac{1}{\sqrt[3]{(x-2-2I)^2}}\,dx + \int_{2+2I}^{4+4I} \frac{1}{\sqrt[3]{(x-2-2I)^2}}\,dx$$

$$= \int_{1+I}^{2+2I} (x-2-2I)^{-2/3}\,dx + \int_{2+2I}^{4+4I} (x-2-2I)^{-2/3}\,dx$$

$$= \lim_{k_1\to 2+2I^-} \int_{1+I}^{k_1} (x-2-2I)^{-2/3}\,dx + \lim_{k_2\to 2+2I^+} \int_{k_2}^{4+4I} (x-2-2I)^{-2/3}\,dx$$

$$= \lim_{k_1\to 2+2I^-} \sqrt[3]{x-2-2I}\Big|_{1+I}^{k_1} + \lim_{k_2\to 2+2I^+} \sqrt[3]{x-2-2I}\Big|_{k_2}^{4+4I}$$

$$= \lim_{k_1 \to 2+2I^-} \sqrt[3]{k_1 - 2 - 2I} - \sqrt[3]{1 + I - 2 - 2I} + \sqrt[3]{4 + 4I - 2 - 2I} - \lim_{k_2 \to 2+2I^+} \sqrt[3]{k_2 - 2 - 2I}$$

$$= \sqrt[3]{1 + I} + \sqrt[3]{2 + 2I} = 1 + I\left[\sqrt[3]{2} - 1\right] + \sqrt[3]{2} + I\left[\sqrt[3]{4} - \sqrt[3]{2}\right]$$

$$= 1 + \sqrt[3]{2} + I\left[\sqrt[3]{4} - 1\right]$$

the improper neutrosophic integral converges.

Remark 2

If $f(x, I)$ is a continuous neutrosophic function over $[a + bI, c + dI]$ except value on in $(a + bI, c + dI)$, then the improper neutrosophic integral of $f(x, I)$ over interval $[a + bI, c + dI]$ is defined as:

$$\int_{a+bI}^{c+dI} f(x, I)\, dx = \int_{a+bI}^{k} f(x, I)\, dx + \int_{k}^{c+dI} f(x, I)\, dx$$

Where k is any real number. The improper neutrosophic integral is convergent if the first term and second term on the right side of the various integral (both of them) are convergent, while the integral is divergent if either term (at least one or both) are divergent.

Example 14

$$\int_{-3-I}^{3+I} \frac{dx}{\sqrt{9 + 7I - x^2}}\, dx = \int_{-3-I}^{0+0I} \frac{dx}{\sqrt{9 + 7I - x^2}} + \int_{0+0I}^{3+I} \frac{dx}{\sqrt{9 - 7I - x^2}}$$

$$= \lim_{k_1 \to -3-I^+} \int_{k_1}^{0+0I} \frac{dx}{\sqrt{9 + 7I - x^2}} + \lim_{k_2 \to 3+I^-} \int_{0+0I}^{k_2} \frac{dx}{\sqrt{9 + 7I - x^2}}$$

$$= \lim_{k_1 \to -3-I^+} sin^{-1}\left(\frac{x}{3 + I}\right)\Big|_{k_1}^{0+0I} + \lim_{k_2 \to 3+I^-} sin^{-1}\left(\frac{x}{3 + I}\right)\Big|_{0+0I}^{k_2}$$

$$= sin^{-1}(0) - \lim_{k_1 \to -3-I^+} sin^{-1}\left(\frac{x}{3 + I}\right) + \lim_{k_2 \to 3+I^-} sin^{-1}\left(\frac{x}{3 + I}\right) - sin^{-1}(0)$$

$$= -sin^{-1}\left(\frac{-3 - I}{3 + I}\right) + sin^{-1}\left(\frac{3 + I}{3 + I}\right) = -sin^{-1}(-1) + sin^{-1}(1) = \pi$$

the improper neutrosophic integral converges.

3. Conclusions

In this paper, we introduced the improper neutrosophic integrals, by providing several definitions that clarify the concept of the improper neutrosophic integrals, and we obtained several results for these integrals, as the improper neutrosophic integral can be convergent, divergent.

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Enhancing Competency-Based Learning with Neutrosophic Regression and the Deming Cycle

[1]Dr, Hortencio Flores Flores, [2] Dr. Irma Reyes Blácido , [3]Dr. Magda Patricia Ramos Cevallos, [4] Dr. Nemesia Hidalgo Penadillo, [5] Mg. Luz Magali Villarroel Nuñez
Universidad Nacional de Educación Enrique Guzmán y Valle, La Cantuta, Chosica, Lima, Perú
E-mail: ireyes@une.edu.pe , hflores@une.edu.pe , mramos@une.edu.pe , nemesiahidalgopenadillo@une.edu.pe , lvillarroel@une.edu.pe

Abstract

This article introduces a novel approach to enhance competency-based learning by combining the Deming Cycle with neutrosophic statistics. Competency-based education focuses on practical skills, but uncertainty in student performance and assessment can hinder its effectiveness. Neutrosophic statistics, unlike traditional methods, explicitly models indeterminacy, providing a more complete picture of uncertainty in educational data. This approach integrates neutrosophic numbers into regression analysis to predict learning outcomes and quantify the confidence level of those predictions. These predictions, with their associated indeterminacy, then inform the Deming Cycle (Plan-Do-Check-Act), enabling educators to dynamically adjust teaching strategies based on data-driven insights. This leads to more informed decision-making, improved accuracy and reliability in predictions, and ultimately fosters continuous improvement in competency-based education.

Keywords: Competency-based learning, Deming Cycle, neutrosophic numbers, regression analysis, uncertainty, educational data, machine learning, continuous improvement

1. Introduction

In the realm of education, uncertainty is an inherent challenge, particularly in competency-based learning where the goal is to equip students with practical skills applicable to real-world scenarios. The dynamic nature of education, coupled with the diverse needs and learning styles of students, introduces complexities that traditional analytical models often struggle to address. This underscores the need for a robust framework that acknowledges and effectively handles uncertainties in educational data, paving the way for continuous improvement in teaching methodologies. This article explores the integration of neutrosophic regression models with the Deming Cycle [1] as a novel approach to enhance the accuracy and effectiveness of competency-based learning.

Competency-based learning has emerged as a crucial pedagogical approach, focusing on developing students' skills and preparing them for the challenges of their future careers. However, uncertainties in assessment and the diverse learning environment often hinder the achievement of desired learning outcomes. Traditional models often oversimplify the educational landscape by neglecting critical factors such as variability in student performance and uncertainties in assessment data. To address this, we propose the use of neutrosophic models, which offer a powerful tool for analytically understanding and managing uncertainty.

The Deming Cycle, widely recognized for its effectiveness in driving continuous improvement in various industries, provides a structured approach to process optimization. By systematically progressing through the stages of planning, doing, checking, and acting, the Deming Cycle enables the identification and resolution of areas for improvement. This study investigates the synergistic combination of neutrosophic regression models with the Deming Cycle to analyze and enhance competency-based learning strategies[2,3].

Using real data from students in the Food Industry and Nutrition program, we demonstrate the practical application of this combined approach in addressing uncertainties in educational data and facilitating data-driven improvements in learning strategies[4]. The findings of this study have significant implications for both educational theorists and practitioners, offering a new perspective on managing indeterminacy and promoting continuous improvement in diverse learning environments[5,6].

This article contributes to the field by presenting a comprehensive framework that integrates pedagogical principles, advanced mathematical tools, and process management techniques. We delve into the theoretical foundations of the proposed approach, analyze the results obtained, and discuss their implications for the future of education. By combining neutrosophic regression models with the Deming Cycle, we aim to provide educators with a robust and adaptable tool for navigating the complexities of competency-based learning and fostering a culture of continuous improvement.

2. Preliminaries.

2.1. Deming cycle.

The Deming Cycle (Figure 1), or PDCA Cycle (Plan, Do, Check, Act), is very important to control quality and always improve a company was created by W. Edwards Deming and draws on the ideas of Walter A. Shewhart [7,8]. The model can be used in many different industries. It involves identifying problems, testing solutions, seeing how they work, and making standard improvements. This process has been very important for companies that want to improve their performance by receiving feedback and making continuous changes to their operations. The cycle begins with the planning stage, where areas of opportunity are identified through a thorough analysis of the situation. During this stage, we set specific goals and create plans based on accurate information. Being able to foresee problems and make practical plans is very important to ensure that our actions are effective. In workplaces such as factories, using this stage helps make processes more accurate and efficient[9,10].

Figure 1. PDCA Cycle for Continuous Improvement

Then, in the Do stage, strategies are put into action on a small, controlled scale to test them. This approach allows you to try out and gather information about how proposals work in real conditions. Although seemingly simple, this phase requires discipline and gathering accurate information to make the analysis that follows easier. The check phase involves critically analyzing the results obtained. Here we compare the actual information with what we initially expected. This step helps us see if what we did worked well and shows us where we can improve. It is important to learn from mistakes and continue to build on successes during this stage. Finally, the Act phase is when successful changes are implemented on a large scale or adjustments are made if the results are not good. This step strengthens organizational learning by solidifying helpful changes and starting the cycle again to find new opportunities for improvement [11].

The Deming Cycle is known for being universally applicable. From manufacturing to education, this model has changed the way we do things by offering an organized method for solving difficult problems. It also promotes a company culture of always wanting to improve, something very important in a competitive environment that is always changing. However, putting an Item into Action has challenges. It requires everyone in the organization to be committed, from managers to day-to-day employees. In addition, gathering and understanding data can be difficult, especially if information systems are poor. However, these problems do not reduce what they can achieve but rather show the importance of using systematic methods to overcome them[12].

In a world that changes rapidly with technology and markets, the Deming Cycle is a flexible and adaptable tool Being able to constantly change strategies and processes helps organizations not only sustain, but also grow in times of uncertainty. This makes it a very useful model for industries with a lot of change and that always need to innovate. In short, the Deming Cycle is not only a way of managing, but it is also an important principle for companies that seek to constantly learn and adapt. Its ability to plan constant improvements makes it a key element in achieving a high level of excellence in operations. Furthermore, in the field of education, the Deming Cycle provides a powerful framework for continuous improvement of teaching methodologies, curriculum development, and learning outcomes, fostering an environment of ongoing adaptation and enhancement in response to the evolving needs of students and society. [13].

2.2. Integrating Neutrosophic Numbers with Machine Learning

In the realm of artificial intelligence and machine learning, we often encounter data and information that is incomplete, uncertain, or imprecise. To address these challenges, various mathematical and computational tools have been developed. One such tool is neutrosophic numbers, introduced by Florentin Smarandache, which extend the concept of fuzzy numbers by including an indeterminacy component[14].

Neutrosophic Numbers expressed as $N = a + bI$, were defined by WB Vasantha Kandasamy and F. Smarandache in 2003, that "a" represents the definite component of N, while "bI" signifies the indeterminate component of N[14]. The indeterminate component I satisfies properties such as$I = I^2$, and for any integers m and n, $mI + nI = (m + n)I$ = (m + n). Additionally, $0 \cdot \mathrm{I} = 0$ and $\mathrm{I}^n = \mathrm{I}$ for any integer $n \geq 1$ [15].

Numerical neutrosophic numbers facilitate the reduction of indeterminacy through operations, whereas intervals exacerbate it (for instance, let $N1 = 5 + 2I$, $N2 = 6 - I$, with indeterminacy I = [0,1]; employing NS yields: N1 + N2 = 11+ I = [11, 12]). Utilizing interval, one obtains: N1 + N2 = [5, 7] + [4, 6] = [9, 13]; therefore, the indeterminacy or the actual data point residing in [9, 13] is greater than the actual data point residing in [11, 12] [16].

Neutrosophic numbers can be used in data preprocessing to handle uncertainty and imprecision. For example, in data cleaning, they can be used to represent missing or inconsistent values. Instead of simply deleting or inputting these values, they can be represented as neutrosophic numbers with a high degree of indeterminacy. This preserves information about the uncertainty in the data, which can be crucial for training more robust models [17].

Indeterminacy can also be incorporated into the training of machine learning models. For instance, in a classification model, neutrosophic numbers can represent uncertainty in class labels. This can be useful in situations where data classification is ambiguous or where there is an overlap between classes. By training the model with neutrosophic numbers, a more robust model can be obtained that is less sensitive to uncertainty in the data[19, 20].

In regression analysis, representing predictions as prediction intervals provides a more complete view of the uncertainty associated with the predictions. A prediction interval gives a range within which we expect the actual value of the dependent variable to fall with a certain probability, typically 95% or 99%. This is particularly useful because it considers variability in the data that might not be captured by the prediction alone [21].

To calculate a prediction interval, both the uncertainty in the regression model estimate and the inherent variability of the data must be considered. The interval is constructed around the predicted value and is typically symmetrical, extending a certain amount above and below the predicted value. This range is determined based on the standard error of the prediction and the residual standard deviation, which reflects the spread of the model's residuals or errors [22].

For example, in a simple linear regression, the prediction interval for a new observation is given by [20]:

$$\hat{y}_0 \pm t_{\propto /2,n-2} \cdot SE \quad (1)$$

Where $\hat{y}_0$ is the predicted value of y from the t distribution for a given confidence level $\propto$and n −2 degrees of freedom, and SE stands for the Standard Error.

Using prediction intervals in regression analysis is beneficial because they offer a realistic spectrum of possible outcomes, which aids in the decision-making process [23]. This recognizes that a single predicted value is not absolute but rather a likely scenario within a range of potential outcomes. This forecasting method effectively incorporates the inherent uncertainties associated with future predictions, providing a more accurate depiction of what to expect. To further refine this model, neutrosophic statistics can be applied, which excel at managing ambiguity and indeterminacy in data. By converting the interval to a neutrosophic number, the traditional interval is enhanced to include a component of indeterminacy. This addition captures the uncertainty and imprecision that are typically present in real-world data, offering a more nuanced understanding of the variability in the data.[20]:

3. Materials and Methods

3.1. Data Source and Augmentation

This study focuses on predicting improvements in competency-based learning, using a dataset that includes information on improvement strategies implemented through the Deming Cycle (PDCA: Plan, Do, Check, Act). These strategies are evaluated based on their impact on learning outcomes, specifically the percentage increase in performance measured through key competencies (on a scale of 0 to 100). The dataset contains information on the resources allocated to the improvement process (in monetary units),

time spent on training (in hours), and the number of PDCA cycles executed from real data from students in the Food Industry and Nutrition program.

To ensure a sufficient dataset for robust model training and evaluation, a data augmentation approach was employed. A synthetic dataset was generated, consisting of 1000 samples. Each sample represents a simulated instance of an improvement process, characterized by the three independent variables mentioned above:

- Resources Allocated: Representing the monetary resources assigned to the improvement process, uniformly varying between 500 and 8000 units.
- Time Spent on Training: Representing the time dedicated to training activities, uniformly varying between 50 and 250 time units.
- Number of PDCA Cycles: Representing the number of iterative cycles of planning, doing, checking, and acting, with values ranging from 5 to 25.

The dependent variable, Performance Improvement, was calculated using a function that linearly combines the independent variables. This function also incorporates a sinusoidal term to simulate potential nonlinear relationships between the independent variables and the performance improvement. Finally, Gaussian noise with a mean of zero and a standard deviation of 300 units was added to represent the inherent variability and unpredictable factors influencing the improvement process.

Data generation was implemented using the NumPy library (v. 1.23.5) [24].

3.2. Data Preprocessing

The dataset was randomly split into training (80%) and testing (20%) sets using the train_test_split function from the scikit-learn library[25] (v. 1.2.1), ensuring reproducibility by setting a random seed (random_state=42). Subsequently, standard scaling was applied to the independent variables using the StandardScaler class from scikit-learn, fitting the scaler to the training set and applying it to both the training and testing sets.

3.3. Regression Models

Five different regression models were implemented and evaluated:

- ElasticNet[25]: A regularized linear regression model that combines L1 and L2 penalties, with a regularization parameter alpha of 0.01 and an l1_ratio parameter of 0.5. Implemented using the ElasticNet class from scikit-learn.
- RANSAC[25]: A robust algorithm for estimating the parameters of a model, used here with a base linear regressor (LinearRegression from scikit-learn). Implemented using the RANSACRegressor class from scikit-learn.

- Partial Least Squares Regression (PLS)[25]: A method that seeks directions in the space of the independent variables that explain the maximum variance in the space of the dependent variable. A model with two components was used. Implemented using the PLSRegression class from scikit-learn.

- Principal Component Analysis (PCA) + Linear Regression[25]: A combination of dimensionality reduction using PCA, retaining two principal components, followed by linear regression. Implemented as a pipeline using the PCA and LinearRegression classes from scikit-learn.

- Quantile Regression[26]: A model that estimates the conditional median of the dependent variable. A model for the 0.5 quantile was fitted using the QuantReg class from the statsmodels library (v. 0.13.5).

3.4. Neutrosophic Model

To integrate the predictions of the individual models (excluding Quantile Regression) and consider the uncertainty associated with each one, a neutrosophic model was proposed using neutrosophic number and interval analysis. This model is based on calculating means predictions of the individual models for each sample in the test set.

The result of the neutrosophic model is represented in neutrosophic form, The neutrosophic form is obtained (X_n) as the sum of $X_l$and X_u, adjusted by the indeterminacy interval I_n:

$$X_N = X_l + X_u I_N;\ I_N \in [I_l, I_u] \quad (2)$$

$I_{l,}$=0, andI_u

$$I_u = \frac{X_u - X_l}{X_u} \quad (3)$$

the neutrosophic approach enhances the robustness of predictions by incorporating uncertainty, leading to more informed and resilient decision-making [27, 28]

3.5. Model Evaluation

The performance of each model was evaluated using three modified metrics :

- Root Mean Squared Error (RMSE) [29]: A measure of the average difference between the predicted values and the actual values. For this purpose, a combined version of the root mean squared error (RMSE) is used, which simultaneously captures the proximity of the lower and upper interval bounds to the observed actual values. The proposed metric is defined as:

  $$RMSE_{interval} = \sqrt{\frac{1}{2n}\sum_{i=1}^{n}\left(\left(X_{l,i} - Y_i\right)^2 + \left(X_{u,i} - Y_i\right)^2\right)} \quad (4)$$

  where :

  $X_{l,i}$ represents the lower bound of the prediction interval for observation i,

  $X_{u,i}$represents the upper bound of the prediction interval for observation i,

$Y_i$is the corresponding observed actual value, and

n is the total number of observations.

- The Mean Absolute Error (MAE) [30]: is a metric that quantifies the average absolute difference between predicted values and actual observed values. When applied to prediction intervals, MAE can be adapted to assess the accuracy of both the lower and upper bounds of these intervals. The mathematical expression for this adapted MAE is:

$$\text{MAE}_{\ interval} = \frac{1}{2n}\sum_{i=1}^{n}\left(\left|X_{l,i} - Y_i\right|^2 + \left|X_{u,i} - Y_i\right|^2\right) \quad (5)$$

where :

$X_{l,i}$ represents the lower bound of the prediction interval for observation i,

$X_{u,i}$represents the upper bound of the prediction interval for observation i,

$Y_i$is the corresponding observed actual value, and

n is the total number of observations.

- Indeterminacy: The Indeterminacy metric is a quantitative measure used to assess the uncertainty or spread of prediction intervals across a dataset. It provides an average of the widths of the prediction intervals, which indicates how indeterminate the model is in its predictions.

$$Indeterminacy = \frac{1}{2}\sum_{i=1}^{n}\left(\left|X_{l,i} - X_{u,i}\right|\right) \quad (6)$$

where :

$X_{l,i}$ represents the lower bound of the prediction interval for observation i,

$X_{u,i}$represents the upper bound of the prediction interval for observation i,

n is the total number of observations.

3.6. Visualization

A graph was generated that represents the predictions of the simple neutrosophic model against the actual values of the test set. An uncertainty interval was included around the predictions of the neutrosophic model, defined by the mean of the predictions ± half the indeterminacy. Additionally, the neutrosophic number graph that represents the prediction interval in the neutrosophic model is visualized. The visualization was performed using the matplotlib library (v. 3.5.1) [31].

4. Results

This study evaluated the performance of five regression models for predicting performance improvement, along with a proposed neutrosophic simple model that combines the predictions of four of the individual models. The models were assessed using RMSE, MAE, and Indeterminacy. The results are presented in both tabular format (Table 1) and visually in Figure 1. In addition, each model's predictions are expressed as intervals and in neutrosophic form.

4.1. Individual Model Performance

Table 1 summarizes the performance metrics for each regression model on the test set.

Table 1. Performance Metrics of Individual and Neutrosophic Models

Model	RMSE	MAE	Indeterminacy
ElasticNet	461.9838	378.6670	378.6670
RANSAC	481.1275	392.3354	392.3354
PLS Regression	462.2177	378.7583	378.7583
PCA + Linear Regression	477.6142	393.9278	393.9278
Quantile Regression	467.2366	382.0711	382.0711
Neutrosophic	460.6323	376.1339	376.1339

As shown in Table 1, all models exhibited relatively strong performance. ElasticNet and PLS Regression demonstrated slightly superior performance compared to the other models, as evidenced by their lower RMSE and MAE values. Conversely, RANSAC and PCA + Linear Regression exhibited higher RMSE and MAE suggesting potentially less precise predictions. Quantile Regression yielded a reasonable performance, comparable to the other models. The table also presents the indeterminacy of each model, providing a more comprehensive view of their performance.

4.2. Neutrosophic Performance

The neutrosophic model, which averages the predictions of the ElasticNet, RANSAC, PLS Regression, and PCA + Linear Regression models, achieved the best overall performance. As shown in Table 1, it obtained an RMSE of 460.6323, and an MAE of 376.1339. The indeterminacy of this model was 376.1339. These results highlight the potential benefits of combining predictions from multiple models, resulting in enhanced accuracy and robustness, as evidenced by the slight improvements in RMSE, MAE, and Indeterminacy measures compared to the best-performing individual model (ElasticNet). The neutrosophic form of the simple model, $748.45 \pm 2362.58I;\ I \in [0, 752.268]$ encapsulates the central tendency and uncertainty associated with its predictions.

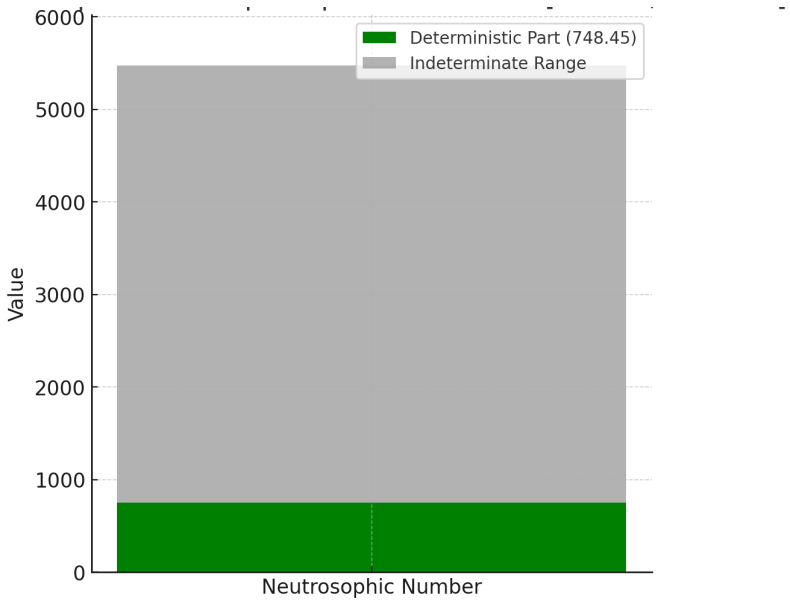

Figure 2. Neutrosophic Bar Graph

The bar graph represents the neutrosophic number within the interval [748.45,5473.60]:

- The green section of the bar indicates the deterministic part, fixed at 748.45
- The gray shaded area represents the indeterminate range, covering the uncertainty between the deterministic value and the upper limit of the interval.

This visualization effectively differentiates the known and uncertain components, offering a clear perspective on the given range.

4.3. Visual Representation of Neutrosophic Model Predictions

Figure 3 visually compares the predictions of the neutrosophic simple model against the actual values in the test set. The uncertainty interval, represented by the shaded gray area, highlights the range of potential values for each prediction, based on the calculated indeterminacy.

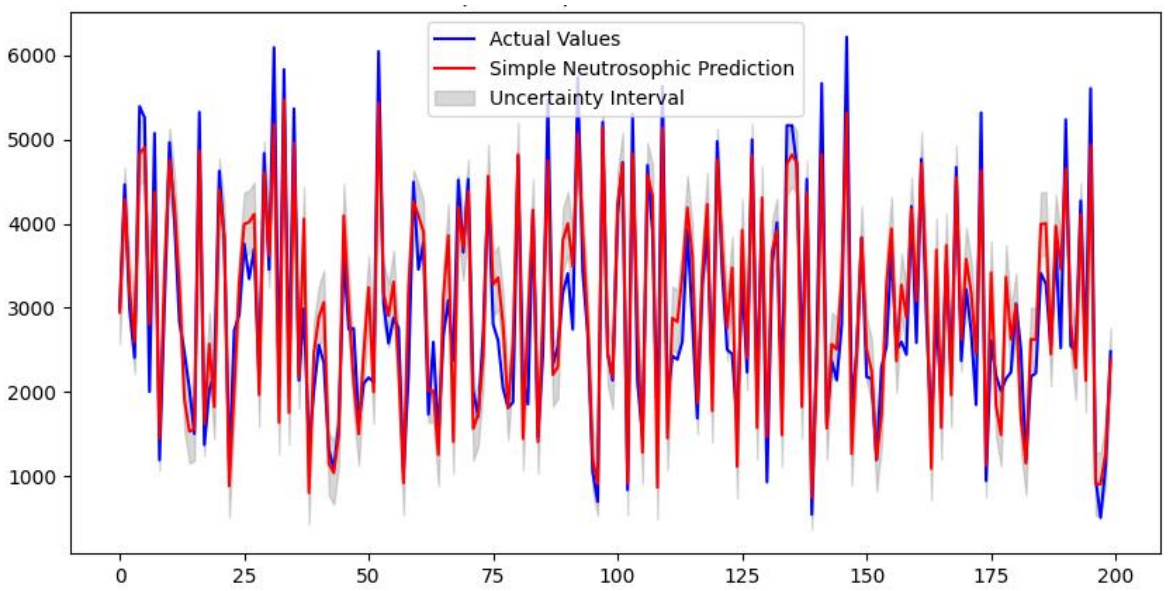

Figure 3. Neutrosophic Prediction with Interval

As depicted in Figure 3, the neutrosophic simple model's predictions closely align with the actual values. The uncertainty interval provides a visual representation of the model's confidence in its predictions, with a narrower interval suggesting higher confidence.

The neutrosophic model, by providing predictions with explicit uncertainty intervals, can significantly enhance the Deming Cycle (Plan-Do-Check-Act) in competency-based education. In the Plan phase, educators can use the predicted range of student performance to design differentiated instruction and set realistic learning goals. The Do phase involves implementing the planned strategies, while the Check phase utilizes the neutrosophic model to monitor student progress and assess the effectiveness of the interventions. Crucially, the uncertainty intervals provide valuable insights into the reliability of the predictions, allowing educators to interpret deviations from the expected outcomes with greater nuance. If a student's performance falls within the predicted range but towards the lower bound, it might signal the need for additional support, even if they are technically "on track." Conversely, exceeding the upper bound could indicate exceptional progress, prompting educators to consider enrichment activities. This continuous feedback loop informs the Act phase, enabling data-driven adjustments to teaching strategies and personalized interventions, ultimately fostering a more adaptive and responsive learning environment.

5. Conclusion

The model developed in this study aims to provide predictions based on PDCA cycles, which are crucial for improving competency-based learning, especially in dynamic and uncertain environments.PDCA cycles and training time are crucial for improving competency-based learning, especially in dynamic and uncertain environments.

This study has successfully integrated neutrosophic numbers into machine learning models, demonstrating their effectiveness in handling uncertainty and indeterminacy in data. The use of neutrosophic numbers in data preprocessing, model training, and evaluation has led to more robust and accurate models compared to traditional methods. The flexibility of neutrosophic numbers allows them to be

adapted to various data types and machine-learning problems, making them a valuable tool for enhancing predictive modeling.

The results of this study have significant implications for the field of machine learning, particularly in applications where uncertainty and indeterminacy are prevalent. By incorporating neutrosophic numbers into machine learning workflows, we can develop models that are more resilient to noisy data and provide more reliable predictions. Future research will focus on exploring the full potential of neutrosophic numbers in machine learning, including their application in deep learning and reinforcement learning.

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Ecological Efficiency Evaluation in Petrochemical Enterprises Using Plithogenic-ARTASI Model

Hailing Chen[1,2*], Dingqun Zhang[3]
[1]School of Biological and Chemical Engineering, Nanyang Institute of Technology, Nanyang, 473004, Henan, China
[2]State Key Laboratory of Motor Vehicle Biofuel Technology, Henan Tianguan Enterprise Group Co., LTD, Nanyang, 473100, Henan, China
[3]School of Intelligent Manufacturing, Nanyang Vocational College of science and Technology, Dengzhou, 474150, Henan, China
*Corresponding author, E-mail: j0104072@nyist.edu.cn

Abstract: The application of the ecological carrying capacity theory to the assessment of petrochemical companies' eco-efficiency is the main goal of this article. This study deviates from previous research that examines company eco-efficiency from a strategic and behavioral incentive standpoint. This study uses the multiple-criteria decision-making (MCDM) approach, considering both qualitative and quantitative factors. Plithogenic sets are used to compute expert importance levels in a decision support system for choosing Petrochemical Enterprises. The suggested plithogenic-based alternative ranking methodology based on adaptive standardized intervals (ARTASI) method is used to determine the ranking of Petrochemical Enterprises. The plithogenic-ARTASI model is a combination of these techniques. For this hybrid model, an algorithm is also created. The plithogenic-ARTASI hybrid model's suitability for Petrochemical Enterprises performance calculations is illustrated through a case study. Tests of robustness are conducted using a variety of sensitivity analysis scenarios. The study's findings show that plithogenic-ARTASI is reliable and relevant. In-depth managerial consequences are discussed and given.

Keywords: Ecological Efficiency; Petrochemical Enterprises; Decision Support; Uncertainty; MCDM Methodology.

1. Introduction

The Canadian Scientific Council (SCC) first put forth the idea of eco-efficiency in the 1970s. Ecological efficiency was added to the global conservation plan in the 1980s by the International Union for Conservation of Nature and Natural Resources (IUCN). Schaltegger and Sturn, two Swiss academics, defined ecological efficiency in 1992.One Several organizations and academics worldwide began researching ecological efficiency in the later 1990s, and they provided a variety of meanings for the term. The World's Sustainable Development of Industrial and Commercial Enterprises Commission (WBCSD) provided the most significant definition of eco-efficiency in 1995[1], [2]. Reducing the intensity of resource and energy usage, increasing the efficiency of using renewable resources, improving substance recovery, and lowering the emission of harmful compounds are the main goals of eco-efficiency[3], [4]. Throughout an enterprise's whole life cycle, the intensity of resource consumption and environmental pollution are decreased to a level consistent with ecological carrying. Three main objectives were established, and the

connotation was further clarified in 2000. Reducing the impact on the environment and consuming less resources are the two goals. Increasing the value of the product is the third[5], [6]. The theory of natural ecology is the source of the idea of carrying capacity, which is the maximum amount of time an organism may survive in each environment. Ecologists first proposed the idea of carrying capacity in the study of human ecology in the 1920s. Park introduced the idea of ecological carrying capacity, or the upper limit of the number of individuals that can exist in a certain habitat and published the carrying capacity principle. As a result, the concept of carrying capacity keeps growing, as does the theory and practice of carrying capacity. The interaction between the industrial economy, resources, environment, and social conditions was then measured using the carrying capacity concept[7], [8].

Decision models by structure serve as the foundation for decision assistance systems. Decision-makers, selection criteria, and alternatives are all included in decision models. Determining the influence of experts on the decision-making process, assessing the significance of criteria, and prioritizing options in this manner are all essential for making the optimal choice. The MCDM approach is used in this study to create a sophisticated decision support system. Figure 1 shows the steps of the MCDM. Furthermore, because plithogenic sets are linguistically based, they are utilized for expert evaluations. Due to its benefits in stabilizing unstable structures in expert evaluations, plithogenic sets are utilized in this area[9], [10], [11].

Plithogenic set is used with the alternative ranking technique based on adaptive standardized intervals (ARTASI) method for the rating of alternatives. Because it offers a two-step standardization option based on absolute maximum and minimum values, the developed plithogenic-ARTASI approach is recommended[12], [13]. A plithogenic-ARTASI hybrid model and a decision support system for ecological efficiency evaluation are therefore established in this study[14], [15].

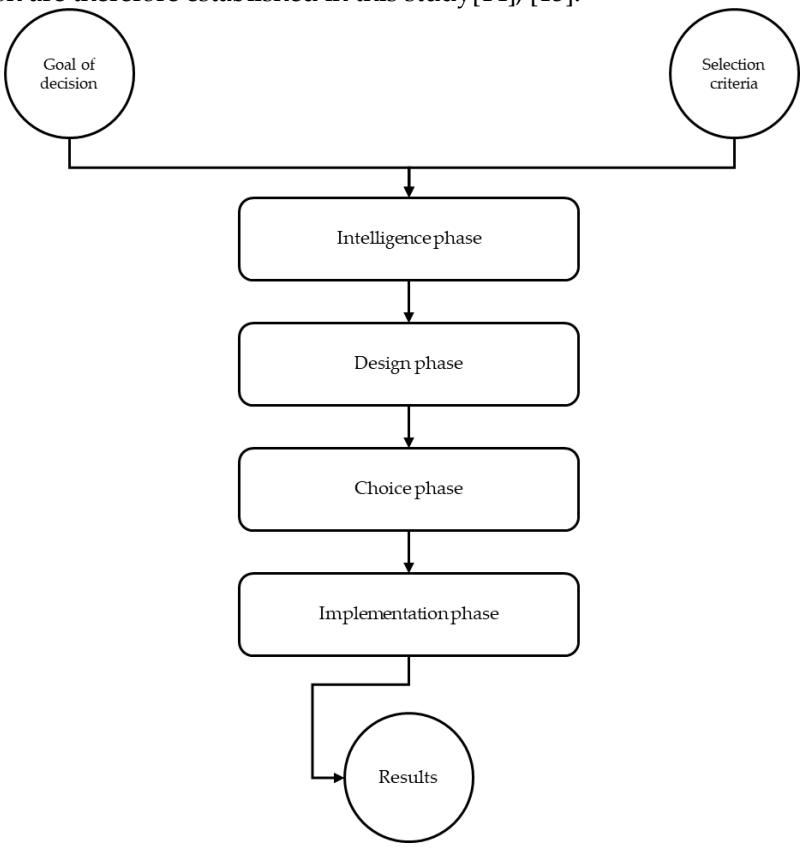

Figure 1. The steps of the MCDM problems.

1.1 Contributions of this study

The following summarizes the research' contributions to the body of literature:

- ✓ Determined a critical need for the creation of a decision support system in the context of ecological efficiency in Petrochemical Enterprises.
- ✓ Underlined the relevance of choosing the appropriate Petrochemical Enterprises to reach the best executive prospects and acknowledged the value of the Petrochemical Enterprises selection process.
- ✓ To raise executives' understanding of the Petrochemical Enterprises selection process, qualitative and quantitative criteria for Petrochemical Enterprises selection were included.
- ✓ Using an MCDM technique and plithogenic sets, the research creatively addressed the decision-making process, setting it apart from previous works.
- ✓ Broadened the decision-making process by combining qualitative and quantitative criteria, offering a more thorough viewpoint than traditional methods that frequently concentrate only on quantitative metrics.
- ✓ Created a new and creative method based on plithogenic sets for Petrochemical Enterprises: the plithogenic-ARTASI hybrid model. This paradigm offers a distinctive framework for decision support.
- ✓ Introduced the plithogenic-ARTASI technique for Petrochemical Enterprises and determining the most effective Petrochemical Enterprises. This approach's two-stage standardization process, which produces more effective results, is credited with its effectiveness.

2. Materials and Methods

Figure 2 shows the methodology used in this study, which was carried out to evaluate the effectiveness of ecological efficiency. The methodological framework was implemented using plithogenic sets and MCDM. Furthermore, the goal was to incorporate both qualitative and quantitative elements into the choice model. Four sections comprised the research approach in this context. The input parameters for measuring the performance of ecological efficiency were established in the first part. Experts, alternative Petrochemical Enterprises, and qualitative and quantitative criteria were among these considerations. The important levels of the experts' contributions to the decision-making process were ascertained in the second section. Alternative rankings were created while taking Petrochemical Enterprises performance levels into account. Furthermore, sensitivity analysis scenarios were used to apply robustness testing for the suggested model[16], [17], [18].

Step 1. Each alternative is evaluated by each expert with respect to each criterion. Initial decision matrices are constructed for the criteria and alternatives.

Step 2. We used the plithogenic operators to aggregate these decision matrices. The combined decision matrix is created in this step.

Step 3. Determine the crisp initial decision matrix between the criteria and alternatives by applying the score function.

Step 4. Compute the maximum and minimum absolute value in the initial decision matrix between the criteria and the alternative.

$$y_j^{max} = \left(\max x_{ij} + \max x_{ij}\right)^{1/m} \quad (1)$$

$$y_j^{min} = \left(\min x_{ij} + \min x_{ij}\right)^{1/m} \quad (2)$$

Step 5. Compute the normalization matrix by two steps such as:

$$R_{ij} = \frac{x_{ij}\left(B^{(u)} - B^{(l)}\right)}{y_j^{max} - y_j^{min}} + \frac{y_j^{max} B^{(l)} - y_j^{min} B^{(u)}}{y_j^{max} - y_j^{min}} \quad (3)$$

$$L_{ij} = \left(-R_{ij} + \max R_{ij} + \min R_{ij}\right) \quad (4)$$

Where $\left[B^{(u)}, B^{(l)}\right]$ is set to values [1,100]

Step 6. Compute the utility level of alternatives for positive and negative criteria such as:

$$P_{ij}^{+} = \left(\frac{L_{ij}}{\max L_{ij}}\right) w_j\left(B^{(u)}\right) \quad (5)$$

$$P_{ij}^{-} = \left(\frac{\min L_{ij}}{L_{ij}}\right) w_j\left(B^{(u)}\right) \quad (6)$$

Step 7. Compute the comprehensive assessment of the utility level of alternatives

$$N_i^{+} = \sum_{j=1}^{n} P_{ij}^{+} \quad (7)$$

$$N_i^{-} = \sum_{j=1}^{n} P_{ij}^{-} \quad (8)$$

Step 8. Compute the final utility function

$$N_i = (N_i^{+} + N_i^{-})(Q)\left(f(N_i^{+})\right)^t + \left((1-Q)\left(f(N_i^{-})\right)^t\right)^{1/t} \quad (9)$$

$$f(N_i^{+}) = \frac{N_i^{+}}{N_i^{+} + N_i^{-}} \quad (10)$$

$$f(N_i^{-}) = \frac{N_i^{-}}{N_i^{+} + N_i^{-}} \quad (11)$$

Where $t \in [1, +\infty]$ and $Q \in [0,1]$

Step 9. Final rank of alternatives.

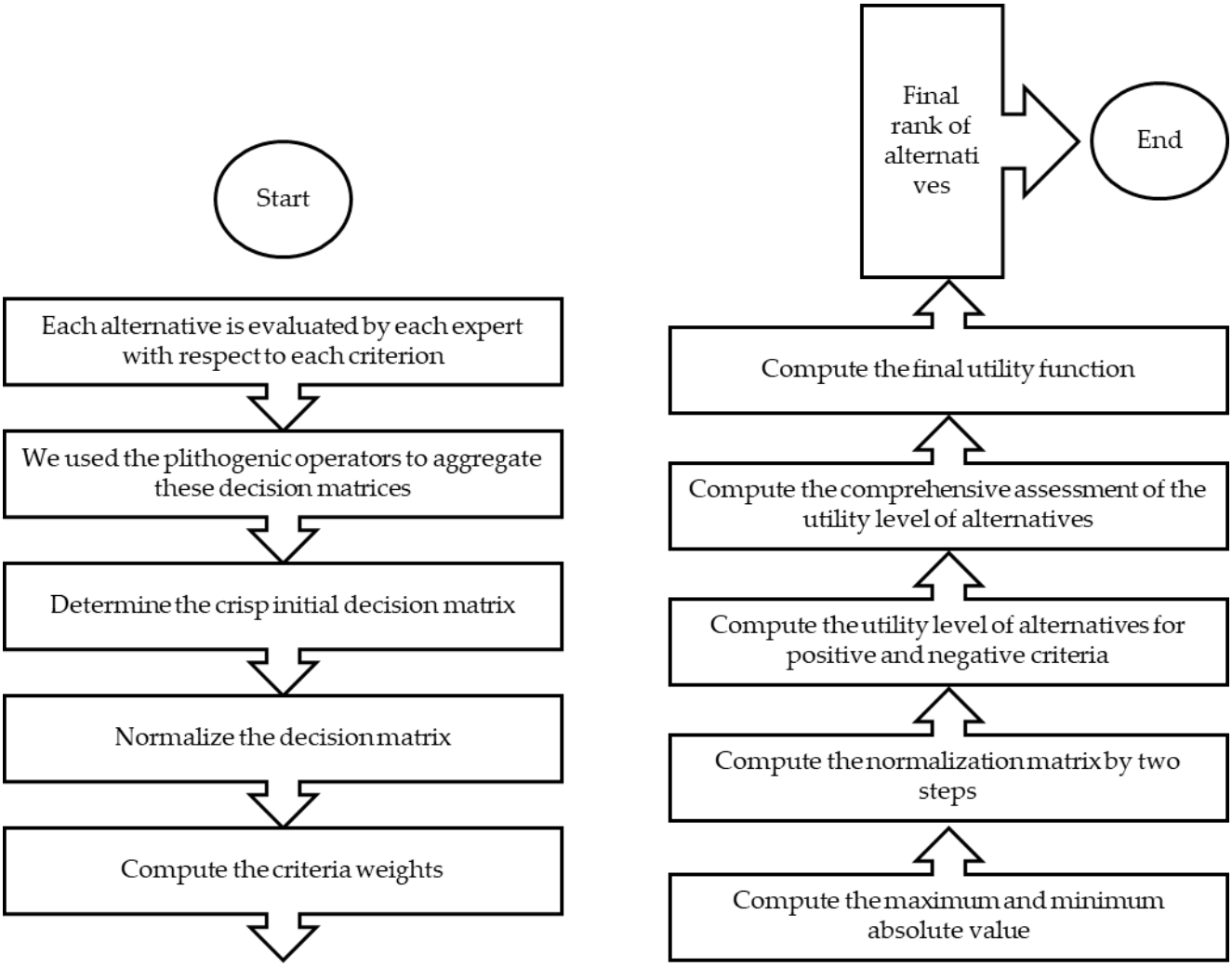
Start
Each alternative is evaluated by each expert with respect to each criterion
We used the plithogenic operators to aggregate these decision matrices
Determine the crisp initial decision matrix
Normalize the decision matrix
Compute the criteria weights
Compute the maximum and minimum absolute value
Compute the normalization matrix by two steps
Compute the utility level of alternatives for positive and negative criteria
Compute the comprehensive assessment of the utility level of alternatives
Compute the final utility function
Final rank of alternatives
End

Figure 2. The Steps of MCDM approach.

3. Results and Discussion

Ecological efficiency evaluation in petrochemical enterprises focuses on promoting sustainable development by balancing economic performance with environmental responsibility. It emphasizes the efficient use of resources and energy while minimizing negative environmental impacts, such as waste generation and pollution. Petrochemical companies play a critical role in global industrial chains, and their operations significantly affect ecosystems and communities. Eco-efficiency evaluation helps these enterprises align with environmental regulations and societal expectations by fostering cleaner production and responsible resource management. Through this approach, companies can enhance their competitiveness, reduce operational risks, and contribute to a circular economy. This process is vital for achieving long-term environmental and economic sustainability in the petrochemical industry. This section evaluates the Petrochemical Enterprises to assess ecological efficiency based on a set of criteria. This study evaluated the criteria and alternatives by four experts and decision makers. Seven criteria and five alternatives (Petrochemical Enterprises) are used in this study as shown in Figure 3. All criteria are positive except the third criterion is negative.

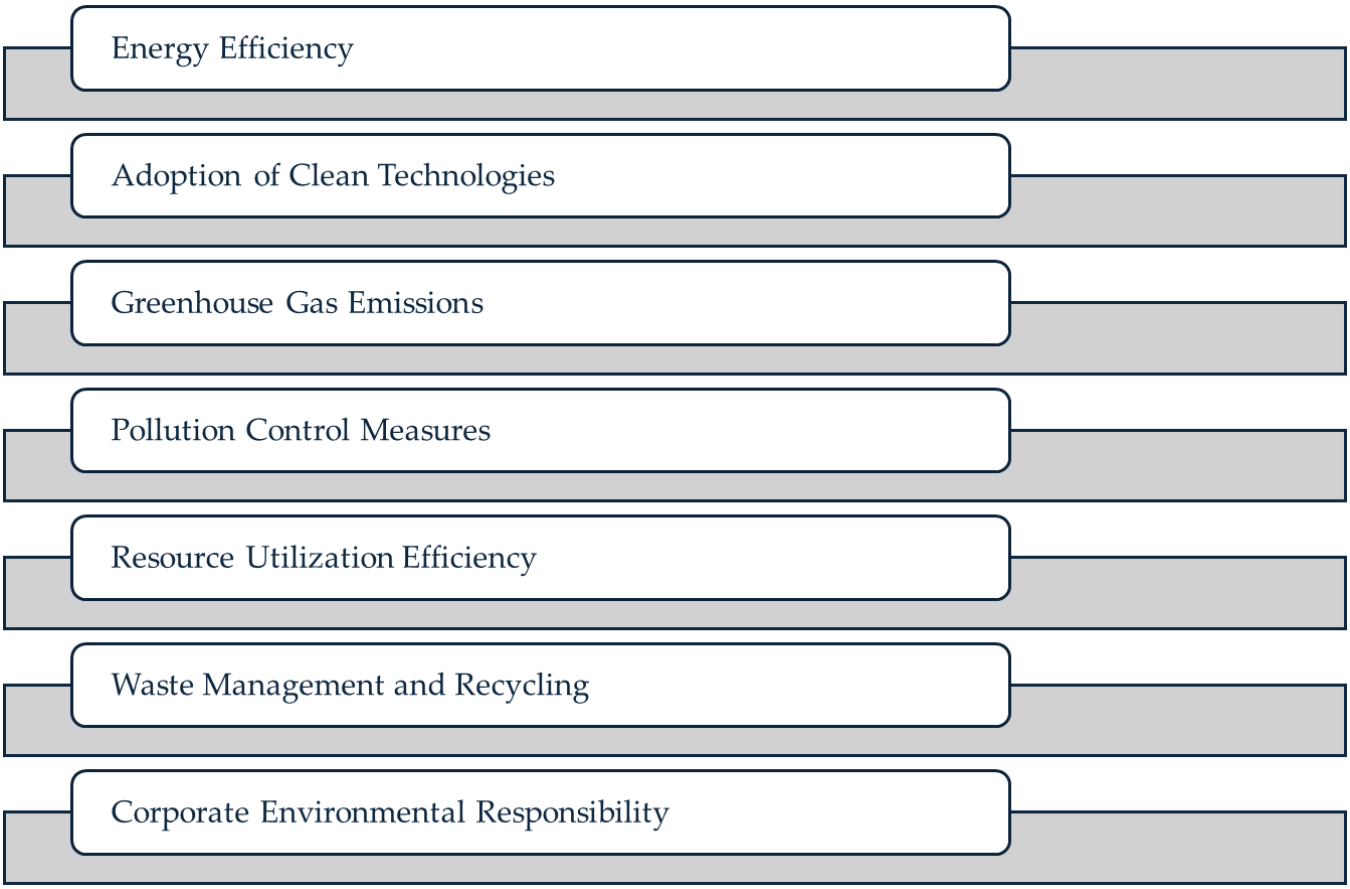

Figure 3. The criteria for assessment the ecological efficiency.

Step 1. Plithogenic numbers are used to evaluate the criteria and alternatives to build the decision matrix as shown in table 1.

Table 1. The decision matrix.

	A_1	A_2	A_3	A_4	A_5
C_1	(0.50, 0.40, 0.60)	(0.30, 0.40, 0.80)	(0.30, 0.40, 0.80)	(0.50, 0.40, 0.60)	(0.70, 0.30, 0.10)
C_2	(0.90, 0.10, 0.10)	(0.10, 0.70, 0.80)	(0.10, 0.70, 0.80)	(0.30, 0.40, 0.80)	(0.50, 0.40, 0.60)
C_3	(0.50, 0.40, 0.60)	(0.90, 0.10, 0.10)	(0.10, 0.70, 0.80)	(0.10, 0.70, 0.80)	(0.50, 0.40, 0.60)
C_4	(0.70, 0.30, 0.10)	(0.50, 0.40, 0.60)	(0.90, 0.10, 0.10)	(0.10, 0.70, 0.80)	(0.10, 0.70, 0.80)
C_5	(0.90, 0.10, 0.10)	(0.90, 0.10, 0.10)	(0.10, 0.70, 0.80)	(0.10, 0.70, 0.80)	(0.30, 0.40, 0.80)
C_6	(0.30, 0.40, 0.80)	(0.50, 0.40, 0.60)	(0.90, 0.10, 0.10)	(0.10, 0.70, 0.80)	(0.10, 0.70, 0.80)
C_7	(0.50, 0.40, 0.60)	(0.90, 0.10, 0.10)	(0.10, 0.70, 0.80)	(0.10, 0.70, 0.80)	(0.10, 0.70, 0.80)
	A_1	A_2	A_3	A_4	A_5
C_1	(0.30, 0.40, 0.80)	(0.30, 0.40, 0.80)	(0.50, 0.40, 0.60)	(0.90, 0.10, 0.10)	(0.10, 0.70, 0.80)
C_2	(0.30, 0.40, 0.80)	(0.30, 0.40, 0.80)	(0.30, 0.40, 0.80)	(0.50, 0.40, 0.60)	(0.30, 0.40, 0.80)
C_3	(0.10, 0.70, 0.80)	(0.30, 0.40, 0.80)	(0.30, 0.40, 0.80)	(0.50, 0.40, 0.60)	(0.10, 0.70, 0.80)
C_4	(0.30, 0.40, 0.80)	(0.30, 0.40, 0.80)	(0.50, 0.40, 0.60)	(0.10, 0.70, 0.80)	(0.70, 0.30, 0.10)
C_5	(0.90, 0.10, 0.10)	(0.30, 0.40, 0.80)	(0.30, 0.40, 0.80)	(0.50, 0.40, 0.60)	(0.90, 0.10, 0.10)
C_6	(0.90, 0.10, 0.10)	(0.10, 0.70, 0.80)	(0.30, 0.40, 0.80)	(0.30, 0.40, 0.80)	(0.50, 0.40, 0.60)

C_7	(0.70, 0.30, 0.10)	(0.30, 0.40, 0.80)	(0.30, 0.40, 0.80)	(0.50, 0.40, 0.60)	(0.90, 0.10, 0.10)
	A_1	A_2	A_3	A_4	A_5
C_1	(0.50, 0.40, 0.60)	(0.70, 0.30, 0.10)	(0.10, 0.70, 0.80)	(0.30, 0.40, 0.80)	(0.10, 0.70, 0.80)
C_2	(0.70, 0.30, 0.10)	(0.50, 0.40, 0.60)	(0.70, 0.30, 0.10)	(0.70, 0.30, 0.10)	(0.30, 0.40, 0.80)
C_3	(0.50, 0.40, 0.60)	(0.70, 0.30, 0.10)	(0.50, 0.40, 0.60)	(0.70, 0.30, 0.10)	(0.70, 0.30, 0.10)
C_4	(0.70, 0.30, 0.10)	(0.50, 0.40, 0.60)	(0.70, 0.30, 0.10)	(0.30, 0.40, 0.80)	(0.10, 0.70, 0.80)
C_5	(0.90, 0.10, 0.10)	(0.70, 0.30, 0.10)	(0.50, 0.40, 0.60)	(0.70, 0.30, 0.10)	(0.50, 0.40, 0.60)
C_6	(0.70, 0.30, 0.10)	(0.50, 0.40, 0.60)	(0.70, 0.30, 0.10)	(0.10, 0.70, 0.80)	(0.70, 0.30, 0.10)
C_7	(0.50, 0.40, 0.60)	(0.70, 0.30, 0.10)	(0.30, 0.40, 0.80)	(0.10, 0.70, 0.80)	(0.10, 0.70, 0.80)
	A_1	A_2	A_3	A_4	A_5
C_1	(0.10, 0.70, 0.80)	(0.70, 0.30, 0.10)	(0.30, 0.40, 0.80)	(0.70, 0.30, 0.10)	(0.90, 0.10, 0.10)
C_2	(0.10, 0.70, 0.80)	(0.10, 0.70, 0.80)	(0.70, 0.30, 0.10)	(0.30, 0.40, 0.80)	(0.30, 0.40, 0.80)
C_3	(0.10, 0.70, 0.80)	(0.70, 0.30, 0.10)	(0.30, 0.40, 0.80)	(0.90, 0.10, 0.10)	(0.90, 0.10, 0.10)
C_4	(0.10, 0.70, 0.80)	(0.10, 0.70, 0.80)	(0.70, 0.30, 0.10)	(0.30, 0.40, 0.80)	(0.90, 0.10, 0.10)
C_5	(0.10, 0.70, 0.80)	(0.70, 0.30, 0.10)	(0.30, 0.40, 0.80)	(0.90, 0.10, 0.10)	(0.70, 0.30, 0.10)
C_6	(0.30, 0.40, 0.80)	(0.10, 0.70, 0.80)	(0.70, 0.30, 0.10)	(0.30, 0.40, 0.80)	(0.90, 0.10, 0.10)
C_7	(0.10, 0.70, 0.80)	(0.70, 0.30, 0.10)	(0.10, 0.70, 0.80)	(0.70, 0.30, 0.10)	(0.30, 0.40, 0.80)

Step 2. Then we combine these numbers into a single matrix using plithogenic operators.
Step 3. Compute crisp values between criteria and alternatives. Then we compute the criteria weights as shown in Figure 4.

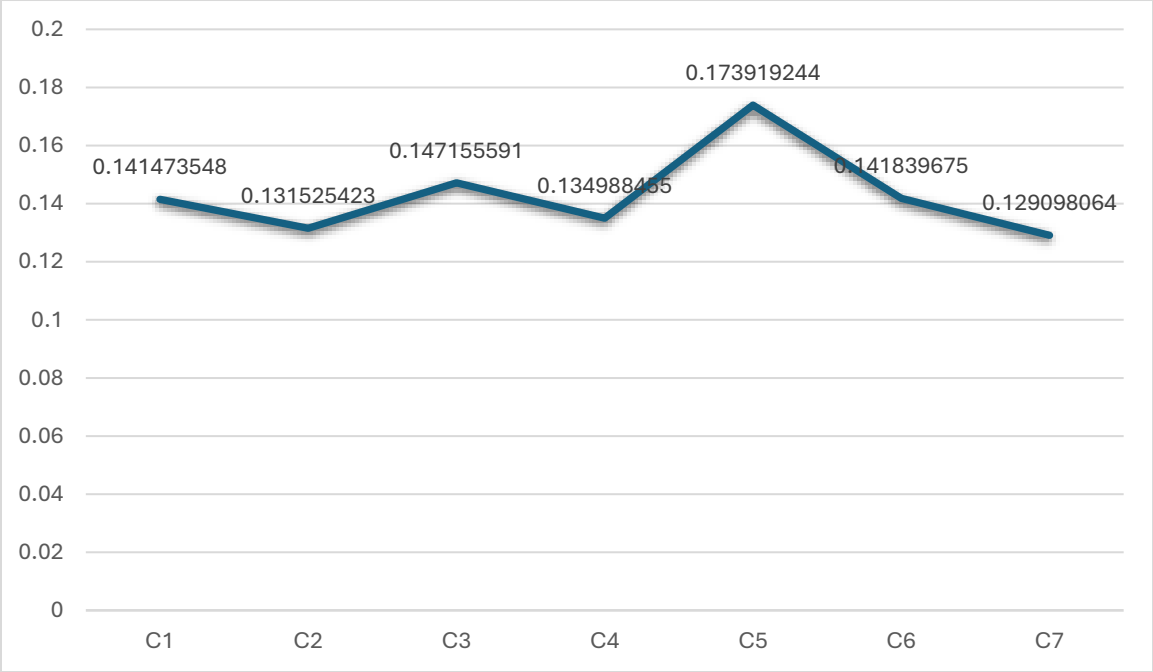

Figure 4. The criteria weights.

Step 4. Eqs. (1 and 2) are used to compute the maximum and minimum values.
Step 5. Eqs. (3 and 4) are used to compute the two-normalization matrix as shown in tables 2 and 3.

Table 2. The first normalization matrix.

	A_1	A_2	A_3	A_4	A_5
C_1	-213.724	21.8161	-275.89	124.2605	-60.4209
C_2	41.63347	-309.464	-21.9484	-10.943	-161.579
C_3	-249.589	155.7683	-235.508	56.25834	56.25834
C_4	3.428192	-91.8483	149.6393	-214.479	-2.10125
C_5	199.5484	142.0548	-225.615	48.54862	101.8798
C_6	86.60537	-141.642	138.548	-220.594	69.76501
C_7	-6.78208	138.548	-220.594	-81.3608	-69.332

Table 3. The second normalization matrix.

	A_1	A_2	A_3	A_4	A_5
C_1	62.09454	-173.446	124.2605	-275.89	-91.2087
C_2	-309.464	41.63347	-245.882	-256.887	-106.251
C_3	155.7683	-249.589	141.6882	-150.079	-150.079
C_4	-68.2684	27.00805	-214.479	149.6393	-62.739
C_5	-225.615	-168.121	199.5484	-74.6149	-127.946
C_6	-168.652	59.59518	-220.594	138.548	-151.811
C_7	-75.2644	-220.594	138.548	-0.68565	-12.7145

Step 6. Then we compute the utility level of alternatives for positive and negative criteria using Eqs. (5 and 6).

Step 7. Then we compute the comprehensive assessment of the utility level of alternatives using Eqs. (7 and 8)

Step 8. Then we compute the final utility function using Eq. (9). We put the value of $Q = 0.5$.

Step 9. Final rank of alternatives is shown in Table 4 and Figure 5.

Table 4. The rank of alternatives.

	N_i^+	N_i^-	$f(N_i^+)$	$f(N_i^-)$	N_i	*Rank*
A_1	-140.795	14.71555907	1.116716	-0.11672	-70.4559	1
A_2	-33.2649	-9.183987339	0.783646	0.216354	-16.5243	5
A_3	-75.1597	16.17790147	1.274286	-0.27429	-37.717	4
A_4	-91.449	-15.27345493	0.856886	0.143114	-45.6529	2
A_5	-77.488	-15.27345493	0.835347	0.164653	-38.6617	3

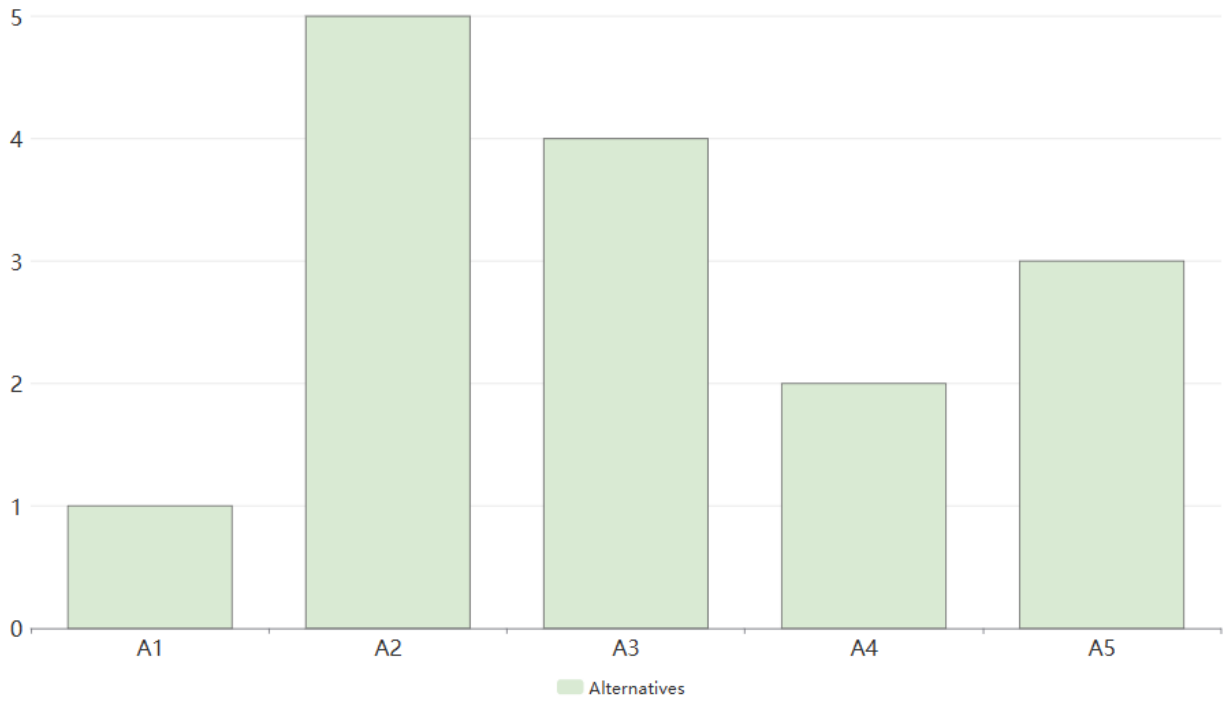

Figure 5. The rank of alternatives.

4. Sensitivity Analysis

This section performed the sensitivity analysis to show the different ranks of the alternatives. Sensitivity analysis shows the stability of the ranks under different cases and changes in each case of the best and worst alternative. This operation ensures the results is not sensitive to change in different cases. This study changes un value of Q between 0 and 1 with eleven cases. Then we applied the proposed approach under these cases as shown in Figure 6. The results show alternative 2 is the best and alternative 1 is the worst. The results show that the rank is stable under different cases.

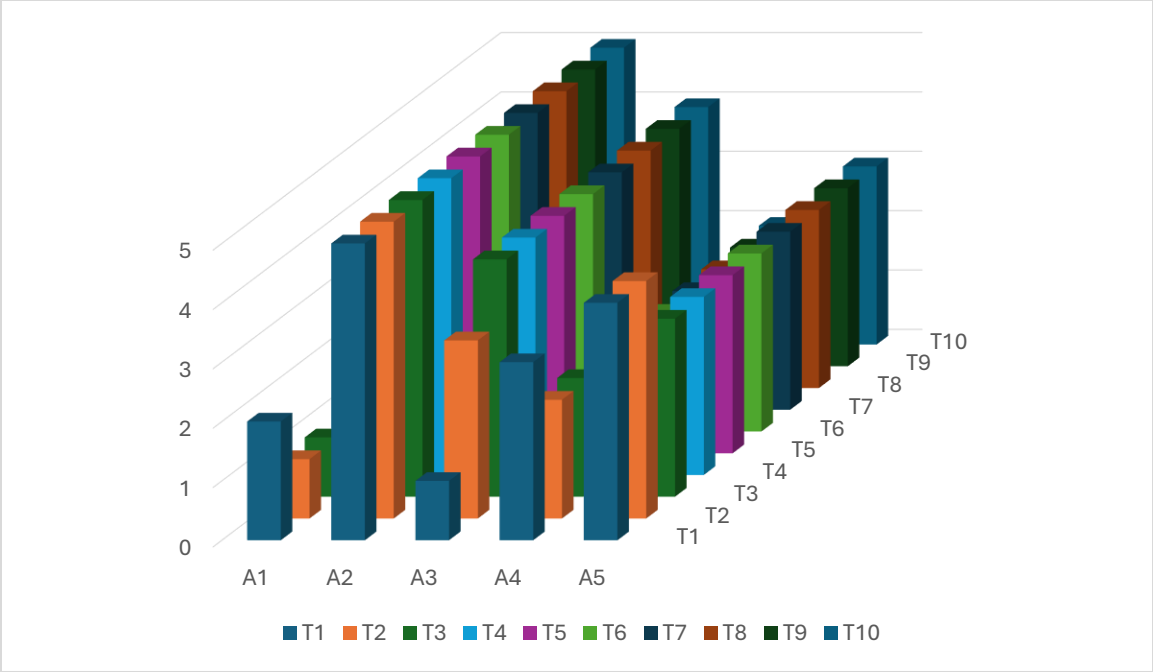

Figure 6. Th sensitivity analysis ranks.

In the second phase in the sensitivity analysis, we change the criteria weights under different weights and then rank the alternatives. Figure 7 shows the criteria weights under different cases. In the first case, we put all criteria weight the same weights. In the second case, we increase the first criterion with 25% and other criteria weights with the same weights.

Then we rank the alternatives under different criteria weights. We show all cases that accepting alternative 2 is the best and alternative 1 is the worst. Case 1, 3, and 4 have identical rank of alternatives. Alternative 2 is the best, followed by alternative 5, alternative 3, alternative 4, and alternative 1 is the worst. Cases 2, 5, 6, 7, and 8 have identical ranks. Alternative 2 is the best, followed by alternative 4, alternative 5, alternative 3, and alternative 1 is the worst. Figure 8 shows the different ranks of alternatives.

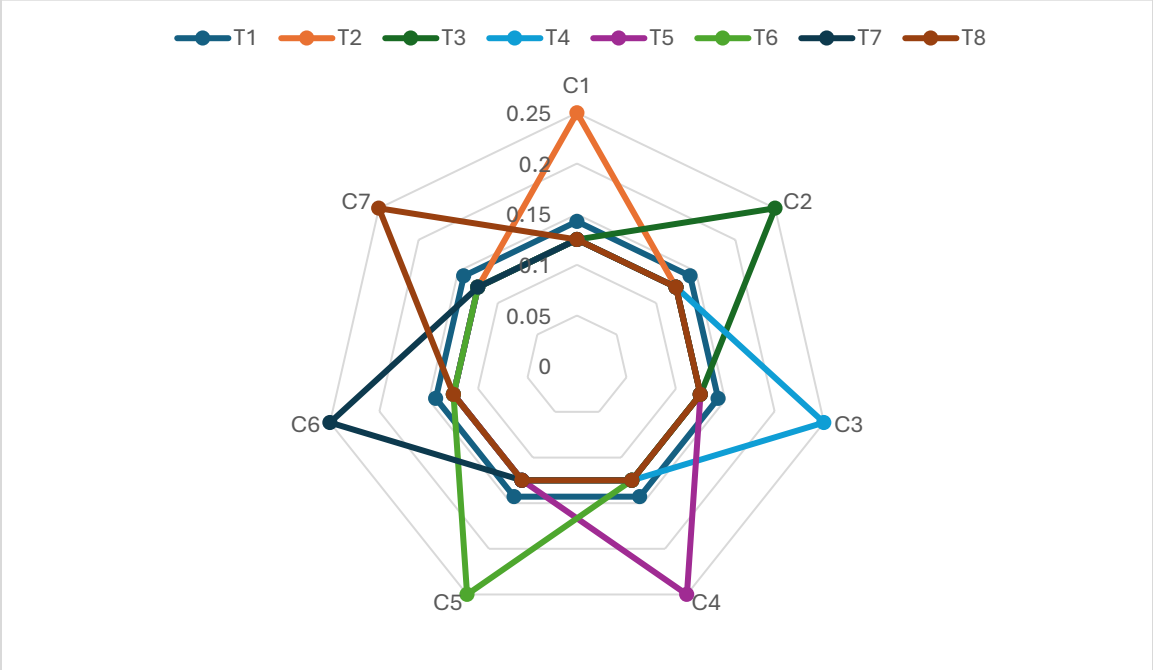

Figure 7. The criteria change under different cases.

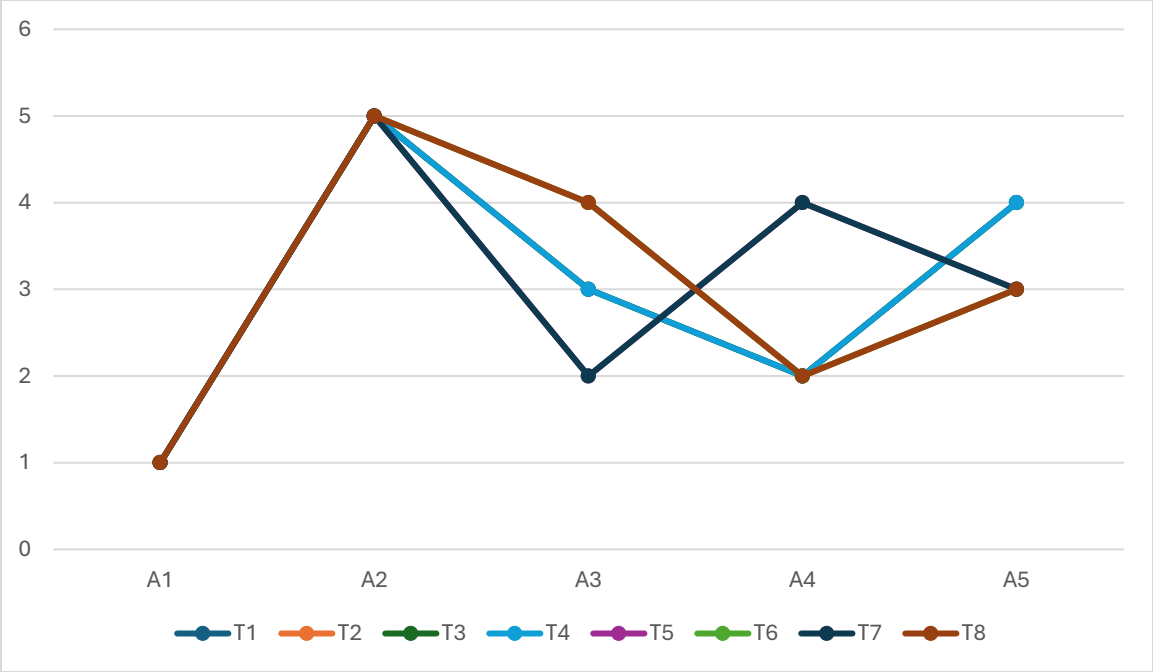

Figure 8. Rank of alternatives under different weights of criteria.

5. Conclusions

With emphasis on the executive selection Petrochemical Enterprises for ecological efficiency, this study tackled the necessity of creating a decision support system within the framework of ecological efficiency evaluation. In contrast to conventional decision support systems, this study offered a thorough viewpoint by integrating qualitative and quantitative criteria in a novel way. For the Petrochemical Enterprises performance analysis topic that was addressed, the suggested plithogenic-ARTASI hybrid model provided a strong foundation. The suggested plithogenic-ARTASI method showed promising results for evaluating Petrochemical Enterprises and determining which one was best. plithogenic aggregation operators were used to determine the weights of the expert opinions that went into evaluating the criterion. Following the presentation of the plithogenic-ARTASI hybrid method's step-by-step development. A case study that examined Petrochemical Enterprises selection for ecological efficiency evaluation was used to test the algorithm's resilience. The results of the study validated the new decision-analytic algorithm's dependability.

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A Concise Introduction to HyperFuzzy, HyperNeutrosophic, HyperPlithogenic, HyperSoft, and HyperRough Sets with Practical Examples

Takaaki Fujita[1]∗, Florentin Smarandache[2],

[1]∗ Independent Researcher, Shinjuku, Shinjuku-ku, Tokyo, Japan. t171d603@gunma-u.ac.jp

[2] University of New Mexico, Gallup Campus, NM 87301, USA. fsmarandache@gmail.com :

Abstract. Various set concepts are widely recognized for effectively a ddressing u ncertainty, i ncluding fuzzy sets, neutrosophic sets, plithogenic sets, rough sets, and soft sets. These concepts have been further extended through hyperstructures (based on powersets) and superhyperstructures (based on n-th powersets, which are sets with repeated power set structures) [5,12–14,32]. These extensions are commonly referred to as HyperUncertain, SuperUncertain, and SuperHyperUncertain Sets/Logics/Probabilities/Statistics, and they were introduced by Smarandache in 2016-2017 [32] as parts of HyperStructures, SuperStructures, and SuperHyperStructures respectively. This paper recalls these concepts along with concrete examples.

Keywords: HyperFuzzy Set, HyperNeutrosophic Set, HyperPlithogenic Set, HyperRough Set, HyperSoft Set

1. Introduction

1.1. *Uncertain Set*

Various types of uncertain sets, such as Fuzzy Set [43], Intuitionistic Fuzzy Set [1], Vague Set [18], Soft Set [22], Rough Set [24], Neutrosophic Set [31], and Plithogenic Set [35], are well-known. For instance:

- A fuzzy set generalizes classical sets by allowing elements to have membership degrees within $[0, 1]$, representing partial truth [43].
- A neutrosophic set extends fuzzy sets by incorporating truth, indeterminacy, and falsity degrees, providing a more nuanced uncertainty model [31].
- A plithogenic set further incorporates multiple attributes, contradictions, and degrees of membership to model complex, contradictory, and multi-valued data [34].
- A soft set is a parameterized family of subsets that models uncertainty by associating attributes with their corresponding approximate elements [22].

- A rough set approximates a set using a pair of lower and upper approximations, capturing uncertainty due to incomplete information [24].

These concepts have been extended to areas like graph theory and topology, leading to various mathematical and applied investigations.

In recent years, these uncertain sets have been generalized into HyperFuzzy, HyperVague, HyperSoft, HyperRough, HyperNeutrosophic, and HyperPlithogenic Sets by utilizing hyperstructures (based on power sets) [5,32]. Furthermore, they have been expanded into SuperHyperFuzzy, SuperHyperVague, SuperHyperSoft, SuperHyperRough, SuperHyperNeutrosophic, and SuperHyperPlithogenic Sets by employing superhyperstructures, which are repeated applications of the power set concept [5].

1.2. *Our Contribution*

This paper revisits the concepts of HyperFuzzy, HyperSoft, HyperRough, HyperNeutrosophic, and HyperPlithogenic Sets, as well as SuperHyperFuzzy, SuperHyperSoft, SuperHyperRough, SuperHyperNeutrosophic, and SuperHyperPlithogenic Sets, as defined in works such as [5, 12–14], providing detailed examples. These sets are also referred to as HyperUncertain Sets and SuperHyperUncertain Sets. This study aims to raise awareness among researchers and contribute to further exploration of HyperUncertain Concepts and SuperHyperUncertain Concepts.

2. **Preliminaries and definitions**

In this section, we present a brief overview of the definitions and notations used throughout this paper. We will specifically cover fundamental concepts related to graphs, including fuzzy graphs, intuitionistic fuzzy graphs, Turiyam Neutrosophic graphs, and neutrosophic graphs.

2.1. *SuperHyperstructure*

Mathematical structures can be systematically extended into Hyperstructures and SuperHyperstructures using the concepts of the power set and n-th powerset. The n-th powerset represents an iterative extension of the powerset concept, where each iteration generates the powerset of the previous level [17, 29, 37]. The definitions and examples of n-th powersets, hyperstructures, and superhyperstructures are provided below.

Definition 2.1 (Set)**.** [20] A *set* is a collection of distinct, well-defined objects, referred to as *elements*. For any object x, it can be determined whether x is an element of a given set. If x belongs to a set A, this is denoted as $x \in A$. Sets are often represented using curly braces.

Definition 2.2 (Subset)**.** [20] A set A is called a *subset* of a set B if every element of A is also an element of B. This is denoted as $A \subseteq B$. Formally:

$$A \subseteq B \iff (\forall x \in A)(x \in B).$$

If $A \subseteq B$ and $A \neq B$, then A is called a *proper subset* of B, denoted as $A \subset B$.

Definition 2.3 (Empty Set)**.** [20] The *empty set*, denoted by $\emptyset$, is the unique set that contains no elements. Formally:

$$\emptyset = \{x \mid x \neq x\}.$$

For any set A, the empty set is a subset of A:

$$\emptyset \subseteq A.$$

Definition 2.4 (Base Set)**.** A *base set* S is the foundational set from which complex structures such as powersets and hyperstructures are derived. It is formally defined as:

$$S = \{x \mid x \text{ is an element within a specified domain}\}.$$

All elements in constructs like $\mathcal{P}(S)$ or $\mathcal{P}_n(S)$ originate from the elements of S.

Definition 2.5 (Powerset)**.** [28] The *powerset* of a set S, denoted $\mathcal{P}(S)$, is the set of all subsets of S, including the empty set and S itself. Formally,

$$\mathcal{P}(S) = \{A \mid A \subseteq S\}.$$

Example 2.6 (Powerset of a Set)**.** Let $S = \{a, b\}$. The powerset $\mathcal{P}(S)$ is:

$$\mathcal{P}(S) = \{\emptyset, \{a\}, \{b\}, \{a, b\}\}.$$

Definition 2.7 (n-th powerset)**.** (cf. [29, 37]) The n-th powerset of H, denoted $P_n(H)$, is defined recursively as:

$$P_1(H) = P(H), \quad P_{n+1}(H) = P(P_n(H)) \quad \text{for } n \geq 1.$$

Similarly, the n-th non-empty powerset of H, denoted $P_n^*(H)$, is defined as:

$$P_1^*(H) = P^*(H), \quad P_{n+1}^*(H) = P^*(P_n^*(H)).$$

Example 2.8 (n-th Powerset of a Set)**.** Let $H = \{a, b\}$. The first powerset $\mathcal{P}_1(H)$ is:

$$\mathcal{P}_1(H) = \{\emptyset, \{a\}, \{b\}, \{a, b\}\}.$$

The second powerset $\mathcal{P}_2(H)$ is the powerset of $\mathcal{P}_1(H)$, given by:

$$\mathcal{P}_2(H) = \{\emptyset, \{\emptyset\}, \{\{a\}\}, \{\{b\}\}, \{\{a, b\}\}, \{\emptyset, \{a\}\}, \ldots, \mathcal{P}_1(H)\}.$$

To establish a formal foundation for the concepts of Hyperstructures and Superhyperstructures, we present the following definitions and propositions.

Definition 2.9 (Classical Structure)**.** (cf. [29, 37]) A *Classical Structure* is a mathematical framework defined on a non-empty set H, equipped with one or more *Classical Operations* that satisfy specified *Classical Axioms*. Specifically:

A *Classical Operation* is a function of the form:

$$\#_0 : H^m \to H,$$

where $m \geq 1$ is a positive integer, and H^m denotes the m-fold Cartesian product of H. Common examples include addition and multiplication in algebraic structures such as groups, rings, and fields.

Definition 2.10 (Hyperoperation)**.** (cf. [27, 39–41]) A *hyperoperation* is a generalization of a binary operation where the result of combining two elements is a set, not a single element. Formally, for a set S, a hyperoperation $\circ$ is defined as:

$$\circ : S \times S \to \mathcal{P}(S),$$

where $\mathcal{P}(S)$ is the powerset of S.

Definition 2.11 (Hyperstructure)**.** (cf. [17, 29, 37]) A *Hyperstructure* extends the notion of a Classical Structure by operating on the powerset of a base set. Formally, it is defined as:

$$\mathcal{H} = (\mathcal{P}(S), \circ),$$

where S is the base set, $\mathcal{P}(S)$ is the powerset of S, and $\circ$ is an operation defined on subsets of $\mathcal{P}(S)$. Hyperstructures allow for generalized operations that can apply to collections of elements rather than single elements.

Example 2.12 (Hyperstructure)**.** Let $S = \{a, b\}$ be a base set. Its powerset is:

$$\mathcal{P}(S) = \{\emptyset, \{a\}, \{b\}, \{a, b\}\}.$$

Define a hyperoperation $\circ$ on $\mathcal{P}(S)$ as follows:

$$X \circ Y = \{x \cup y \mid x \in X, y \in Y\}, \quad \text{for } X, Y \subseteq \mathcal{P}(S).$$

For example:

$$\{a\} \circ \{b\} = \{\{a, b\}\}, \quad \{a\} \circ \emptyset = \{\{a\}\}.$$

This structure $\mathcal{H} = (\mathcal{P}(S), \circ)$ is a hyperstructure where the operation $\circ$ maps pairs of subsets to subsets of the powerset.

Definition 2.13 (SuperHyperOperations)**.** (cf. [37]) Let H be a non-empty set, and let $\mathcal{P}(H)$ denote the powerset of H. The n-th powerset $\mathcal{P}^n(H)$ is defined recursively as follows:

$$\mathcal{P}^0(H) = H, \quad \mathcal{P}^{k+1}(H) = \mathcal{P}(\mathcal{P}^k(H)), \quad \text{for } k \geq 0.$$

A *SuperHyperOperation* of order (m, n) is an m-ary operation:

$$\circ^{(m,n)} : H^m \to \mathcal{P}_*^n(H),$$

where $\mathcal{P}_*^n(H)$ represents the n-th powerset of H, either excluding or including the empty set, depending on the type of operation:

- If the codomain is $\mathcal{P}_*^n(H)$ excluding the empty set, it is called a *classical-type* (m, n)-*SuperHyperOperation*.
- If the codomain is $\mathcal{P}^n(H)$ including the empty set, it is called a *Neutrosophic* (m, n)-*SuperHyperOperation*.

These SuperHyperOperations are higher-order generalizations of hyperoperations, capturing multi-level complexity through the construction of n-th powersets.

Definition 2.14 (n-Superhyperstructure)**.** (cf. [29,37]) An *n-Superhyperstructure* further generalizes a Hyperstructure by incorporating the n-th powerset of a base set. It is formally described as:

$$\mathcal{SH}_n = (\mathcal{P}_n(S), \circ),$$

where S is the base set, $\mathcal{P}_n(S)$ is the n-th powerset of S, and $\circ$ represents an operation defined on elements of $\mathcal{P}_n(S)$. This iterative framework allows for increasingly hierarchical and complex representations of relationships within the base set.

Example 2.15 (SuperHyperstructure)**.** Let $S = \{a, b\}$. The first powerset is:

$$\mathcal{P}_1(S) = \mathcal{P}(S) = \{\emptyset, \{a\}, \{b\}, \{a, b\}\}.$$

The second powerset is:

$$\mathcal{P}_2(S) = \mathcal{P}(\mathcal{P}(S)) = \{\emptyset, \{\emptyset\}, \{\{a\}\}, \{\{b\}\}, \{\{a, b\}\}, \{\emptyset, \{a\}\}, \ldots\}.$$

Define a superhyperoperation $\circ$ on $\mathcal{P}_2(S)$ as:

$$X \circ Y = \{\mathcal{P}(x \cup y) \mid x \in X, y \in Y\}, \quad \text{for } X, Y \subseteq \mathcal{P}_2(S).$$

For example:

$$\{\{a\}\} \circ \{\{b\}\} = \{\mathcal{P}(\{a, b\})\} = \{\emptyset, \{a\}, \{b\}, \{a, b\}\}.$$

The structure $\mathcal{SH}_2 = (\mathcal{P}_2(S), \circ)$ is a 2-Superhyperstructure, which incorporates the second powerset and a higher-order operation $\circ$.

3. **Review Result in this paper**

This paper revisits the HyperUncertain and SuperHyperUncertain versions of Fuzzy Sets, Neutrosophic Sets, Plithogenic Sets, Soft Sets, and Rough Sets.

3.1. *Fuzzy Set*

The Fuzzy Set is a well-known concept used to address uncertainty in set theory. These sets can be extended into Hyperfuzzy Sets and SuperHyperfuzzy Sets using hyperstructures and superhyperstructures. The definition is provided below [43].

Definition 3.1 (Fuzzy Set)**.** [43] A *fuzzy set* τ in a non-empty universe Y is a mapping $\tau : Y \to [0,1]$. A *fuzzy relation* on Y is a fuzzy subset δ in $Y \times Y$. If τ is a fuzzy set in Y and δ is a fuzzy relation on Y, then δ is called a *fuzzy relation on* τ if

$$\delta(y,z) \leq \min\{\tau(y), \tau(z)\} \quad \text{for all } y, z \in Y.$$

Example 3.2 (Fuzzy Set: Temperature Assessment)**.** Consider a universe Y representing daily temperatures in degrees Celsius, ranging from -30 to 50. Define a fuzzy set τ called *Hot*, where:

$$\tau(t) = \begin{cases} 0 & \text{if } t \leq 20, \\ \frac{t-20}{15} & \text{if } 20 < t < 35, \\ 1 & \text{if } t \geq 35. \end{cases}$$

Here, $\tau(t)$ indicates the degree of membership of temperature t in the concept "Hot." Temperatures at or below 20°C are not considered hot ($\tau(t) = 0$), those at or above 35°C are fully hot ($\tau(t) = 1$), and intermediate values transition linearly between 0 and 1.

Definition 3.3 (HyperFuzzy Set)**.** [19, 21, 38] Let X be a non-empty set. A *hyperfuzzy set* over X is defined as a mapping:

$$\tilde{\mu} : X \to \mathcal{P}([0,1]) \setminus \{\emptyset\},$$

where $\mathcal{P}([0,1]) \setminus \{\emptyset\}$ represents the family of all non-empty subsets of the interval $[0,1]$.

For each element $x \in X$, $\tilde{\mu}(x)$ assigns a non-empty subset of $[0,1]$, representing the possible membership degrees of x in the hyperfuzzy set. This definition generalizes classical fuzzy sets by allowing the membership degree of each element to be a range (set of values) instead of a single scalar value.

Example 3.4 (HyperFuzzy Set: Interval-Based Uncertainty)**.** Let X be the set of daily temperatures $\{-30, -29, \ldots, 49, 50\}$. Define a hyperfuzzy set $\tilde{\mu}$ called *Hot-Interval*, where:

$$\tilde{\mu}(t) \subseteq [0,1],$$

and each $\tilde{\mu}(t)$ is an interval representing uncertainty about how hot the day is. For instance:

$$\tilde{\mu}(20) = [0, 0.1], \quad \tilde{\mu}(25) = [0.3, 0.5], \quad \tilde{\mu}(35) = [0.9, 1].$$

Thus, rather than assigning a single membership degree, we assign a set of possible degrees, reflecting incomplete or imprecise data on temperature perception.

Definition 3.5 (n-SuperHyperFuzzy Set)**.** [5, 7] Let X be a non-empty set. The n-SuperHyperFuzzy Set is a recursive generalization of fuzzy sets, hyperfuzzy sets, and superhyperfuzzy sets. It is defined as:

$$\tilde{\mu}_n : \tilde{\mathcal{P}}_n(X) \to \tilde{\mathcal{P}}_n([0,1]),$$

where:

- $\tilde{\mathcal{P}}_1(X) = \tilde{\mathcal{P}}(X)$, and for $k \geq 2$,
$$\tilde{\mathcal{P}}_k(X) = \tilde{\mathcal{P}}(\tilde{\mathcal{P}}_{k-1}(X)),$$
represents the k-th nested family of non-empty subsets of X.
- $\tilde{\mathcal{P}}_n([0,1])$ is similarly defined for the interval $[0,1]$.
- $\tilde{\mu}_n$ assigns to each element $A \in \tilde{\mathcal{P}}_n(X)$ a non-empty subset $\tilde{\mu}_n(A) \subseteq [0,1]$, representing the degrees of membership associated with A at the n-th level.

Example 3.6 (n-SuperHyperFuzzy Set: Multi-Level Uncertainty)**.** Suppose X again represents daily temperatures, and we extend our uncertainty to multiple layers. Let

$$\tilde{\mathcal{P}}_1(X) = \tilde{\mathcal{P}}(X), \quad \tilde{\mathcal{P}}_2(X) = \tilde{\mathcal{P}}(\tilde{\mathcal{P}}(X)), \quad \ldots, \quad \tilde{\mathcal{P}}_n(X) = \tilde{\mathcal{P}}(\tilde{\mathcal{P}}_{n-1}(X)),$$

where each $\tilde{\mathcal{P}}(\cdot)$ denotes the family of non-empty subsets at each level. Define

$$\tilde{\mu}_n : \tilde{\mathcal{P}}_n(X) \to \tilde{\mathcal{P}}_n([0,1]),$$

so that for an element $A \in \tilde{\mathcal{P}}_n(X)$, $\tilde{\mu}_n(A)$ is a non-empty subset of $[0,1]$ capturing multiple layers of uncertainty. Concretely, if A is a second-level set describing "moderately hot" or "extremely hot" temperature ranges, then $\tilde{\mu}_n(A)$ might itself be a family of intervals in $[0,1]$ that represent various degrees of membership for these subcategories. This approach models complex, nested uncertainties about temperature evaluations across several interpretive levels.

3.2. *Neutrosophic Set*

This section provides the definitions of Neutrosophic Set, HyperNeutrosophic Set, and SuperHyperNeutrosophic Set. A Neutrosophic Set models uncertainty using three membership functions: truth (T), indeterminacy (I), and falsity (F), which satisfy:

$$0 \leq T + I + F \leq 3.$$

[4, 16, 30, 31, 42]. These sets can be extended into HyperNeutrosophic Sets and SuperHyperNeutrosophic Sets using hyperstructures and superhyperstructures.

Definition 3.7 (Neutrosophic Set)**.** [30, 31] Let X be a non-empty set. A *Neutrosophic Set (NS)* A on X is characterized by three membership functions:

$$T_A : X \to [0,1], \quad I_A : X \to [0,1], \quad F_A : X \to [0,1],$$

where for each $x \in X$, the values $T_A(x)$, $I_A(x)$, and $F_A(x)$ represent the degrees of truth, indeterminacy, and falsity, respectively. These values satisfy the following condition:

$$0 \leq T_A(x) + I_A(x) + F_A(x) \leq 3.$$

Example 3.8 (Neutrosophic Set: Public Opinion)**.** Imagine a public poll about a new policy, where X is the set of all citizens. For each person $x \in X$, define:

$$T_A(x) = \text{degree of support for the policy},$$

$$I_A(x) = \text{degree of uncertainty},$$

$$F_A(x) = \text{degree of opposition}.$$

Each value lies in $[0, 1]$, and we have $0 \leq T_A(x) + I_A(x) + F_A(x) \leq 3$. For instance, if a citizen is mostly supportive yet somewhat undecided, we might have $\big(T_A(x), I_A(x), F_A(x)\big) =$ (0.7, 0.2, 0.1).

Example 3.9 (Neutrosophic Set: Product Quality Evaluation)**.** Let X be a collection of manufactured items. For each product $x \in X$, define:

$$T_A(x) = \text{degree of “high quality”},$$

$$I_A(x) = \text{degree of “uncertain/untested”},$$

$$F_A(x) = \text{degree of “defective”}.$$

Suppose a certain product is partially tested but with ambiguous data. One might assign $\big(T_A(x), I_A(x), F_A(x)\big) = (0.5, 0.4, 0.2)$, indicating moderate confidence in its quality, some unverified aspects, and a small chance of being defective.

Definition 3.10 (HyperNeutrosophic Set)**.** [5,10,32] Let X be a non-empty set. A mapping $\tilde{\mu} : X \to \tilde{P}([0, 1]^3)$ is called a *HyperNeutrosophic Set* over X, where $\tilde{P}([0, 1]^3)$ denotes the family of all non-empty subsets of the unit cube $[0, 1]^3$. For each $x \in X$, $\tilde{\mu}(x) \subseteq [0, 1]^3$ represents a set of neutrosophic membership degrees, each consisting of truth (T), indeterminacy (I), and falsity (F) components, satisfying:

$$0 \leq T + I + F \leq 3.$$

Example 3.11 (HyperNeutrosophic Set: Ranges of Public Sentiment)**.** Let X be the set of all citizens, and let a *HyperNeutrosophic Set* $\tilde{\mu}$ map each person $x \in X$ to a non-empty subset of $[0, 1]^3$. Instead of a single triple (T, I, F), assign a set of possible triples. For example, a person's support for a policy might lie in:

$$\tilde{\mu}(x) = \big\{(0.6, 0.2, 0.2),\ (0.7, 0.1, 0.2)\big\} \subseteq [0, 1]^3,$$

reflecting slight uncertainty in whether the person's true stance is $(T = 0.6, I = 0.2, F = 0.2)$ or $(T = 0.7, I = 0.1, F = 0.2)$, with each triple still obeying $0 \leq T + I + F \leq 3$.

Example 3.12 (HyperNeutrosophic Set: Product Quality with Interval Uncertainty)**.** Let X be a set of products, and define $\tilde{\mu}(x) \subseteq [0,1]^3$ for each product x. Suppose a batch of items has variable but uncertain test outcomes. We might assign:

$$\tilde{\mu}(x) = \{(T, I, F) \mid T \in [0.6, 0.8], I \in [0.1, 0.3], F \in [0, 0.2]\},$$

indicating a continuous range of plausible truth, indeterminacy, and falsity values for its quality. All points in that set satisfy $0 \leq T + I + F \leq 3$.

Definition 3.13 (n-SuperHyperNeutrosophic Set)**.** [5, 32] Let X be a non-empty set. An n-SuperHyperNeutrosophic Set is a recursive generalization of Neutrosophic Sets, HyperNeutrosophic Sets, and SuperHyperNeutrosophic Sets. It is defined as:

$$\tilde{A}_n : \tilde{\mathcal{P}}_n(X) \to \tilde{\mathcal{P}}_n([0,1]^3),$$

where:

- $\tilde{\mathcal{P}}_1(X) = \tilde{\mathcal{P}}(X)$, and for $k \geq 2$,

  $$\tilde{\mathcal{P}}_k(X) = \tilde{\mathcal{P}}(\tilde{\mathcal{P}}_{k-1}(X)),$$

  represents the k-th nested family of non-empty subsets of X.
- $\tilde{\mathcal{P}}_n([0,1]^3)$ is similarly defined for the unit cube $[0,1]^3$.
- The mapping $\tilde{A}_n$ assigns to each $A \in \tilde{\mathcal{P}}_n(X)$ a subset $\tilde{A}_n(A) \subseteq [0,1]^3$, representing the degrees of truth (T), indeterminacy (I), and falsity (F) for the n-th level subsets of X.

For each $A \in \tilde{\mathcal{P}}_n(X)$ and $(T, I, F) \in \tilde{A}_n(A)$, the following condition is satisfied:

$$0 \leq T + I + F \leq 3,$$

where T, I, and F represent the truth, indeterminacy, and falsity degrees, respectively.

Example 3.14 (n-SuperHyperNeutrosophic Set: Hierarchical Polling)**.** Let X be a population divided into regions, cities, and neighborhoods. Define $\tilde{\mathcal{P}}_1(X)$ as the family of non-empty subsets of X, $\tilde{\mathcal{P}}_2(X)$ as the family of non-empty subsets of $\tilde{\mathcal{P}}_1(X)$, and so on, up to $\tilde{\mathcal{P}}_n(X)$. An n-SuperHyperNeutrosophic Set $\tilde{A}_n$ assigns each nested subset $A \in \tilde{\mathcal{P}}_n(X)$ a set of triples in $[0,1]^3$. For example, at the second level ($n = 2$), a city-wide subset $A \subseteq X$ might map to

$$\tilde{A}_2(A) = \big\{(0.65, 0.20, 0.15),\ (0.70, 0.15, 0.15)\big\},$$

reflecting possible truth (support), indeterminacy, and falsity (opposition) degrees about that city's collective opinion.

Example 3.15 (n-SuperHyperNeutrosophic Set: Multi-Level Product Testing)**.** Consider a company testing components, sub-assemblies, and final products. Define X as all individual components, with $\tilde{\mathcal{P}}_n(X)$ nesting up to n levels (e.g., combining components into sub-assemblies, then products). The map $\tilde{A}_n$ assigns each multi-level subset $A \in \tilde{\mathcal{P}}_n(X)$ a set of (T, I, F) triples in $[0, 1]^3$. If a sub-assembly is tested but still uncertain, we may record:

$$\tilde{A}_n(A) = \{(0.4, 0.5, 0.2), (0.5, 0.4, 0.2)\},$$

allowing multiple possible truth, indeterminacy, and falsity distributions for quality, each adhering to $0 \leq T + I + F \leq 3$.

3.3. *Plithogenic Set*

A Plithogenic Set is a mathematical framework that incorporates multi-valued degrees of appurtenance and contradictions, making it suitable for complex decision-making processes. Various studies have been conducted on Plithogenic Sets [3, 11, 15, 34, 35]. The definition is presented below.

Definition 3.16 (Plithogenic Set)**.** [34,35] Let S be a universal set, and $P \subseteq S$. A *Plithogenic Set* PS is defined as:

$$PS = (P, v, Pv, pdf, pCF)$$

where:

- v is an attribute.
- Pv is the range of possible values for the attribute v.
- $pdf : P \times Pv \to [0, 1]^s$ is the *Degree of Appurtenance Function (DAF).*
- $pCF : Pv \times Pv \to [0, 1]^t$ is the *Degree of Contradiction Function (DCF).*

These functions satisfy the following axioms for all $a, b \in Pv$:

(1) *Reflexivity of Contradiction Function*:

$$pCF(a, a) = 0$$

(2) *Symmetry of Contradiction Function*:

$$pCF(a, b) = pCF(b, a)$$

Example 3.17 (Plithogenic Set: Product Evaluation)**.** Consider a universal set S of products, and let $P \subseteq S$ be the subset of products currently sold in a store. Suppose we focus on a single attribute v called "Flavor," with possible values

$$Pv = \{\texttt{Sweet}, \texttt{Sour}, \texttt{Bitter}, \texttt{Salty}\}.$$

Define the Degree of Appurtenance Function

$$pdf : P \times Pv \to [0,1]^1$$

that assigns to each product $x \in P$ and each flavor $\alpha \in Pv$ a membership degree $pdf(x,\alpha) \in [0,1]$. For example, if product x is a "Lemon Candy," we might have

$$pdf(x,\texttt{Sour}) = 0.8, \quad pdf(x,\texttt{Sweet}) = 0.6, \quad pdf(x,\texttt{Bitter}) = 0.0, \quad pdf(x,\texttt{Salty}) = 0.0.$$

The Degree of Contradiction Function

$$pCF : Pv \times Pv \to [0,1]^1$$

measures how contradictory two flavor values are. For instance,

$$pCF(\texttt{Sweet},\texttt{Bitter}) = 0.7$$

and

$$pCF(\texttt{Sweet},\texttt{Sweet}) = 0,$$

reflecting that identical flavor values are not contradictory (0) and sweet versus bitter has moderate contradiction (0.7). Together, these functions form a Plithogenic Set $\big(P, v, Pv, pdf, pCF\big)$ for product evaluation.

These definitions establish the foundational framework necessary for exploring the HyperPlithogenic Set and the SuperHyperPlithogenic Set. The definitions of the HyperPlithogenic Set and the SuperHyperPlithogenic Set are presented below [5, 7, 15].

Definition 3.18 (HyperPlithogenic Set)**.** [5, 7, 15] Let X be a non-empty set, and let A be a set of attributes. For each attribute $v \in A$, let Pv be the set of possible values of v. A *HyperPlithogenic Set* HPS over X is defined as:

$$HPS = (P, \{v_i\}_{i=1}^n, \{Pv_i\}_{i=1}^n, \{\tilde{pdf}_i\}_{i=1}^n, pCF)$$

where:

- $P \subseteq X$ is a subset of the universe.
- For each attribute v_i, Pv_i is the set of possible values.
- For each attribute v_i, $\tilde{pdf}_i : P \times Pv_i \to \tilde{P}([0,1]^s)$ is the *Hyper Degree of Appurtenance Function (HDAF)*, assigning to each element $x \in P$ and attribute value $a_i \in Pv_i$ a set of membership degrees.
- $pCF : (\bigcup_{i=1}^n Pv_i) \times (\bigcup_{i=1}^n Pv_i) \to [0,1]^t$ is the *Degree of Contradiction Function (DCF).*

Example 3.19 (HyperPlithogenic Set: Interval Uncertainty in Product Ratings)**.** Let X be the set of all products in a large online marketplace, and $P \subseteq X$ be the subset of currently active items. Suppose we have attributes $v_1 = \texttt{Quality}$ and $v_2 = \texttt{PriceRange}$. Define:

$$Pv_1 = \{\texttt{Low}, \texttt{Medium}, \texttt{High}\}, \quad Pv_2 = \{\texttt{Budget}, \texttt{Standard}, \texttt{Premium}\}.$$

For each attribute v_i, we introduce a *Hyper Degree of Appurtenance Function*:

$$\tilde{pdf}_i : P \times Pv_i \to \tilde{P}([0,1]^s),$$

where $\tilde{P}([0,1]^s)$ is the set of non-empty subsets of $[0,1]^s$. For instance, for a certain product $x \in P$,

$$\tilde{pdf}_1(x, \texttt{Medium}) = \{[0.4, 0.5], [0.5, 0.7]\},$$

meaning the *Quality* could be in the interval $[0.4, 0.5]$ or $[0.5, 0.7]$ with some uncertainty about how well x meets medium quality. The *PriceRange* might similarly assign sets of possible membership intervals. The *Degree of Contradiction Function* pCF still measures contradictions between any two values from $Pv_1 \cup Pv_2$, for example:

$$pCF(\texttt{High}, \texttt{Budget}) = 0.8,$$

indicating a strong mismatch between "High quality" and "Budget" price. Thus, the tuple

$$\left(P, \{v_i\}_{i=1}^2, \{Pv_i\}_{i=1}^2, \{\tilde{pdf}_i\}_{i=1}^2, pCF\right)$$

constitutes a HyperPlithogenic Set capturing interval-based uncertainties in product ratings.

Definition 3.20 (n-SuperHyperPlithogenic Set)**.** [5,7,15] Let X be a non-empty set, and let $V = \{v_1, v_2, \ldots, v_n\}$ be a set of attributes, each associated with a set of possible values P_{v_i}. An n-SuperHyperPlithogenic Set ($SHPS_n$) is defined recursively as:

$$SHPS_n = (P_n, V, \{P_{v_i}\}_{i=1}^n, \{\tilde{pdf}_i^{(n)}\}_{i=1}^n, pCF^{(n)}),$$

where:

- $P_1 \subseteq X$, and for $k \geq 2$,
$$P_k = \tilde{\mathcal{P}}(P_{k-1}),$$
represents the k-th nested family of non-empty subsets of P_1.
- For each attribute $v_i \in V$, P_{v_i} is the set of possible values of the attribute v_i.
- For each k-th level subset P_k, $\tilde{pdf}_i^{(n)} : P_n \times P_{v_i} \to \tilde{\mathcal{P}}([0,1]^s)$ is the *Hyper Degree of Appurtenance Function (HDAF)*, assigning to each element $x \in P_n$ and attribute value $a_i \in P_{v_i}$ a subset of $[0,1]^s$.
- $pCF^{(n)} : \bigcup_{i=1}^n P_{v_i} \times \bigcup_{i=1}^n P_{v_i} \to [0,1]^t$ is the *Degree of Contradiction Function (DCF)*, satisfying:
  (1) Reflexivity: $pCF^{(n)}(a,a) = 0$ for all $a \in \bigcup_{i=1}^n P_{v_i}$,

(2) Symmetry: $pCF^{(n)}(a,b) = pCF^{(n)}(b,a)$ for all $a, b \in \bigcup_{i=1}^{n} P_{v_i}$.

- s and t are positive integers representing the dimensions of the membership degrees and contradiction degrees, respectively.

Example 3.21 (n-SuperHyperPlithogenic Set: Multi-Level Inventory Analysis)**.** Consider a retail chain managing products (P_1), their packaging combinations (P_2), and potential distribution bundles (P_3), and so on. Each P_k is formed by taking non-empty subsets of P_{k-1}, leading to

$$P_k = \tilde{\mathcal{P}}(P_{k-1}).$$

Let $V = \{v_1, v_2\}$ be two attributes: $v_1 = \texttt{DemandLevel}$ and $v_2 = \texttt{ShipmentSize}$. Suppose

$$P_{v_1} = \{\texttt{Low}, \texttt{Moderate}, \texttt{High}\}, \quad P_{v_2} = \{\texttt{Small}, \texttt{Medium}, \texttt{Large}\}.$$

At the n-th nesting, each element of P_n might be a complex grouping of lower-level sets. A *Hyper Degree of Appurtenance Function* $\tilde{pdf}_i^{(n)}$ assigns to each (x, a_i) a subset in $[0,1]^s$. For instance, an entire distribution bundle $x \in P_n$ could be associated with

$$\tilde{pdf}_1^{(n)}(x, \texttt{High}) = \big\{\,(0.7, 0.8)\big\},$$

indicating it has a strong membership in "High DemandLevel." The *Degree of Contradiction Function*

$$pCF^{(n)} : \bigcup_{i=1}^{2} P_{v_i} \times \bigcup_{i=1}^{2} P_{v_i} \to [0,1]^t$$

remains reflexive and symmetric. For example,

$$pCF^{(n)}(\texttt{High}, \texttt{Low}) = 0.9, \quad pCF^{(n)}(\texttt{Small}, \texttt{Small}) = 0.$$

Putting these pieces together, the n-SuperHyperPlithogenic Set

$$SHPS_n = \big(P_n, V, \{P_{v_i}\}_{i=1}^{2}, \{\tilde{pdf}_i^{(n)}\}_{i=1}^{2}, pCF^{(n)}\big)$$

captures multiple hierarchical levels of inventory grouping, attribute values, and their possible contradictions. This structure helps a large retailer navigate uncertainties in both demand and shipment size across layered distribution networks.

3.4. *Soft Set*

A soft set is a mathematical structure that associates subsets of a universal set with parameters, providing a systematic framework for modeling uncertainty [22,23]. A hypersoft set extends the concept of soft sets by mapping Cartesian products of multiple attribute domains to subsets of a universal set [33]. A superhypersoft set further generalizes hypersoft sets by mapping combinations of power sets of attribute domains to subsets of a universal set [2,6,36]. The definitions and concrete examples of these concepts are provided below.

Definition 3.22 (Soft Set)**.** [22, 23] Let U be a universal set and A be a set of attributes. A soft set over U is a pair $(\mathcal{F}, S)$, where $S \subseteq A$ and $\mathcal{F} : S \to \mathcal{P}(U)$. Here, $\mathcal{P}(U)$ denotes the power set of U. Mathematically, a soft set is represented as:

$$(\mathcal{F}, S) = \{(\alpha, \mathcal{F}(\alpha)) \mid \alpha \in S, \mathcal{F}(\alpha) \in \mathcal{P}(U)\}.$$

Each $\alpha \in S$ is called a parameter, and $\mathcal{F}(\alpha)$ is the set of elements in U associated with α.

Example 3.23 (Soft Set: Car Dealership)**.** Let U be the set of all cars in a dealership, and let $A = \{\texttt{Color}, \texttt{Brand}, \texttt{FuelType}\}$ be a set of attributes. Suppose we select $S = \{\texttt{Color}, \texttt{Brand}\} \subseteq A$. A soft set $(\mathcal{F}, S)$ can be defined by specifying, for each attribute in S, the subset of cars having that property:

$$\mathcal{F}(\texttt{Color}) = \{\text{cars that are red}\}, \quad \mathcal{F}(\texttt{Brand}) = \{\text{cars from Toyota}\}.$$

Thus, $\mathcal{F}(\texttt{Color})$ might include any red car in the dealership, and $\mathcal{F}(\texttt{Brand})$ might include every Toyota model on the lot. This soft set captures which cars match each chosen attribute.

Example 3.24 (Soft Set: Student Enrollment)**.** Consider U to be all students at a university, and let $A = \{\texttt{Major}, \texttt{Year}, \texttt{Club}\}$. Suppose we pick $S = \{\texttt{Major}, \texttt{Year}\} \subseteq A$. Define

$$\mathcal{F}(\texttt{Major}) = \{\text{all students in Mathematics}\}, \quad \mathcal{F}(\texttt{Year}) = \{\text{all second-year students}\}.$$

Each attribute in S is a parameter, and $\mathcal{F}(\alpha)$ identifies the subset of students satisfying α. This helps administrators see which students share the same major or academic year.

Definition 3.25 (Hypersoft Set)**.** [33] Let U be a universal set, and let $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_m$ be attribute domains. Define $\mathcal{C} = \mathcal{A}_1 \times \mathcal{A}_2 \times \cdots \times \mathcal{A}_m$, the Cartesian product of these domains. A hypersoft set over U is a pair $(G, \mathcal{C})$, where $G : \mathcal{C} \to \mathcal{P}(U)$. The hypersoft set is expressed as:

$$(G, \mathcal{C}) = \{(\gamma, G(\gamma)) \mid \gamma \in \mathcal{C}, G(\gamma) \in \mathcal{P}(U)\}.$$

For an m-tuple $\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_m) \in \mathcal{C}$, where $\gamma_i \in \mathcal{A}_i$ for $i = 1, 2, \ldots, m$, $G(\gamma)$ represents the subset of U corresponding to the combination of attribute values $\gamma_1, \gamma_2, \ldots, \gamma_m$.

Example 3.26 (Hypersoft Set: Apartment Hunting)**.** Let U be the set of all available apartments in a city. Suppose we have three attribute domains:

$$\mathcal{A}_1 = \{\texttt{1-Bedroom}, \texttt{2-Bedroom}\}, \quad \mathcal{A}_2 = \{\texttt{Downtown}, \texttt{Suburb}\}, \quad \mathcal{A}_3 = \{\texttt{LowRent}, \texttt{HighRent}\}.$$

The Cartesian product $\mathcal{C} = \mathcal{A}_1 \times \mathcal{A}_2 \times \mathcal{A}_3$ consists of all 3-tuples specifying (Size, Location, RentLevel). Define $G : \mathcal{C} \to \mathcal{P}(U)$ by:

$$G(\texttt{1-Bedroom}, \texttt{Downtown}, \texttt{LowRent}) = \{\text{all 1-BR downtown apartments with relatively low rent}\}.$$

Each element $\gamma = (\gamma_1, \gamma_2, \gamma_3) \in \mathcal{C}$ maps to the set of apartments matching that combination of features. This hypersoft set enables a real estate agency to group apartments by multiple attributes at once.

Example 3.27 (Hypersoft Set: Course Selection)**.** Let U be all courses offered at a university, and let the attribute domains be

$$\mathcal{A}_1 = \{\texttt{Morning}, \texttt{Afternoon}\}, \quad \mathcal{A}_2 = \{\texttt{Undergraduate}, \texttt{Graduate}\}, \quad \mathcal{A}_3 = \{\texttt{In-Person}, \texttt{Online}\}.$$

The Cartesian product $\mathcal{C} = \mathcal{A}_1 \times \mathcal{A}_2 \times \mathcal{A}_3$ contains all possible combinations of (TimeSlot, Level, Mode). A hypersoft set $(G, \mathcal{C})$ is defined by

$$G(\gamma) \subseteq U,$$

where $G(\gamma)$ is the subset of courses that match each combination γ. For example,

$$G(\texttt{Morning}, \texttt{Undergraduate}, \texttt{Online})$$

could be all online, morning classes offered at the undergraduate level.

Definition 3.28 (SuperHyperSoft Set)**.** [36] Let U be a universal set, and let $\mathcal{P}(U)$ denote the power set of U. Consider n distinct attributes $a_1, a_2, \ldots, a_n$, where $n \geq 1$. Each attribute a_i is associated with a set of attribute values A_i, satisfying the property $A_i \cap A_j = \emptyset$ for all $i \neq j$.

Define $\mathcal{P}(A_i)$ as the power set of A_i for each $i = 1, 2, \ldots, n$. Then, the Cartesian product of the power sets of attribute values is given by:

$$\mathcal{C} = \mathcal{P}(A_1) \times \mathcal{P}(A_2) \times \cdots \times \mathcal{P}(A_n).$$

A SuperHyperSoft Set over U is a pair $(F, \mathcal{C})$, where:

$$F : \mathcal{C} \to \mathcal{P}(U),$$

and F maps each element $(\alpha_1, \alpha_2, \ldots, \alpha_n) \in \mathcal{C}$ (with $\alpha_i \in \mathcal{P}(A_i)$) to a subset $F(\alpha_1, \alpha_2, \ldots, \alpha_n) \subseteq U$. Mathematically, the SuperHyperSoft Set is represented as:

$$(F, \mathcal{C}) = \{(\gamma, F(\gamma)) \mid \gamma \in \mathcal{C}, F(\gamma) \in \mathcal{P}(U)\}.$$

Here, $\gamma = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in \mathcal{C}$, where $\alpha_i \in \mathcal{P}(A_i)$ for $i = 1, 2, \ldots, n$, and $F(\gamma)$ corresponds to the subset of U defined by the combined attribute values $\alpha_1, \alpha_2, \ldots, \alpha_n$.

Example 3.29 (SuperHyperSoft Set: Customized Tour Packages)**.** Let U be the set of all available tourist destinations in a travel agency's database. Suppose we have three distinct attributes:

$$a_1 = \texttt{Sightseeing}, \quad a_2 = \texttt{Adventure}, \quad a_3 = \texttt{Luxury}.$$

Each attribute a_i is linked with a set of possible values A_i. For instance,

$$A_1 = \{\texttt{Museums}, \texttt{HistoricSites}\},\ A_2 = \{\texttt{Hiking}, \texttt{Kayaking}\},\ A_3 = \{\texttt{FiveStarHotels}, \texttt{Resorts}\}.$$

We then consider the power sets $\mathcal{P}(A_1)$, $\mathcal{P}(A_2)$, $\mathcal{P}(A_3)$ and form the Cartesian product

$$\mathcal{C} = \mathcal{P}(A_1) \times \mathcal{P}(A_2) \times \mathcal{P}(A_3).$$

A SuperHyperSoft Set $(F, \mathcal{C})$ is defined by a mapping

$$F : \mathcal{C} \to \mathcal{P}(U),$$

where each tuple $\gamma = (\alpha_1, \alpha_2, \alpha_3)$ with $\alpha_i \subseteq A_i$ represents a combination of chosen sightseeing options, adventure activities, and luxury services. Then $F(\gamma)$ is the subset of destinations that satisfy the selected combination at once. For instance,

$$F\big(\{\texttt{Museums}, \texttt{HistoricSites}\}, \{\texttt{Hiking}\}, \{\texttt{Resorts}\}\big)$$

could be all travel packages offering museum visits, hiking excursions, and resort accommodations in a single itinerary.

Example 3.30 (SuperHyperSoft Set: Laboratory Testing)**.** Let U be the set of experimental protocols in a research lab. Suppose there are $n = 2$ attributes:

$$a_1 = \texttt{EquipmentUsed}, \quad a_2 = \texttt{SampleType}.$$

Then A_1 might include $\{\texttt{Microscope}, \texttt{Centrifuge}, \texttt{Spectrometer}\}$, and A_2 might include $\{\texttt{BloodSample}, \texttt{TissueSample}\}$. We take the power sets $\mathcal{P}(A_1)$ and $\mathcal{P}(A_2)$:

$$\mathcal{C} = \mathcal{P}(A_1) \times \mathcal{P}(A_2).$$

A SuperHyperSoft Set $(F, \mathcal{C})$ assigns to each pair

$$(\alpha_1, \alpha_2) \in \mathcal{P}(A_1) \times \mathcal{P}(A_2)$$

a subset of laboratory protocols in $\mathcal{P}(U)$. For example, if $\alpha_1 = \{\texttt{Microscope}, \texttt{Centrifuge}\}$ and $\alpha_2 = \{\texttt{BloodSample}\}$, then

$$F(\alpha_1, \alpha_2) = \{\text{Protocols requiring a microscope } \textit{and} \text{ centrifuge on blood samples}\}.$$

This approach systematically encodes possible experimental setups and the corresponding protocols.

3.5. *Rough Set*

A Rough Set approximates a subset using lower and upper bounds based on equivalence classes, capturing certainty and uncertainty in membership [24, 26]. The definitions are provided below.

Definition 3.31 (Rough Set Approximation)**.** [25] Let X be a non-empty universe of discourse, and let $R \subseteq X \times X$ be an equivalence relation (or indiscernibility relation) on X. The equivalence relation R partitions X into disjoint equivalence classes, denoted by $[x]_R$ for $x \in X$, where:

$$[x]_R = \{y \in X \mid (x, y) \in R\}.$$

For any subset $U \subseteq X$, the *lower approximation* $\underline{U}$ and the *upper approximation* $\overline{U}$ of U are defined as follows:

(1) *Lower Approximation* $\underline{U}$:

$$\underline{U} = \{x \in X \mid [x]_R \subseteq U\}.$$

The lower approximation $\underline{U}$ includes all elements of X whose equivalence classes are entirely contained within U. These are the elements that *definitely* belong to U.

(2) *Upper Approximation* $\overline{U}$:

$$\overline{U} = \{x \in X \mid [x]_R \cap U \neq \emptyset\}.$$

The upper approximation $\overline{U}$ contains all elements of X whose equivalence classes have a non-empty intersection with U. These are the elements that *possibly* belong to U.

The pair $(\underline{U}, \overline{U})$ forms the *rough set* representation of U, satisfying the relationship:

$$\underline{U} \subseteq U \subseteq \overline{U}.$$

Example 3.32 (Rough Set: Quality Control in a Factory)**.** Suppose X is a collection of manufactured items, and R is an equivalence relation indicating that two items are "indiscernible" if they have the same measured characteristics (e.g., weight and dimensions). For a subset $U \subseteq X$ representing "items potentially defective," the lower approximation $\underline{U}$ contains items whose equivalence classes lie entirely within U (i.e., definitely defective), and the upper approximation $\overline{U}$ includes items whose equivalence classes intersect with U (i.e., possibly defective). If item x shares exactly the same measurements as other defective items, then $[x]_R \subseteq U$ and x is in $\underline{U}$. If item y's measurements partially match a defective class, then $[y]_R \cap U \neq \emptyset$ and y belongs to $\overline{U}$. This helps the factory refine which items definitely or possibly need rechecking or further tests.

The concept of a *HyperRough Set* extends the framework of the HyperSoft Set [33] to Rough Set theory. Furthermore, this has been generalized using superhyperstructures, leading to the notion of the *SuperHyperRough Set*. The formal definition is provided below.

Definition 3.33 (HyperRough Set)**.** [5, 8, 9] Let X be a non-empty finite universe, and let $T_1, T_2, \ldots, T_n$ be n distinct attributes with respective domains $J_1, J_2, \ldots, J_n$. Define the Cartesian product of these domains as:

$$J = J_1 \times J_2 \times \cdots \times J_n.$$

Let $R \subseteq X \times X$ be an equivalence relation on X, where $[x]_R$ denotes the equivalence class of x under R.

A *HyperRough Set* over X is a pair (F, J), where:

- $F : J \to \mathcal{P}(X)$ is a mapping that assigns a subset $F(a) \subseteq X$ to each attribute value combination $a = (a_1, a_2, \ldots, a_n) \in J$.
- For each $a \in J$, the rough set $(\underline{F(a)}, \overline{F(a)})$ is defined as:

$$\underline{F(a)} = \{x \in X \mid [x]_R \subseteq F(a)\}, \quad \overline{F(a)} = \{x \in X \mid [x]_R \cap F(a) \neq \emptyset\}.$$

The *lower approximation* $\underline{F(a)}$ represents the set of elements in X whose equivalence classes are entirely contained within $F(a)$, while the *upper approximation* $\overline{F(a)}$ includes elements whose equivalence classes have a non-empty intersection with $F(a)$.

Additionally, the following properties hold:

- $\underline{F(a)} \subseteq \overline{F(a)}$ for all $a \in J$.
- If $F(a) = \varnothing$, then $\underline{F(a)} = \overline{F(a)} = \varnothing$.
- If $F(a) = X$, then $\underline{F(a)} = \overline{F(a)} = X$.

Example 3.34 (HyperRough Set: Customer Segmentation With Composite Attributes)**.** Let X be the set of customers at a retail chain, and define attributes $T_1 =$ `AgeGroup` (e.g., "Under30," "30-60," "Over60") and $T_2 =$ `SpendingLevel` (e.g., "Low," "Medium," "High"). Their domains are J_1 and J_2, and the Cartesian product $J = J_1 \times J_2$ lists all pairs (`AgeGroup`, `SpendingLevel`). Let R be an equivalence relation on X such that two customers are indiscernible if they have the same purchase history pattern. A *HyperRough Set* is formed by $F : J \to \mathcal{P}(X)$. For instance,

$$F(\texttt{Under30}, \texttt{High}) = \{\text{customers under 30 who might spend at a high level}\}.$$

Its rough approximations $\underline{F(a)}, \overline{F(a)}$ identify which customers definitely or possibly belong to that group, based on equivalence classes. This approach refines marketing strategies by clarifying which customers are unequivocally in a target demographic versus those whose purchase patterns only partially fit.

Definition 3.35 (n-SuperHyperRough Set)**.** [5, 8, 9] Let X be a non-empty finite universe, and let $T_1, T_2, \ldots, T_n$ be n distinct attributes with respective domains $J_1, J_2, \ldots, J_n$. For each attribute T_i, let $\mathcal{P}(J_i)$ denote the power set of J_i. Define the set of all possible attribute value combinations as the Cartesian product of these power sets:

$$J = \mathcal{P}(J_1) \times \mathcal{P}(J_2) \times \cdots \times \mathcal{P}(J_n).$$

Let $R \subseteq X \times X$ be an equivalence relation on X, where $[x]_R$ denotes the equivalence class of x under R.

An *n-SuperHyperRough Set* over X is a pair (F, J), where:

- $F : J \to \mathcal{P}(X)$ is a mapping that assigns a subset $F(A) \subseteq X$ to each attribute value combination $A = (A_1, A_2, \ldots, A_n) \in J$, where $A_i \subseteq J_i$ for all i.
- For each $A \in J$, the rough set $(\underline{F(A)}, \overline{F(A)})$ is defined as:

$$\underline{F(A)} = \{x \in X \mid [x]_R \subseteq F(A)\}, \quad \overline{F(A)} = \{x \in X \mid [x]_R \cap F(A) \neq \emptyset\}.$$

The *lower approximation* $\underline{F(A)}$ represents the set of elements in X whose equivalence classes are entirely contained within $F(A)$, while the *upper approximation* $\overline{F(A)}$ includes elements whose equivalence classes have a non-empty intersection with $F(A)$.

Properties:

- $\underline{F(A)} \subseteq \overline{F(A)}$ for all $A \in J$.
- If $F(A) = \varnothing$, then $\underline{F(A)} = \overline{F(A)} = \varnothing$.
- If $F(A) = X$, then $\underline{F(A)} = \overline{F(A)} = X$.
- For any $A, B \in J$:

$$\underline{F(A \cap B)} \subseteq \underline{F(A)} \cap \underline{F(B)}, \quad \overline{F(A \cup B)} \supseteq \overline{F(A)} \cup \overline{F(B)}.$$

Example 3.36 (n-SuperHyperRough Set: Multi-Level Data Analysis)**.** Consider a scenario where X is a set of documents in a large digital library. Let $T_1, T_2, \ldots, T_n$ be attributes, such as `Topic`, `Author`, `PublicationYear`, etc., each with a domain J_i. For each i, $\mathcal{P}(J_i)$ is the power set of possible attribute values. The Cartesian product

$$J = \mathcal{P}(J_1) \times \mathcal{P}(J_2) \times \cdots \times \mathcal{P}(J_n)$$

lists all combinations of chosen topics, authors, publication years, and so on. Define an equivalence relation R where two documents are indiscernible if they share the same metadata (e.g., identical classification tags). An n-SuperHyperRough Set (F, J) assigns each combination $A = (A_1, A_2, \ldots, A_n)$ a subset $F(A) \subseteq X$, and obtains rough approximations $\underline{F(A)}$ and $\overline{F(A)}$. For instance, if

$$A_1 = \{\texttt{Mathematics}, \texttt{Physics}\}, \quad A_2 = \{\texttt{Smith}, \texttt{Johnson}\},$$

then

$F(A) = \{$documents concerning Mathematics or Physics written by Smith or Johnson$\}$.

Its lower approximation consists of documents whose equivalence classes lie entirely in that set, and its upper approximation includes all documents whose equivalence classes intersect with it. This multi-level approach enables more nuanced filtering, revealing which documents definitely or possibly match complex criteria across multiple attributes.

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Data Availability

This paper does not involve any data analysis.

Ethical Approval

This article does not involve any research with human participants or animals.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

Disclaimer

This study primarily focuses on theoretical aspects, and its application to practical scenarios has not yet been validated. Future research may involve empirical testing and refinement of the proposed methods. The authors have made every effort to ensure that all references cited in this paper are accurate and appropriately attributed. However, unintentional errors or omissions may occur. The authors bear no legal responsibility for inaccuracies in external sources, and readers are encouraged to verify the information provided in the references independently. Furthermore, the interpretations and opinions expressed in this paper are solely those of the authors and do not necessarily reflect the views of any affiliated institutions.

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Conjugation of Hypersoft Set and Triangular Neutrosophic to Evaluate Cognitive Digital Twins' Contribution to Smart Manufacturing

Mona Mohamed [1], Nurhan Alaa[2]

[1]Higher Technological Institute, 10th of Ramadan City 44629, Egypt,

Email: Mona.fouad@hti.edu.eg;

[2] Department of Computer Science, Faculty of Computers and Informatics, Zagazig University, Zagazig 44519, Egypt,

Email: nohrhan.alaa@hti.edu.eg.

* Correspondence: nohrhan.alaa@hti.edu.eg

Abstract: The latest innovation in digital twin technology is called Cognitive Digital Twins (CDT). The sophisticated and autonomous activities made possible by this technology have the potential to revolutionize manufacturing. An overview of CDT in manufacturing is given in this paper, along with an examination of their main features, components, and possible uses. CDT can learn from data, reason about difficult circumstances, and make well-informed judgments by combining artificial intelligence, machine learning, and knowledge representation approaches. The advantages of CDT in smart manufacturing are covered in the paper. Hence, the objective of this paper is evaluating the smart manufacturing that adopts CDT in its operations and practices to be smart. Multi-Criteria Decision Making (MCDM) as CRiteria Importance Through Intercriteria Correlation and multi-objective optimization based on simple ratio analysis (MOOSRA) are leveraged to construct soft decision models. This model can trat with vague and incomplete information through harnessing uncertainty theory especially Triangular Neurosophic Number (TriNN). Also, Hypersoft set is utilized with MOOSRA to rank alternatives of smart manufacturing

Keywords: smart manufacturing; cognitive digital twin; CRITIC technique; MOOSRA technique; Hpersoft set;Triangular Neurosophic Number.

1. Introduction

Manufacturing is one of the foremost sectors that has an immense influence on a nation's economy and development [1] . As well as [2] making products that meet consumer demand, is one of the sustainable manufacturing objectives. The notion of sustainable manufacturing (SusM) [3]has gained widespread support in business, particularly in industry, and is receiving more attention in the research community within the broader topic of sustainability.

From the perspective[4] adopting sustainability ideas and technologies that impact manufacturing's performance is crucial for practitioners, policymakers, and manufacturers.

Researchers, governments, society, and the manufacturing industrial sector have all shown a great deal of interest in sustainability according to [5]. This heightened emphasis on sustainability is motivated by the need to remain competitive in a globalized economy and adjust to quick changes in consumer needs. Whereby [6] linked the concept of sustainability with influential aspects such as environmentally: diminishing the influence on the environment by cutting back on pollution, waste production, and resource consumption [7]; Economically: Putting into practice economical,

sustainable methods that don't substantially raise manufacturing costs [8]; Ingenuity :Putting expenditures on research and development to develop new environmentally friendly technology and procedures[9]. Socially: emphasizes the welfare of employees, local communities, and society [10].
Surveying prior scholars' perspectives as[11],[12], [13] demonstrated that deploying the notion of sustainability in manufacturing has a favorable impact. In this context, Figure 1 compiled these perspectives on bolstering aspects of sustainability in manufacturing.

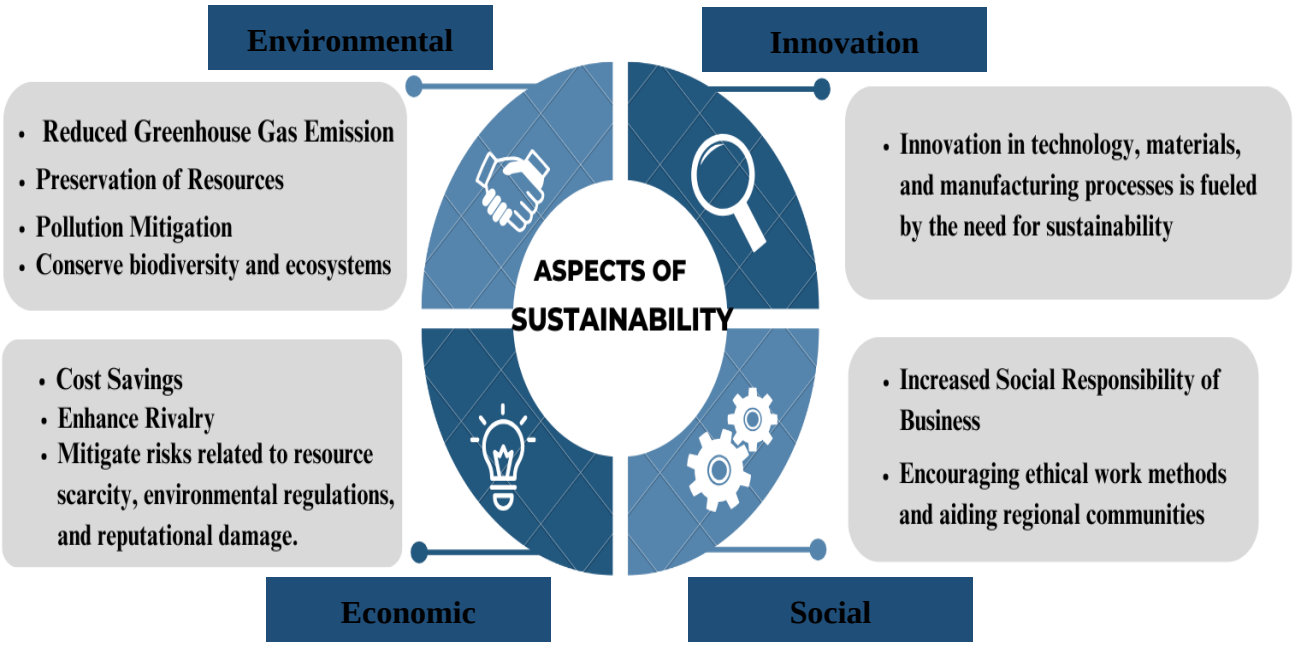
Environmental
Innovation
Reduced Greenhouse Gas Emission
Preservation of Resources
Pollution Mitigation
Conserve biodiversity and ecosystems
Innovation in technology, materials, and manufacturing processes is fueled by the need for sustainability
ASPECTS OF SUSTAINABILITY
Cost Savings
Enhance Rivalry
Mitigate risks related to resource scarcity, environmental regulations, and reputational damage.
Increased Social Responsibility of Business
Encouraging ethical work methods and aiding regional communities
Economic
Social

Figure 1. Sustainability in Manufacturing

On other side, [14] stated that a significant contributor to climate change and one of the biggest worldwide sources of carbon emissions is the manufacturing sector. Accordingly manufacturers are coming under more pressure to lower their carbon impact and actively promote environmental goals[15]. Hence, many stakeholders have started to put pressure on firms to implement sustainable practices after realizing the impact on the environment.
The sustainability of manufacturing is hampered by several obstacles, despite its significance. These obstacles discussed in [16] that entailed in domestic disturbances to worldwide calamities as diseases (Covid 19), floods that threaten the sustainability of manufacturing.
Scholars in [17] assumed another obstacles as accurate decision making based on predicting equipment failures through analyzing real-time data to avoid malfunctions that can cause interruptions in production and problems with quality [18].
To fix this obstacle, scholars in [19] utilized digital technologies as artificial intelligence (AI) and its subset machine learning(ML) and deep learning(DL) to enhance demand forecasts after that, allowing producers to maximize inventory control and production scheduling.
Generally, deploying digital technologies, industry 4.0, and industry 5.0 enable manufacturing to be automated, digital and smart. These technologies [20] reduce manual labor, increasing production speed, and minimizing errors.

Smart manufacturing is a contemporary strategy that integrates cutting-edge technology like cloud computing, artificial intelligence, and the Internet of Things (IoT) into manufacturing processes in tandem with sustainable production[21] .Manufacturing systems can now independently sense their surroundings, make the best judgments, and carry out jobs with accuracy thanks to this connection. Thus, smart manufacturing minimizes errors and waste while increasing production precision and efficiency[22]

By enabling production that is more flexible, adaptive, and customized to meet the demands of individual customers, this transition toward smart manufacturing opens new prospects for industries worldwide [23]. Whereas creates a digital replica of a physical object or process by digital twins (DTs) enabling better understanding, optimization, and improved lifecycle management. Falekas et al.[24] described DTs as virtual mirrors of physical systems that allow for real-time monitoring, analysis, and optimization, have been developed to improve smart manufacturing. DTs can increase the efficiency and quality of item production, forecast equipment breakdowns, and simulate production processes in smart manufacturing

Digital twins give businesses comprehensive insights into the performance and behavior of physical assets using data analytics and machine learning, enabling them to foresee problems and make data-driven modifications [25].The paper emphasizes that digital twins are vital to future technology advancements across industries, offering potential cost savings, improved decision-making, and enhanced operational efficiency. Scholas [26] showcases an advanced form of a digital twin that integrates cognitive computing capabilities, this form entailed in cognitive digital twin (CDT). In the same vein [27] CDT defined as an expansion of pre-existing digital twins, cognitive digital twins include three levels access, analytics, and cognition. These layers include extra communication, analytics, and intelligence capabilities. In light of CDT and cognitive computing, ML , DL are utilized to analyze data from the physical twin and other sources to identify patterns, trends. Based on knowledge gained from analyzing data, it can forecast outcomes, offer suggestions, and even manage some physical system components on its own. Accordingly, various studies employed CDT in various purposes as [28] that leveraged CDT in smart manufacturing to Identify inefficiencies and bottlenecks in production data analysis, then suggest adjustments to improve the process.

In this context, this paper involves CDT producing accurate results as it acts as human being, by incorporating human cognition through AI and Semantic Web technologies into the design of autonomous manufacturing. wherein CDT [29] aims to improve autonomous manufacturing by enabling manufacturing resources to think, learn, and comprehend the dynamics of industrial environments.

The importance of CDT in manufacturing to be smart and sustainable is the objective of our study. Hence, we evaluate the role of CDT in manufacturing to be sustainable through constructing a robust soft decision model. The evaluation is conducted based on a set of criteria.

To objectively evaluate and assign priority levels, the CRITIC technique is leveraged in our soft decision model to offer a quantitative approach by examining the correlations and differences between criteria. This guarantees that the criteria's weighting is objective and data-driven, reflecting actual significance rather than personal opinion[30] .One major advantage of this technique,

particularly the suggested methodology based on CRITIC principles, lies in its objectivity and structured computation process. This method uses a systematic, straightforward calculation process to generate criteria weights, ensuring clarity and reducing complexity. Consequently, CRITIC method enables decision-makers to effectively prioritize criteria, which can be especially useful in fields where decision accuracy and consistency are crucial. This objective framework allows for repeatable, reliable results that are essential for robust decision-making and analysis in complex scenarios.

MOOSRA technique is a powerful multi-criteria decision-making tool that ranks alternatives based on performance across multiple criteria [31]. It accounts for both maximizing and minimizing attributes and is particularly effective in handling significant variations in criteria values.

These MCDM techniques are combined with TriNN and Hypersoft sets to bolster decisions of decision makers in vague situations and incomplete information.

2. Literature review

Decision-making in smart manufacturing has advanced significantly because of automation and data integration technology. The ability of manufacturing systems to make decisions is being shaped by linked devices, data analytics, and artificial intelligence (AI), according to recent research. The move to data-centric, AI-driven frameworks is a significant trend that enables manufacturers to use real-time data to make better education, predictive decisions to reduce downtime and improve operational efficiency,this strategy aids in the identification of bottlenecks, production optimization, and proactive equipment maintenance management.

Here is a related work which focuses on smart manufacturing and decision-making. These papers cover topics like the use of AI, IoT, big data analytics, and decision support systems in enhancing decision-making capabilities within smart manufacturing:

Implementing the Internet of Things (IoT) in smart manufacturing within sustainable supply chain management (SSCM) offers several issues, especially regarding scalability, energy consumption, security, and privacy. Although these obstacles have been highlighted in earlier research, a methodical strategy to overcome them in smart manufacturing settings is still developing. To evaluate and prioritize various IoT issues, this research presents a decision-making framework that integrates q-Rung Orthopair Fuzzy sets, CRITIC, and VIKOR methodologies. This paradigm improves resource allocation and decision-making precision in smart manufacturing by enabling accurate prioritizing. The model can be used to solve IoT complications in SSCM for smart manufacturing environments because of its reliability, which has been validated by sensitivity analysis and comparisons with alternative approaches [32].

Since they enable more customization, efficiency, and productivity, Industry 4.0 technologies like IoT, big data, and CPS are crucial to the growth of smart manufacturing. However, utilizing these technologies requires both technological innovation and a trained workforce. They go over important Industry 4.0 subjects and worker competencies needed for future smart factories to aid with the transition to intelligent production [33].

Sanusi et al. [34]looked at the adoption of smart manufacturing in Indonesian SMEs and identified several obstacles, including a lack of funding, poor infrastructure, and a lack of skilled workers. They show how even little investments in automation may significantly increase productivity and cut costs in their case studies of South Sulawesi SMEs.The report emphasizes the importance of business and governmental cooperation in removing adoption hurdles, highlighting Indonesia's "Making Indonesia 4.0" effort. According to Sanusi et al., stakeholders should work together to encourage SME digital transformation in developing nations by offering financial and policy assistance.

A digital twin-assisted model was created to maximize collaborative production in smart systems. Effective resource allocation and proactive equipment management are made possible by the model's combination of digital twin technology and predictive diagnostics using Elman neural networks and IVIF-TOPSIS. To improve supply chain KPIs like efficiency and adaptability and enables businesses to respond quickly to market demands, it also integrates a value co-creation framework [35].

A key component of smart manufacturing, digital twins improve output through data-driven optimization, simulation, and real-time monitoring. By building virtual versions of actual processes, digital twins facilitate flexible and responsive decision-making in the context of smart manufacturing. Their effects on lowering downtime, enhancing quality, and anticipating maintenance requirements—all crucial components of a smart manufacturing system—have been shown in earlier research. Digital twins support smart manufacturing's emphasis on flexibility and efficiency by helping producers find inefficiencies and make changes without interfering with real operations by mimicking production settings.

The application of CDT in manufacturing is examined with a focus on how they can convert conventional digital twins into sentient, intelligent beings. Access, analytics, and cognition are the three levels that make up CDTs. providing a path for completely autonomous, cross-domain manufacturing systems that make use of knowledge graphs, semantic reasoning, and artificial intelligence (AI) technologies, as well as a prototype that has been evaluated using production line performance statistics [36].

3. Methodology

3.1 preliminaries and definitions

In this subsection, a set of definitions and preliminaries related to HyperSoft and utilized techniques that contributed to constructing soft decision model.

The HyperSoft defined and proposed by Smarandache [37]:

- Let μ be a universe of discourse, (μ) the power set of μ, and A a set of attributes. Then, the pair (F, μ), F: $A\rightarrow$ (μ) is called a Soft Set over
- $Let\ a1, a1\ , \dots\ an\ for\ n \geq 9$, be n distinct attributes,

Whose corresponding attributes are respectively the set $A1\ , A1\ , \dots\ An\ with\ Ai \cap Aj = \emptyset, for\ i \neq j, and\ i,j \in \{9,1, \dots\ n\}$.

- Then the pair (F: $A1 \times A1 \times \dots\ An \rightarrow (\mu)$) is called a HyperSoft over μ.

3.2 Soft Decision Model: Procedures of Proposed Model

This subsection provides the techniques we use to find the weights for each criterion the techniques for ranking the alternatives of smart manufacturing that adopt CDT in its operations and manufacturing to choose the suitable alternative with the best results.

3.2.1 Create the decision matrix

1. Determining the alternatives and criteria for the problem
2. Evaluating the alternatives based on the criteria by formed expert panel
3. Create the decision matrix according to the scales mentioned in Table 1.
4. use Eq.1 to convert the triangular neutrosophic scale into deneutrosophic value.

$$\text{Score}(Qij) = \frac{\text{lij+mij+uij}}{9} * (2 + T - I - F) \quad (1)$$

Where: i=1,2,3,…m; n=1,2,3,….. j; l, m, u refer to the lover, middle, and upper values and T,I, F refer to truth, indeterminacy and false respectively.

5. Combine the decision matrices of each decision maker into a single decision matrix by using Eq. 2.

$$xi=\frac{\sum_{j=1}^{N}(\text{Qij})}{\text{N}} \quad (2)$$

Where Qij is the value of each criterion and N is number of decision makers.

Table 1.Linguistic triangular neutrosophic scale

Crisp Scale	Explanation	TriNN Scale
1	Equally Essential	<<1,1,1>;0.5,0.5,0.5>>
2	Slightly Moderately	<<1,2,3>;0.4,0.6,0.65>>
3	slightly Essential	<<2,3,4>;0.3,0.75,0.7>>
4	Minor To Strong	<<3,4,5>;0.35,0.6,0.4>>
5	Mighty Essential	<<4,5,6>;0.8,0.15,0.2>>
6	Slightly Strong Essential	<<5,6,7>;0.7,0.25,0.3>>
7	High Strong Essential	<<6,7,8>;0.9,0.1,0.1>>
8	Very High Strong Essential	<<7,8,9>;0.85,0.1,0.15>>
9	Absolutely High Essential	<<9,9,9>;0.1,0.0,0.0>>

3.2.2 CRITIC Method

6. Normalize the aggragated matrix using Eq.3, Eq.4

$$\text{Rij}=\frac{xij-\min(xij)}{\max(xij)-\min(xij)} \text{ for benefit criteria} \quad (3)$$

$$\text{Rij}=\frac{\max(xij)-xij}{\max(xij)-\min(xij)} \text{ for non benefit criteria} \quad (4)$$

7. Calculate the standard deviation (sd) of the matrix with Eq. 5

$$sd=\sqrt{\frac{\sum_{i=1}^{m}(Rij-\bar{R}j)}{m}},\ \bar{R}=\frac{1}{m}\sum_{i=0}^{m}(Rij) \quad (5)$$

8. Calculate correlation between each criteria using Eq. 6

$$cjk=\frac{\sum_{j=1}^{m}(Rij-\bar{R}j)(Rik-\bar{R}k)}{=\sqrt{\sum_{j=1}^{m}(Rij-\bar{R}j)^2\ \sum_{j=1}^{m}(Rik-\bar{R}k)^2}} \quad (6)$$

9. Calculate the convert degree rij using Eq.7

$$rj=\sum_{i=1}^{m}(1-cjk) \quad (7)$$

10. calculate the weight using Eq.8

$$wj=sd*(\frac{ri}{\sum_{j=1}^{n}ri}) \quad (8)$$

3.2.3 MOOSRA Method

1. Normalize the aggregated decision matrix by Eq. 9

$$Xij*=\frac{\sum_{j=1}^{m}Xij}{=\sqrt{\sum_{i=1}^{n}(Xij)^2}} \quad (9)$$

2. Create weighted normalize decision matrix by Eq 10

$$z=Xij*wj \quad (10)$$

3. Rank the alternatives by Eq. 11

$$si=\frac{\sum benefit\ value}{\sum non\ benefit\ value} \quad (11)$$

4 Real case study

To validate the accuracy of constructed soft decision model, we communicate with manufacturing that adopting CDT in its operations and manufacturing. Hence, in our study four smart manufacturing contributed to our evaluation process to be alternatives. The evaluation is conducted based on five criteria and five attributes.

4.1 identify attributes, set of criteria and alternatives.

Herein five criteria, four alternatives and the attributes related to the criteria ae determined.

We have 5 criteria and 4 alternatives (SM) in this study depending on decision makers' opinions we use 4 attribute values (with 14 sub-attributes) for all criteria

The criteria we identified:

C1= production (benefit)

C2= efficiency (benefit)

C3= Flexibility (benefit)

C4= complexity (non- benefit)

C5= costs (non- benefit)

The attributes values are:

A1={{A1-1=high}, {A1-2=moderate}, ({A1-3=low}}

A2={{A2-1=>55%},{A2-2=25-55%},{A2-3= <30%}}

A3={{A3-1=>Acceptable},{A3-2=<Non-acceptable}}

A4={{A4-1=high}, {A4-2=moderate}, {A4-3=low}}

A5={{5-1=<2 million $},{5-2=2-5 million $},{A5-3=>5 million $}}

4.2 valuating criteria and attribute values

- Valuating Criteria Weight: CRITIC-TriNN

We applied critic method with TriNN to obtain weight

Table 2. Normalize aggregated matrix

	C1	**C2**	**C3**	**C4**	**C5**
SM1	0.376984	0.061047	1	0	0.222656
SM2	1.178571	1	0.588235	0.612774	1
SM3	0	0.22093	0.066176	1	0.841797
SM4	1	0	0	0.0499	0

Table 3. Finding correlation between each criterion

	C1	**C2**	**C3**	**C4**	**C5**
C1	1	0.503829	0.017161	-0.36938	-0.07155
C2	0.503829	1	0.20446	0.453134	0.806848
C3	0.017161	0.20446	1	-0.38594	0.018747
C4	-0.36938	0.453134	-0.38594	1	0.864689
C5	-0.07155	0.806848	0.018747	0.864689	1

Table 4. Convert the degree (rij)

	C1	**C2**	**C3**	**C**4	**C5**
C1	0	0.496171	0.982839	1.369376	1.071552
C2	0.496171	0	0.79554	0.546866	0.193152
C3	0.982839	0.79554	0	1.385939	0.981253
C4	1.369376	0.546866	1.385939	0	0.135311
C5	1.071552	0.193152	0.981253	0.135311	0

Table 5. Final Criteria Weights

	sd	rij	Cj	wj	wj%
C1	0.547245	3.919938	2.145167	0.274061	27%
C2	0.462483	2.03173	0.93964	0.120046	12%
C3	0.471215	4.14557	1.953455	0.249569	25%
C4	0.478491	3.437492	1.644809	0.210137	21%

C5	0.480523	2.381268	1.144253	0.146187	15%
Sum			7.827324	1	100%

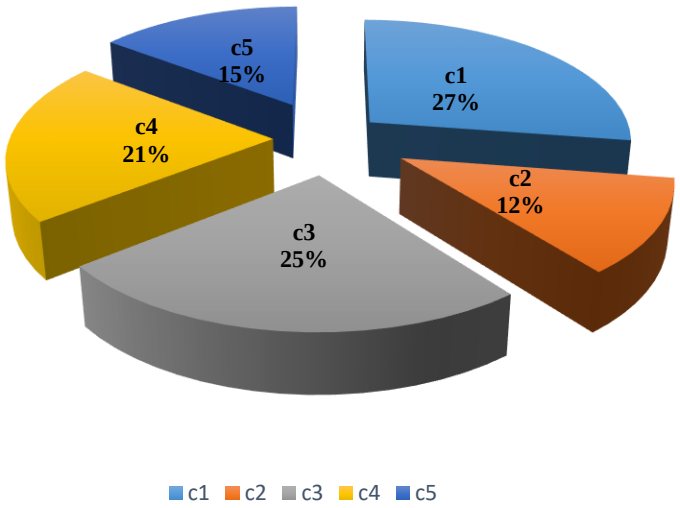

Figure 2. Weights of the criteria using CRITIC

4.3 Ranking alternatives using (MOOSRA-Hypersoft Set) and TirNN

Since C=C1,C2,C3,C4,C5 are contributed to ranking process for four alternatives. We deploy hypersoft set methodology with TriNN in MOOSRA to rank four alternatives. Hence, we are leveraging attributes (A1-1:A5-3), then we choose A1-1,A2-1,A3-1,A4-3,A5-1.

Table 6. square the decision matrix

	A1-1	**A2-1**	**A3-1**	**A4-3**	**A5-1**
SM1	5.867160494	2.46025034	5.653827	21.10892	4.85631
SM2	21.77777778	4.69444444	3.853224	8.345679	2.151111
SM3	1.867777778	2.79013717	2.065075	3.280123	2.613611
SM4	17.36111111	2.3397668	1.877915	19.85198	5.831331
sum	46.87382716	12.2845988	13.45004	52.5867	15.45236
root	6.846446316	3.5049392	3.66743	7.251669	3.930949

Table 7. Normalize decision matrix.

	A1-1	**A2-1**	**A3-1**	**A4-3**	**A5-1**
SM1	0.856964361	0.7019381	1.541632	2.910905	1.235404
SM2	3.180887832	1.33937971	1.05066	1.150863	0.547224
SM3	0.272809819	0.79605865	0.563085	0.452327	0.66488
SM4	2.535784305	0.66756274	0.512052	2.737573	1.483441
Wj	0.274061343	0.12004614	0.249569	0.210137	0.146187

Table 8. Weighted normalized decision matrix

	A1-1	A2-1	A3-1	A4-3	A5-1
SM1	0.234860804	0.08426496	0.384743	0.611688	0.1806
SM2	0.871758392	0.16078736	0.262212	0.241839	0.079997
SM3	0.074766625	0.09556377	0.140528	0.09505	0.097197
SM4	0.694960453	0.08013833	0.127792	0.575265	0.21686

Table 9. ranking the alternatives.

	sum of benefits	sum of non-benefits	Si	Ranking
SM1	0.703868797	0.79228838	0.8884	4
SM2	1.294757686	0.32183585	4.023038	1
SM3	0.310858812	0.19224739	1.616973	2
SM4	0.902890927	0.79212478	1.139834	3

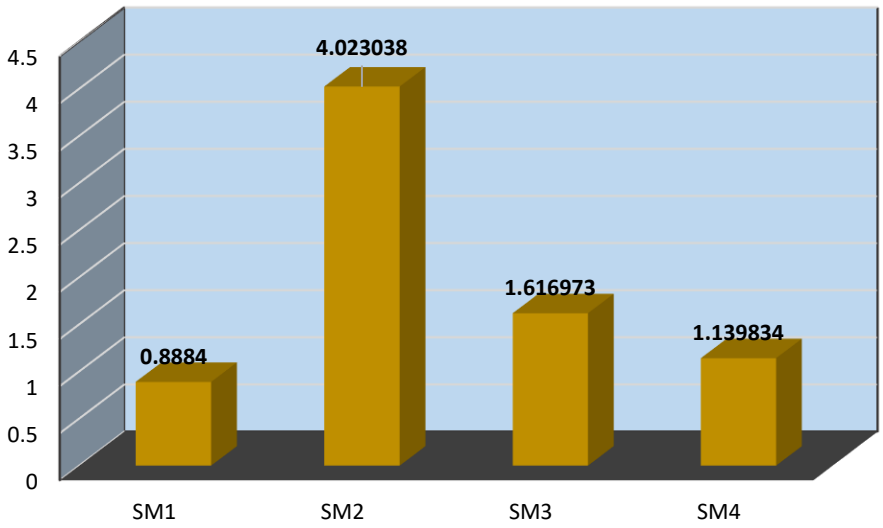

Figure 3. Ranking of Alternatives

5 Conclusion

This paper has investigated the optimization of multi-robot collaborative systems within smart manufacturing through the integration of cognitive digital twins and advanced decision-making methodologies. We present a framework for selecting the most suitable machine for smart manufacturing based on several factors and criteria. The main criteria considered include Production (C1), Efficiency (C2), Survivability (C3), Complexity (C4), and Costs (C5). Four different machine alternatives were evaluated in this paper.

The proposed MCDM techniques are working under authority of Hypersoft set and TriNN for selecting the most suitable alternative of smart manufacturing. By leveraging the CRITIC method to determine the weights of criteria and the MOOSRA technique to rank the alternatives, this study ensures an objective and data-driven evaluation process. The results demonstrate that the proposed

approach provides more reliable and actionable insights compared to traditional methods, offering a practical guide for managers to prioritize strategies effectively. This research reinforces the importance of integrating Industry 4.0 technologies to enhance efficiency, sustainability, and decision-making in manufacturing to be smart. The findings of the constructed soft decision model indicated that SM2 ia the optimal alternative otherwise SM1 is the worst alternative.

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On the category of neutrosophic submodules

Saeid Jafari[1], Ernesto Lax[2], Neelamegarajan Rajesh[3] and Florentin Smarandache [4]

[1] Mathematical and Physical Science Foundation, Sidevej 5, 4200 Slagelse, Denmark; jafaripersia@gmail.com; saeidjafari@topositus.com

[2] Department of mathematics and computer sciences, physics and earth sciences, University of Messina, Viale Ferdinando Stagno d'Alcontres 31, 98166 Messina, Italy; erlax@unime.it

[3] Department of Mathematics, Rajah Serfoji Government College, Thanjavur-613005, Tamilnadu, India; nrajesh_topology@yahoo.co.in

[4] Professor Emeritus, PhD, PostDocs, University of New Mexico, Mathematics, Physics and Natural Sciences Division 705 Gurley Ave., Gallup, NM 87301, USA; smarand@unm.edu

Abstract. The purpose of this paper is to introduce the important and useful notions of inverse and direct systems in the category of neutrosophic modules and present some of fundamental properties of them.

Keywords: inverse and direct systems; inverse and direct limits; exact sequence; fuzzy chain complex; fuzzy module; intuitionistic fuzzy module.

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1. Introduction

In 1965, Zadeh [6] laid the foundation of fuzzy set theory by introducing the concept of degree of membership (or truth) t and defining the fuzzy set. Building on this, Smarandache introduced the term *neutrosophic* to highlight a significant extension of fuzzy logic. The term *neutrosophic* derives from a combination of the French *neutre* (Latin *neuter*) (neutral), and the Greek *sophia* (skill/wisdom) emphasizing knowledge that incorporates neutrality or indeterminacy. This neutrality represents a pivotal distinction between fuzzy and neutrosophic logic. Unlike fuzzy sets, which involve only truth (membership) and falsehood (non-membership) components, neutrosophic sets include an additional, independent component: *indeterminacy* (or *neutrality*). This concept, introduced by Smarandache in 1995 (formally published in 1998), allowed the definition of *neutrosophic set* using three independent components (t, i, f), corresponding to truth, indeterminacy, and falsehood, respectively.

Smarandache's neutrosophic set theory generalizes classical and fuzzy set theories, providing a more flexible and comprehensive framework. It encompasses the notion of interval-valued neutrosophic sets, further broadening its applicability. The versatility of neutrosophic sets has led to their application across diverse fields (as documented on the website `http://fs.gallup.unm.edu/neutrosophy.htm`).

In this paper, we explore the notions of inverse and direct systems within the category of neutrosophic modules, extending the ideas presented in [2]. We also examine their fundamental properties, providing a deeper insight into the theoretical landscape of neutrosophic module systems.

2. Preliminaries

Definition 2.1. Let X be a nonempty set. A *neutrosophic set* [4] A on X is a structure

$$A = \{\langle x, \varsigma_A(x), \varpi_A(x), \nu_A(x)\rangle \ : \ x \in X\}, \tag{1}$$

where $\varsigma_A : X \to [0,1]$ is a truth membership function, $\varpi_A : X \to [0,1]$ is an indeterminate membership function, and $\nu_A : X \to [0,1]$ is a false membership function. The neutrosophic set in (1) will be simply denoted by $A = (\varsigma_A, \varpi_A, \nu_A)$.

Building on the general concept of neutrosophic sets, it is possible to describe integrate it within the context of algebraic structures, especially modules over a ring.

Definition 2.2. Let R be a ring and let F be a R-module. A neutrosophic set $F_1 = (\varsigma_{F_1}, \varpi_{F_1}, \nu_{F_1})$ in F is called a *neutrosophic submodule* of F if the following conditions are satisfied

(1) $\varsigma_{F_1}(0) = 1$,
(2) $\min\{\varsigma_{F_1}(x), \varsigma_{F_1}(y)\} \leq \varsigma_{F_1}(x-y)$ for all $x, y \in F$,
(3) $\varsigma_{F_1}(x) \leq \varsigma_{F_1}(r \cdot x)$ for all $x \in F$ and $r \in R$,
(4) $\varpi_{F_1}(0) = 1$,
(5) $\min\{\varpi_{F_1}(x), \varpi_{F_1}(y)\} \leq \varpi_{F_1}(x-y)$ for all $x, y \in F$,
(6) $\varpi_{F_1}(x) \leq \varpi_{F_1}(r \cdot x)$ for all $x \in F$ and $r \in R$,
(7) $\nu_{F_1}(0) = 0$,
(8) $\nu_{F_1}(x-y) \leq \max\{\nu_{F_1}(x), \nu_{F_1}(y)\}$ for all $x, y \in F$,
(9) $\nu_{F_1}(r \cdot x) \leq \nu_{F_1}(x)$ for all $x \in F$ and $r \in R$.

Definition 2.3. Given an inverse sequence of abelian groups $G_1 \xleftarrow{\rho} G_2 \xleftarrow{\rho} G_3 \cdots$ (in short $\{G_i\}$). Denote by Π the direct product of the groups G_i, and consider the map $\delta : \Pi \to \Pi$ defined as

$$\delta(g_1, g_2, \cdots) = (g_1 - \rho g_2, g_2 - \rho g_3, g_3 - \rho g_4, \cdots).$$

The kernel of δ is called the *inverse limit* of the sequence $\{G_i\}$ and we denote it by $\mathcal{L}_{inv}\{G_i\}$.

3. Some properties of neutrosophic modules

In what follows, a neutrosophic submodule $F_1 = (\varsigma, \varpi, \nu)$ of F will be denoted by $(\varsigma, \varpi, \nu)_F$.

Definition 3.1. A function $h : (\varsigma, \varpi, \nu)_F \to (\varsigma', \varpi', \nu')_{F'}$ is called a homomorphism of neutrosophic modules if the conditions $\varsigma_1(h(x)) \geq \varsigma(x)$, $\varpi_1(h(x)) \geq \varpi(x)$ and $\nu_1(h(x)) \leq \nu(x)$ are satisfied.

Observe that, neutrosophic modules and their morphisms form a category which we denote by N-Mod.

Let F,E be two R-modules and let $(\varsigma, \varpi, \nu)_F$ be a neutrosophic submodule of F. Suppose $h : F \to E$ is an R-module homomorphism. We can define a neutrosophic module structure on E *via* the map h has it follows:

$$\varsigma^h(y) = \sup\{\varsigma(x) \ : \ h(x) = y\},$$
$$\varpi^h(y) = \sup\{\varpi(x) \ : \ h(x) = y\},$$
$$\nu^h(y) = \inf\{\nu(x) \ : \ h(x) = y\}.$$

It is clear that $(\varsigma^h, \varpi^h, \nu^h)_E$ is a neutrosophic submodule of E and the function $h : (\varsigma, \varpi, \nu)_F \to (\varsigma^h, \varpi^h, \nu^h)_E$ is a homomorphism of neutrosophic modules.

Conversely, let $(\eta, \theta, \nu)_E$ be a neutrosophic submodule of E and $h : F \to E$ a homomorphism of R-modules. It is possible to define a neutrosophic module structure in F by setting

$$\eta^h(x) = \eta(h(x)),$$
$$\theta^h(x) = \theta(h(x)),$$
$$\nu^h(x) = \nu(h(x)).$$

Hence, $(\eta^h, \theta^h, \nu^h)_F$ is a neutrosophic module and $h : (\eta^h, \theta^h, \nu^h)_F \to (\eta, \theta, \nu)_E$ is a homomorphism of neutrosophic modules.

We state the following lemma.

Lemma 3.2. *Let F and E be R-modules and $h : F \to E$ be a homomorphism of R-modules. The following conditions are true*

(1) *If $(\varsigma, \varpi, \nu)_F$ is a neutrosophic module, then there exist modular grade functions ς^h,ϖ^h, ν^h on E such that for any modular grade function $B = (\alpha, \beta, \gamma)$ on E, $h : (\varsigma, \varpi, \nu)_F \to (\alpha, \beta, \gamma)_E$ is a neutrosophic homomorphism if and only if $\alpha \geq \varsigma^h$, $\beta \geq \varpi^h$, $\gamma \leq \nu^h$.*

(2) *If $(\alpha, \beta, \gamma)_E$ is a neutrosophic module, then there exists a modular grade function $(\alpha^h, \beta^h, \gamma^h)$ on F such that for every neutrosophic module $(\varsigma, \varpi, \nu)_F$, $h : (\varsigma, \varpi, \nu)_F \to (\alpha, \beta, \gamma)_E$ is a neutrosophic homomorphism if and only if $\varsigma \leq \alpha^h$, $\varpi \leq \beta^h$, $\nu \geq \gamma^h$.*

Lemma 3.3. (1) *Let $\{F_i\}_{i\in\Delta}$, E are modules and a family of R-homomorphisms $\eta = \{h_i : F_i \to E\}_{i\in\Delta}$. If $\{(\varsigma_i, \varpi_i, \nu_i)_{F_i}\}_{i\in\Delta}$ is a family of neutrosophic modules, then there exists the smallest grade functions $\alpha = \varsigma^\eta = \varsigma^{\{h_i\}}$, $\beta = \varpi^\eta = \varpi^{\{h_i\}}$, $\gamma = \nu^\eta = \nu^{\{h_i\}}$ such that, for all $i \in \Delta$, $\tilde{h} : (\varsigma_i, \varpi_i, \nu_i)_{F_i} \to (\alpha, \beta, \gamma)_E$ is a neutrosophic homomorphism.*

(2) *Let $F, \{E_i\}_{i\in\Delta}$ be modules and $B = \{g_i : F \to E_i\}_{i\in\Delta}$ be a family of R-homomorphisms. If $\{(\alpha_i, \beta_i, \gamma_i)_{E_i}\}_{i\in\Delta}$ is a family of neutrosophic modules, then there exist the largest grade functions $\varsigma = \alpha_B = \alpha_{\{h_i\}}$, $\varpi = \beta_B = \beta_{\{h_i\}}$, $\nu = \gamma_B = \gamma_{\{h_i\}}$ such that, for all $i \in \Delta$, $h : (\varsigma, \varpi, \nu)_F \to (\alpha_i, \beta_i, \gamma_i)_{E_i}$ is a neutrosophic homomorphism.*

Proof. (1). Let $\alpha = \varsigma^{\overline{\lambda}} = \bigvee\limits_{\alpha\in\Delta} \mu_\alpha^{f_\alpha}, \theta = \xi^\eta = \bigvee\limits_{i\in\Delta} \xi_i^{h_i}, \gamma = \nu^\eta = \bigwedge\limits_{\alpha\in\Delta} \nu_i^{h_i}$.

(2). Let $\varsigma = \alpha_B = \bigwedge\limits_{i\in\Delta} (\alpha_i)_{h_i}, \varpi = \beta_B = \bigwedge\limits_{i\in\Delta} (\beta_i)_{h_i}, \nu = \gamma_B = \bigvee\limits_{i\in\Delta} (\gamma_i)_{h_i}$. □

We are now in position to define submodule, quotient module, product and co-product operations in the category of neutrosophic modules, by using the above lemma.

If $(\varsigma, \varpi, \nu)_F$ is a neutrosophic module on F and $E \subset F$ is a submodule, then $(\varsigma|_E, \varpi|_E, \nu|_E)_E$ is called a *neutrosophic submodule* of $(\varsigma, \varpi, \nu)_F$.

If $(\varsigma, \varpi, \nu)_F$ is a neutrosophic module and $h_c : F \to F/\sim$ is a canonical homomorphism, then $(\varsigma^{h_c}, \varpi^{h_c}, \nu^{h_c})_{F/\sim}$ is called a *quotient module* of $(\varsigma, \varpi, \nu)_F$.

Hence, for each homomorphism of neutrosophic modules $h : (\varsigma, \varpi, \nu)_F \to (\alpha, \beta, \gamma)_E$, the neutrosophic submodules $(\varsigma|_{\operatorname{Ker} h}, \varpi|_{\operatorname{Ker} h}, \nu|_{\operatorname{Ker} h})_{\operatorname{Ker} h}$ and the neutrosophic quotient module $(\alpha^\pi, \beta^\pi, \gamma^\pi)_{E/\operatorname{Im} h}$ are obtained, where $\pi : E \to E/\operatorname{Im} f$ is a canonical homomorphism.

If $\{(\varsigma_i, \varpi_i, \nu_i)_{F_i}\}_{i\in\Delta}$ is a family of neutrosophic modules, then we define the *product* of this family by $(\varsigma_D, \varpi_D, \nu_D)_{\prod\limits_{i\in\Delta} F_i}$, where $D = \left\{\pi_i : \prod\limits_{i\in\Delta} F_i \to F_i\right\}_{i\in\Delta}$ is a family of the usual projection maps. Moreover, the *co-product* of the family is defined as $(\varsigma^H, \varpi^H, \nu^H)_{\sum F_i}$, where $H = \left\{j_i : F_i \to \sum\limits_{i\in\Delta} F_i\right\}_{i\in\Delta}$ is a family of the usual injections.

Mind that, the category of neutrosophic modules has zero objects, sums, products, kernels and cokernels.

Definition 3.4. A functor $\Phi : \Lambda^{op} \to \mathrm{N-Mod}$ ($\Phi : \Lambda \to \mathrm{N-Mod}$), where Λ is a directed set (considered as a category), is called an *inverse* (resp. *direct*) *system* of neutrosophic modules, the limit of Φ is called a *limit* of the inverse (resp. direct) system.

Consider an inverse system of neutrosophic modules

$$(\underline{\varsigma}, \underline{\varpi}, \underline{\nu})_{\underline{F}} = \left\{(\varsigma_i, \varpi_i, \nu_i)_{F_i}, \underline{h_c}_{i'i}\right\}_{i\in\Delta}. \tag{2}$$

Let $D = \left\{\pi_i, \prod\limits_{i\in\Delta} F_i \to F_i\right\}_{i\in\Delta}$ be a family of projections and let $(\varsigma_D, \varpi_D, \nu_D)_{\prod\limits_{i\in\Delta} F_i}$, the direct product of the neutrosophic modules $(\varsigma_i, \varpi_i, \nu_i)_{F_i}$. Then, we obtain a neutrosophic submodule $(\varsigma_D|_{\varprojlim F_i}, \varpi_D|_{\varprojlim F_i}, \nu_D|_{\varprojlim F_i})_{\varprojlim F_i}$, where $\varprojlim F_i$ is a limit of an inverse system of modules $\{F_i\}_{i\in\Delta}$.

Theorem 3.5. *Every inverse system* (2) *has a limit in the category* N − Mod*, which is equal to the a neutrosophic submodule* $(\varsigma_D|_{\varprojlim F_i}, \varpi_D|_{\varprojlim F_i}, \nu_D|_{\varprojlim F_i})_{\varprojlim F_i}$.

Proof. Our aim is to show that there exists a unique homomorphism of neutrosophic modules $\overline{\theta} : (\alpha, \beta, \gamma)_E \to (\varsigma_D|_{\varprojlim F_i}, \varpi_D|_{\varprojlim F_i}, \nu_D|_{\varprojlim F_i})_{\varprojlim F_i}$, which makes the following diagram

$$\begin{array}{ccc} (\alpha,\beta,\gamma)_E & \xrightarrow{\quad\overline{\varphi}_i\quad} & (\varsigma_i,\varpi_i,\nu_i)_{F_i} \\ \Big\downarrow{\scriptstyle \overline{\theta}} & \nearrow & \\ (\varsigma_D|_{\varprojlim F_i}, \varpi_D|_{\varprojlim F_i}, \nu_D|_{\varprojlim F_i})_{\varprojlim F_i} & & \end{array}$$

commutative. Observe that, for every neutrosophic module $(\alpha, \beta, \gamma)_E$ it is true that $\{\overline{\varphi}_i : (\alpha, \beta, \gamma)_E \to (\varsigma_i, \varpi_i, \nu_i)_{F_i}\}_{i\in\Delta}$ is a family of homomorphism of neutrosophic modules, providing the commutativity of the diagram

$$\begin{array}{ccc} (\alpha,\beta,\gamma)_E & \xrightarrow{\quad\overline{\varphi}_i\quad} & (\varsigma_i,\varpi_i,\nu_i)_{F_i} \\ \Big\downarrow{\scriptstyle \overline{\varphi}_{i'}} & \nearrow{\scriptstyle \overline{h}^{i,i'}} & \\ (\varsigma_{i'},\varpi_{i'},\nu_{i'})_{F_{i'}} & & \end{array}$$

Also, $\overline{\pi_i} : (\varsigma_D|_{\varprojlim F_i}, \varpi_D|_{\varprojlim F_i}, \nu_D|_{\varprojlim F_i})_{\varprojlim F_i} \to (\varsigma_i, \varpi_i, \nu_i)_{F_i}$ is a canonical projection. We define the map $\theta : E \to \varprojlim F_i$ as $f(x) = \{\varphi_i(x)\}_{i\in\Delta}$, for every $x \in F$, which is a module homomorphism. We show that $\overline{\theta} : (\alpha, \beta, \gamma)_E \to (\varsigma_D|_{\varprojlim F_i}, \varpi_D|_{\varprojlim F_i}, \nu_D|_{\varprojlim F_i})_{\varprojlim F_i}$ is a homomorphism of neutrosophic modules. Since $\overline{\varphi_i} : (\alpha, \beta, \gamma)_F \to (\varsigma_i, \varpi_i, \nu_i)_{F_i}$ is a homomorphism of neutrosophic modules for every $i \in \Delta$, the conditions $\varsigma_i(\varphi_i(x)) \geq \alpha(x)$, $\varpi_i(\varphi_i(x)) \geq \beta(x)$ and $\nu_i(\varphi_i(x)) \leq \gamma(x)$ are satisfied for every $x \in E$. Therefore, we obtain

$$\begin{aligned} \varsigma_D(\{\varphi_i(x)\}) &= \bigwedge_{i\in\Delta} \varsigma_i(\varphi_i(x)) \geq \alpha(x), \\ \varpi_D(\{\varphi_i(x)\}) &= \bigwedge_{i\in\Delta} \varpi_i(\varphi_i(x)) \geq \beta(x), \\ \nu_A(\{\varphi_i(x)\}) &= \bigvee_{i\in\Delta} \nu_i(\varphi_i(x)) \leq \gamma(x). \end{aligned}$$

Hence, $\overline{\theta}$ is a homomorphism of neutrosophic modules. The uniqueness of θ follows from the uniqueness of $\overline{\theta}$.

It is conspicuous to show that $\varprojlim$ is a functor from the category of inverse system of neutrosophic modules to the category of neutrosophic modules. □

Let us now focus on the problem of exact limit for inverse systems of exact sequences.

Definition 3.6. A sequence

$$\cdots \to (\varsigma_{n-1}, \varpi_{n-1}, \nu_{n-1})_{F_{n-1}} \xrightarrow{\check{h}_{n-1}} (\varsigma_n, \varpi_n, \nu_n)_{F_n} \xrightarrow{\check{h_n}} (\varsigma_{n+1}, \varpi_{n+1}, \nu_{n+1})_{F_{n+1}} \to \cdots$$

of neutrosophic modules is said *neutrosophic exact* if

$$(\varsigma_n|_{\operatorname{Im} h_{n-1}}, \varpi_n|_{\operatorname{Im} h_{n-1}}, \nu_n|_{\operatorname{Im} h_{n-1}}) = (\varsigma_n|_{\operatorname{Ker} h_n}, \varpi_n|_{\operatorname{Ker} h_n}, \nu_n|_{\operatorname{Ker} h_n}), \qquad \text{for every } n \in \mathbb{Z}.$$

Remark 3.7. Observe that, given a sequence of R-modules

$$\cdots \to F_{n-1} \xrightarrow{h_{n-1}} F_n \xrightarrow{h_n} F_{n+1} \to \cdots \tag{3}$$

it is clear that if (3) is exact, then the induced sequence of neutrosophic modules

$$\cdots \to (\varsigma_{n-1}, \varpi_{n-1}, \nu_{n-1})_{F_{n-1}} \xrightarrow{\overline{h_{n-1}}} (\varsigma_n, \varpi_n, \nu_n)_{F_n} \xrightarrow{\overline{h_n}} (\varsigma_{n+1}, \varpi_{n+1}, \nu_{n+1})_{F_{n+1}} \to \cdots \tag{4}$$

is neutrosophic exact, with

$$\varsigma_n = \chi_{\{0\}}, \quad \varpi_n = \chi_{\{0\}}, \quad \nu_n = 1 - (\varsigma_n + \xi_n).$$

Exactness of (4) implies the exactness of (3), since the equality of two neutrosophic sets is just the equality of their respective maps, which implies the equality of their corresponding domains (that is, $\operatorname{Ker} h_n = \operatorname{Im} h_{n-1}$). On the other hand, the exactness of (3) does not necessarily imply the exactness of (4). Namely, the exactness of R-modules sequences is not derived from the exactness of sequences of a neutrosophic modules.

Example 3.8. For all $n \in \mathbb{N}$, consider the $\mathbb{Z}$-modules $F_n = \mathbb{Z}$, $F'_n = \mathbb{Z}$, $F''_n = \mathbb{Z}_2$. Then,

$$\underline{F} = (\{F_n\}_{n\in E}, \{p_{n+1n}(m) = 3m\}),$$
$$\underline{F'} = (\{F'_n\}_{n\in E}, \{q_{n+1n}(m) = 3m\}),$$
$$\underline{F''} = (\{F''_n\}_{n\in E}, \{r_{n+1n}(m) = [m]\})$$

are inverse systems of modules. Hence, we consider the sequence $0 \to \underline{F'} \xrightarrow{h} \underline{F} \xrightarrow{g} \underline{F''} \to 0$, with $h = \{h_n : F'_n \to F_n \ : \ h_n(m) = 2m\}$ and $g = \{g_n : F_n \to F''_n \ : \ g_n(m) = [m]\}$, which is a short exact sequence of inverse systems of $\mathbb{Z}$-modules. Then, the following sequence $\overline{0} \to (\varsigma'_n, \varpi'_n, \nu'_n)_{F'_n} \xrightarrow{\overline{h}_n} (\varsigma_n, \varpi_n, \nu_n)_{F_n} \xrightarrow{\overline{g}_n} (\varsigma''_n, \varpi''_n, \nu''_n)_{F''_n} \to \overline{0}$ is a short exact sequence of neutrosophic modules, where $\varsigma_n = (\chi(0))_{F_n}$, $\varpi_n = (\chi(0))_{F_n}$, $\nu_n = 1-(\varsigma_n+\varpi_n)$, $\varsigma'_n = (\chi(0))_{F'_n}$, $\varpi'_n = (\chi(0))_{F'_n}$, $\nu'_n = 1 - (\varsigma'_n + \varpi'_n)$, $\varsigma''_n = (\chi(0))_{F''_n}$, $\varpi''_n = (\chi(0))_{F''_n}$, $\nu''_n = 1 - (\varsigma''_n + \varpi''_n)$. Therefore, the sequence $\overline{0} \to (\underline{\varsigma}', \underline{\varpi}', \underline{\nu}')_{\underline{F}} \xrightarrow{\overline{h_n}} (\underline{\varsigma}, \underline{\varpi}, \underline{\nu})_{\underline{F}} \xrightarrow{\overline{g}} (\underline{\varsigma}'', \underline{\varpi}'', \underline{\nu}'')_{\underline{F''}} \to \overline{0}$ is a short exact

sequence of inverse systems of neutrosophic modules. Nevertheless, taking the limit of this sequence, we obtain $0 \to 0 \to 0 \to (\varsigma'', \varpi'', \nu'')_{Z_2} \to 0$, which is not exact.

The limit of inverse system of exact sequence of neutrosophic modules is not exact, hence we need to introduce the notion of derivative functor of inverse limit functor in $\mathrm{N}-\mathrm{Mod}$.

Consider an inverse system of neutrosophic modules as in (2). Define also the R-modules homomorphism $d : \prod_{\alpha} F_\alpha \to \prod_{\alpha} F_\alpha$ as $d(\{x_\alpha\}) = \{x_\alpha - p_{\alpha'\alpha}(x_{\alpha'})\}_{\alpha \prec \alpha'}$. We want to show that d is a homomorphism of neutrosophic modules. Indeed,

$$\begin{aligned}
\varsigma_D(d(\{x_i\})) &= \varsigma_D(x_i - p_{i'i}(x_{i'})) \\
&= \bigwedge_i \varsigma_i(x_i - p_{i'i}(x_{i'})) \\
&\geq \bigwedge_\alpha \min\{\varsigma_i(x_i), \varsigma_i(p_{i'i}(x_{i'}))\} \\
&\geq \bigwedge_i \min\{\varsigma_i(x_i), \varsigma_i'(x_{i'})\} \ (\because \ \varsigma_i(p_{i'i}(x_{i'})) \geq \varsigma_{i'}(x_{i'})) \\
&= \bigwedge_i (\varsigma_i(x_i) \wedge \varsigma_i'(x_{i'})) \\
&= \bigwedge_i \varsigma_i(x_i) \\
&= \varsigma_D(x_i),
\end{aligned}$$

$$\begin{aligned}
\varpi_D(d(\{x_i\})) &= \varpi_D(x_i - p_{i'i}(x_{i'})) \\
&= \bigwedge_i \varpi_i(x_i - p_{i'i}(x_{i'})) \\
&\geq \bigwedge_i \min\{\varpi_i(x_i), \varpi_i(p_{i'i}(x_{i'}))\} \\
&\geq \bigwedge_i \min\{\varpi_i(x_i), \varpi_i'(x_{i'})\} \ (\because \ \varpi_i(p_{i'i}(x_{i'})) \geq \varpi_{i'}(x_{i'})) \\
&= \bigwedge_i (\varpi_i(x_i) \wedge \varpi_i'(x_{i'})) \\
&= \bigwedge_i \varpi_i(x_i) \\
&= \varpi_D(x_i),
\end{aligned}$$

$$\begin{aligned}
\nu_D(d(\{x_i\})) &= \nu_D(x_i - p_{i'i}(x_{i'})) \\
&= \bigvee_i \nu_i(x_i - p_{i'i}(x_{i'})) \\
&\leq \bigvee_i \max\{\nu_\alpha(x_i), \nu_i(p_{i'i}(x_{i'}))\} \ (\because\ \nu_i(p_{i'i}(x_{i'})) \leq \nu_{i'}(x_{i'})) \\
&\leq \bigvee_i \max\{\nu_i(x_i), \nu i'(x_{i'})\} \\
&= \bigvee_i (\nu_i(x_i) \vee \nu_i'(x_{i'})) \\
&= \bigvee_i \nu_i(x_i) \\
&= \nu_D(x_i).
\end{aligned}$$

This shows that d is a homomorphism of neutrosophic modules. Thus,

$$(\varsigma_D|_{\operatorname{Ker} d}, \varpi_D|_{\operatorname{Ker} d}, \nu_D|_{\operatorname{Ker} d})_{\operatorname{Ker} d}, \text{ and } ((\varsigma_D)^p, (\varpi_D)^p, (\nu_D)^p)_{\operatorname{Coker} d}$$

are defined.

For inverse system of R-modules $\{F_i, p_{i'i}\}_{i\in\Delta}$, we have $\varprojlim^{(1)} F_i = \prod_i / \operatorname{Im} d$, where $\varprojlim^{(1)}$ is the derivative functor of $\varprojlim$ [3].

If $\pi : \prod_i \to \varprojlim^{(1)} F_i$ is a canonical homomorphism, then we are able to define a neutrosophic module by $((\varsigma_D)^\pi), (\varpi_D)^\pi, (\nu D)^\pi)_{\varprojlim^{(1)} F_i}$.

Definition 3.9. $((\varsigma_D)^\pi, (\varpi_D)^\pi, (\nu_D)^\pi)_{\varprojlim^{(1)} F_i}$ is called the *first derived functor* of the limit of the inverse system of neutrosophic modules (2).

Proposition 3.10. $\varprojlim^{(1)}$ *is a functor.*

Proof. We are done if we show that for each morphism of neutrosophic chain complexes $\check{h} = \left(\rho : H \to D, \left\{\check{h_j} : (\varsigma_{\rho(j)}, \varpi_{\rho(j)}, \nu_{\rho(j)})_{F_{\rho(j)}} \to (i_l, j_l, k_l)_{N_l}\right\}_{j\in H}\right)$, the map $\varprojlim^{(1)}\check{h} : ((\varsigma_D)^\pi, (\varpi_D)^\pi, (\nu_D)^\pi)_{\varprojlim^{(1)} F_i} \to ((\alpha_H)^\pi, (\beta_H)^\pi, (\gamma_H)^\pi)_{\varprojlim^{(1)} N_j}$ is a homomorphism of neutrosophic modules. We have

$$\begin{aligned}
(\varsigma_D)^\pi(x + \operatorname{Im} d) &= \sup_{z\in\operatorname{Im} d} \varsigma_D(x + z) \\
&\leq \sup_{z\in\operatorname{Im} d} \alpha_D(h(x + z)) \\
&= \sup_{z\in\operatorname{Im} d} \alpha_D(h(x) + h(z)) \\
&= \sup_{y=h(z)} \alpha_D(h(x) + y) \\
&\leq \sup_{y\in\operatorname{Im} d} \alpha_D(h(x) + y) \\
&= (\alpha_D)^\pi(\varprojlim^{(1)} h(x + \operatorname{Im} d)),
\end{aligned}$$

$$\begin{aligned}(\varpi_D)^{\pi}(x+\operatorname{Im} d) &= \sup_{z\in \operatorname{Im} d} \varpi_D(x+z)\\ &\leq \sup_{z\in \operatorname{Im} d} \beta_D(h(x+z))\\ &= \sup_{z\in \operatorname{Im} d} \beta_D(f(x)+h(z))\\ &= \sup_{y=h(z)} \beta_D(h(x)+y)\\ &\leq \sup_{y\in \operatorname{Im} d} \beta_D(h(x)+y)\\ &= (\beta_D)^{\pi}(\varprojlim{}^{(1)} h(x+\operatorname{Im} d)),\end{aligned}$$

$$\begin{aligned}(\nu_D)^{\pi}(x+\operatorname{Im} d) &= \inf_{z\in \operatorname{Im} d} \nu_D(x+z)\\ &\geq \inf_{z\in \operatorname{Im} d} \gamma_H(h(x+z))\\ &= \inf_{z\in \operatorname{Im} d} \gamma_H(h(x)+h(z))\\ &= \inf_{y=h(z)} \gamma_H(h(x)+y)\\ &\geq \inf_{y\in \operatorname{Im} d} \gamma_D(h(x)+y)\\ &= (\gamma_D)^{\pi}(\varprojlim{}^{(1)} h(x+\operatorname{Im} d)).\end{aligned}$$

Hence, $\varprojlim^{(1)}$ is a functor, as stated. □

We intend to investigate the properties of the functor $\varprojlim^{(1)}$. Therefore, we introduce the category of chain (cochain) complexes of neutrosophic modules. This category is defined in the same lines as in [1].

Definition 3.11. A neutrosophic chain complex

$$(\varsigma, \varpi, \nu)_F = \{(\varsigma_n, \varpi_n, \nu_n)_{F_n}, \overline{\partial_n}\}_{n\in\mathbb{Z}}$$

is an object in N-Mod along with a neutrosophic endomorphism $\overline{\partial} : (\varsigma, \varpi, \nu)_F \to (\varsigma, \varpi, \nu)_F$ of degree -1 such that $\overline{\partial\partial} = \overline{0}$.

Definition 3.12. A morphism $\overline{\varphi} : (\varsigma, \varpi, \nu)_F \to (\alpha, \beta, \gamma)_E$ of neutrosophic chain complexes is a morphism $\overline{\varphi} = \{\overline{\varphi}_n : (\varsigma_n, \varpi_n, \nu_n)_{F_n} \to (\alpha_n, \beta_n, \gamma_n)_{E_n}\}$, which has a degree 0 such that $\overline{\varphi_{n-1}} \circ \overline{\partial_n} = \overline{\partial'_n}\overline{\varphi_n}$, where $\overline{\partial}$ denotes the neutrosophic differential in $(\alpha, \beta, \gamma)_E$.

Definition 3.13. Let $(\varsigma, \varpi, \nu)_F = (\varsigma_n, \varpi_n, \nu_n)_{F_n}, \overline{\partial_n}\}_{n\in\mathbb{Z}}$ be a neutrosophic chain complex. The condition $\overline{\partial} \circ \overline{\partial} = \overline{0}$ implies that $\operatorname{Im} \overline{\partial_{n+1}} \subset \operatorname{Ker} \overline{\partial_n}$, for $n \in \mathbb{Z}$. Hence, we can associate a neutrosophic graded module with $(\varsigma, \varpi, \nu)_F$ $H((\varsigma, \varpi, \nu)_F) = \{H_n(\varsigma, \varpi, \nu)_F\}$, where

$$H_n((\varsigma, \varpi, \nu)_F) = \frac{\left(\varsigma_n|_{\operatorname{Ker} \overline{\partial_n}}, \varpi_n|_{\operatorname{Ker} \overline{\partial_n}}, \nu_n|_{\operatorname{Ker} \overline{\partial_n}}\right)_{\operatorname{Ker} \overline{\partial_n}}}{\left(\varsigma_n|_{\operatorname{Im} \overline{\partial_{n+1}}}, \varpi_n|_{\operatorname{Im} \overline{\partial_{n+1}}}, \nu_n|_{\operatorname{Im} \overline{\partial_{n+1}}}\right)_{\operatorname{Im} \overline{\partial_{n+1}}}}.$$

$H((\varsigma, \varpi, \nu)_F)$ is called the *neutrosophic homology module* of the neutrosophic complex $(\varsigma, \varpi, \nu)_F$.

Analogously, one can define cochain neutrosophic complex and neutrosophic cohomology module.

Let $\overline{\varphi}, \overline{\psi} : (\varsigma, \varpi, \nu)_F \to (\alpha, \beta, \gamma)_E$ be morphisms of neutrosophic chain complexes.

Definition 3.14. A *neutrosophic homotopy* $\overline{\Sigma} : (\varsigma, \varpi, \nu)_F \to (\alpha, \beta, \gamma)_E$ between $\overline{\varphi}$ and $\overline{\psi}$ is a neutrosophic morphism of degree $+1$ such that $\overline{\psi} - \overline{\varphi} = \overline{\partial} \circ \overline{\Sigma} + \overline{\Sigma} \circ \overline{\partial}$.

The morphism $\overline{\varphi}$, $\overline{\psi}$ are said to be *neutrosophic homotopic*, if there exists a neutrosophic homotopy between them.

Theorem 3.15. *The neutrosophic homotopy relation is an equivalence relation. Moreover, neutrosophic homology and cohomology modules are invariant with respect of this relation.*

Observe the following neutrosophic cochain complex

$$\overline{0} \to (\varsigma_D, \varpi_D, \nu_D)_{\prod F_\alpha} \xrightarrow{\overline{d}} (\varsigma_D, \varpi_D, \nu_D)_{\prod F_\alpha} \to \overline{0}.$$

Neutrosophic cohomology modules of this complex are $\operatorname{Ker} \overline{d}$ and $\operatorname{Coker} \overline{d}$.

Lemma 3.16. $\varprojlim^{(1)}(\varsigma_i, \varpi_i, \nu_i)_{F_i} = \operatorname{Ker} \overline{d}$ *and* $\varprojlim^{(1)}(\mu_\alpha, \xi_\alpha, \nu_i)_{F_i} = \operatorname{Coker} \overline{d}$.

Proof. Obvious. □

In what follows, we consider the set of natural numbers as index set of inverse systems.

Theorem 3.17. *For each infinite subsequence of the inverse sequence of neutrosophic modules*

$$(F_1, \varsigma_1, \varpi_1, \nu_1)_{F_1} \xleftarrow{\overline{p}_{12}} (F_2, \varsigma_2, \varpi_2, \nu_2)_{F_2} \xleftarrow{\overline{p}_{23}} \cdots,$$

$\varprojlim^{(1)}$ *does not change.*

Proof. Let $S = \{i, j, k, \cdots\}$ be an infinite subsequence of natural numbers. By Lemma 3.16, $\varprojlim^{(1)}$ is defined by the following homomorphism of neutrosophic modules subsequence S

$$\overline{d}' : \Big(\bigwedge_{s\in S} \varsigma_s, \bigwedge_{s\in S} \varpi_s, \bigvee_{s\in S} \nu_s\Big)_{\prod_{s\in S} F_s} \to \Big(\bigwedge_{s\in S} \varsigma_s, \bigwedge_{s\in S} \varpi_s, \bigvee_{s\in S} \nu_s\Big)_{\prod_{s\in S} F_s}.$$

We also define two module homomorphisms $h_0, h_1 : \prod_{s\in S} F_s \to \prod_{s\in S} F_s$ as

$$h_0(x_i, x_j, x_k, \cdots) = (p_{1i}(x_i), p_{2i}(x_i), \cdots, p_{i-1i}(x_i), x_i, p_{i+1j}(x_j), \cdots, p_{j-1j}(x_j), x_j, \cdots),$$
$$h_1(x_i, x_j, x_k, \cdots) = (0, 0, \cdots, x_i, 0, \cdots, x_j, 0, \cdots, x_k, 0, \cdots).$$

Moreover, $\left(\bigwedge_{n\in N} \varsigma_n\right)(p_{1i}(x_i), p_{2i}(x_i), \cdots, p_{i-1i}(x_i), x_i, p_{i+1j}(x_j), \cdots, p_{j-1j}(x_j), x_j, \cdots)$

$$\begin{aligned}
&= \varsigma_1(p_{1i}(x_i)) \wedge \cdots \wedge \varsigma_{i-1}(p_{i-1i}(x_i)) \wedge \varsigma_i(x_i) \wedge \varsigma_{i+1}(p_{i+1j}(x_j)) \wedge \cdots \wedge \varsigma_j(x_j) \wedge \cdots \\
&\geq [\mu_i(x_i) \wedge \cdots \wedge \varsigma_i(x_i) \wedge \varsigma_i(x_i)] \wedge [\varsigma_j(x_j) \wedge \cdots \wedge \varsigma_j(x_j)] \wedge \cdots \\
&= \varsigma_i(x_i) \wedge \varsigma_j(x_j) \wedge \cdots \\
&= \bigwedge_{s\in S} \varsigma_s(x_s),
\end{aligned}$$

$\left(\bigwedge_{n\in N} \xi_n\right)(p_{1i}(x_i), p_{2i}(x_i), \cdots, p_{i-1i}(x_i), x_i, p_{i+1j}(x_j), \cdots, p_{j-1j}(x_j), x_j, \cdots)$

$$\begin{aligned}
&= \varpi_1(p_{1i}(x_i)) \wedge \cdots \wedge \varpi_{i-1}(p_{i-1i}(x_i)) \wedge \varpi_i(x_i) \wedge \varpi_{i+1}(p_{i+1j}(x_j)) \wedge \cdots \wedge \varpi_j(x_j) \wedge \cdots \\
&\geq [\varpi_i(x_i) \wedge \cdots \wedge \varpi_i(x_i) \wedge \varpi_i(x_i)] \wedge [\varpi_j(x_j) \wedge \cdots \wedge \varpi_j(x_j)] \wedge \cdots \\
&= \varpi_i(x_i) \wedge \varpi_j(x_j) \wedge \cdots \\
&= \bigwedge_{s\in S} \varpi_s(x_s),
\end{aligned}$$

$\left(\bigvee_{n\in N} \nu_n\right)(p_{1i}(x_i), p_{2i}(x_i), \cdots, p_{i-1i}(x_i), x_i, p_{i+1j}(x_j), \cdots, p_{j-1j}(x_j), x_j, \cdots)$

$$\begin{aligned}
&= \nu_1(p_{1i}(x_i)) \vee \cdots \vee \nu_{i-1}(p_{i-1i}(x_i)) \vee \nu_i(x_i) \vee \nu_{i+1}(p_{i+1j}(x_j)) \vee \cdots \vee \nu_j(x_j) \vee \cdots \\
&\geq [\nu_i(x_i) \vee \cdots \vee \nu_i(x_i) \vee \nu_i(x_i)] \vee [\nu_j(x_j) \vee \cdots \vee \nu_j(x_j)] \vee \cdots \\
&= \nu_i(x_i) \vee \nu_j(x_j) \vee \cdots \\
&= \bigvee_{s\in S} \nu_s(x_s)
\end{aligned}$$

$$\begin{aligned}
\left(\bigwedge_{n\in N} \varsigma_n\right)(0, 0, \cdots, x_i, 0, \cdots, x_j, 0, \cdots) &= \varsigma_1(0) \wedge \cdots \wedge \varsigma_i(x_i) \wedge \varsigma_{i+1}(0) \wedge \cdots \wedge \varsigma_j(x_j) \wedge \cdots \\
&= \varsigma_i(x_i) \wedge \varsigma_j(x_j) \wedge \cdots \\
&= \bigwedge_{s\in S} \varsigma_s(x_s)
\end{aligned}$$

$$\left(\bigwedge_{n\in N}\varpi_n\right)(0,0,\cdots,x_i,0,\cdots,x_j,0,\cdots)=\varpi_1(0)\wedge\cdots\wedge\varpi_i(x_i)\wedge\varpi_{i+1}(0)\wedge\cdots\wedge\varpi_j(x_j)\wedge\cdots$$
$$=\varpi_i(x_i)\wedge\varpi_j(x_j)\wedge\cdots$$
$$=\bigwedge_{s\in S}\varpi_s(x_s)$$

$$\left(\bigvee_{n\in N}\nu_n\right)(0,0,\cdots,x_i,0,\cdots,x_j,0,\cdots)=\nu_1(0)\vee\cdots\vee\nu_i(x_i)\vee\nu_{i+1}(0)\vee\cdots\vee\nu_j(x_j)\vee\cdots$$
$$=\nu_i(x_i)\vee\nu_j(x_j)\vee\cdots$$
$$=\bigvee_{s\in S}\nu_s(x_s).$$

Then, $\check{h}_0,\check{h}_1:\left(\bigwedge_{s\in S}\varsigma_s,\bigwedge_{s\in S}\varpi_s,\bigvee_{s\in S}\nu_s\right)_{\prod_{s\in S}F_s}\to\left(\bigwedge_{n\in N}\varsigma_n,\bigwedge_{n\in N}\varpi_n,\bigvee_{n\in N}\nu_n\right)_{\prod_{n\in N}F_n}$ are homomorphisms of neutrosophic modules. Observe that, the following diagram is commutative:

$$\begin{array}{ccc}\left(\bigwedge_{s\in S}\varsigma_s,\bigwedge_{s\in S}\varpi_s,\bigvee_{s\in S}\nu_s\right)_{\prod_{s\in S}F_s} & \longrightarrow & \left(\bigwedge_{n\in N}\varsigma_n,\bigwedge_{n\in N}\varpi_n,\bigvee_{n\in N}\nu_n\right)_{\prod_{n\in N}F_n}\\ \overline{d}'\downarrow & & \downarrow\overline{d}\\ \left(\bigwedge_{s\in S}\varsigma_s,\bigwedge_{s\in S}\varpi_s,\bigvee_{s\in S}\nu_s\right)_{\prod_{s\in S}F_s} & \longrightarrow & \left(\bigwedge_{n\in N}\varsigma_n,\bigwedge_{n\in N}\varpi_n,\bigvee_{n\in N}\nu_n\right)_{\prod_{n\in N}F_n}\end{array}$$

That is, $\{\check{h}_0,\check{h}_1\}$ are morphisms of neutrosophic cochain complexes. Now, we define two homomorphisms $g_0,g_1:\prod_{n\in N}F_n\to\prod_{s\in S}F_s$ as $g_0(x_1,x_2,x_3,\cdots)=(x_i,x_j,x_k,\cdots)$, $g_1(x_1,x_2,x_3,\cdots)=(x_i+p_{ii+1}(x_{i+1})+\cdots+p_{ij-1}(x_{j-1}),x_j+p_{jj+1}(x_{j+1})+\cdots+p_{jk-1}(x_{k-1}),\cdots)$. Moreover,

$$\Big(\bigwedge_{s\in S}\varsigma_s\Big)(x_i,x_j,x_k,\cdots)=\varsigma_i(x_i)\wedge\varsigma_j(x_j)\wedge\cdots\geq\bigwedge_{n\in N}\varsigma_n(x_n),$$
$$\Big(\bigwedge_{s\in S}\varpi_s\Big)(x_i,x_j,x_k,\cdots)=\varpi_i(x_i)\wedge\xi_j(x_j)\wedge\cdots\geq\bigwedge_{n\in N}\varpi_n(x_n),$$
$$\Big(\bigvee_{s\in S}\nu_s\Big)(x_i,x_j,x_k,\cdots)=\nu_i(x_i)\vee\nu_j(x_j)\vee\cdots\leq\bigvee_{n\in N}\nu_n(x_n),$$

Therefore, we have

$$\left(\bigwedge_{s\in S}\varsigma_s\right)(x_i+p_{ii+1}(x_{i+1})+\cdots+p_{ij-1}(x_{j-1}),x_j+\cdots+p_{jk-1}(x_{k-1}),\cdots)$$

$$=\varsigma_i(x_i+p_{ii+1}(x_{i+1})+\cdots+p_{ij-1}(x_{j-1}))\wedge\varsigma_j(x_j+\cdots+p_{jk-1}(x_{k-1}))\wedge\cdots$$

$$\geq\min\{\varsigma_i(x_i),\varsigma_i(p_{ii+1}(x_{i+1})),\cdots,\varsigma_i(p_{ij-1}(x_{j-1}))\}\wedge\min\{\varsigma_j(x_j),\cdots,\varsigma_j(p_{jk-1}(x_{k-1}))\}\wedge\cdots$$

$$\geq\min\{\varsigma_i(x_i),\varsigma_{i+1}(x_{i+1}),\cdots,\varsigma_{j-1}(x_{j-1})\}\wedge\min\{\varsigma_j(x_j),\varsigma_{j+1}(x_{j+1}),\cdots,\varsigma_{k-1}(x_{k-1})\}\wedge\cdots$$

$$=\bigwedge_{m\in M}(x_m)$$

$$\geq\bigwedge_{n\in N}(x_n)$$

$$\left(\bigwedge_{s\in S}\varpi_s\right)(x_i+p_{ii+1}(x_{i+1})+\cdots+p_{ij-1}(x_{j-1}),x_j+\cdots+p_{jk-1}(x_{k-1}),\cdots)$$

$$=\varpi_i(x_i+p_{ii+1}(x_{i+1})+\cdots+p_{ij-1}(x_{j-1}))\wedge\varpi_j(x_j+\cdots+p_{jk-1}(x_{k-1}))\wedge\cdots$$

$$\geq\min\{\varpi_i(x_i),\varpi_i(p_{ii+1}(x_{i+1})),\cdots,\varpi_i(p_{ij-1}(x_{j-1}))\}\wedge\min\{\varpi_j(x_j),\cdots,\xi_j(p_{jk-1}(x_{k-1}))\}\wedge\cdots$$

$$\geq\min\{\varpi_i(x_i),\xi_{i+1}(x_{i+1}),\cdots,\varpi_{j-1}(x_{j-1})\}\wedge\min\{\varpi_j(x_j),\varpi_{j+1}(x_{j+1}),\cdots,\varpi_{k-1}(x_{k-1})\}\wedge\cdots$$

$$=\bigwedge_{m\in M}(x_m)$$

$$\geq\bigwedge_{n\in N}(x_n)$$

$$\left(\bigvee_{s\in S}\nu_s\right)(x_i+p_{ii+1}(x_{i+1})+\cdots+p_{ij-1}(x_{j-1}),x_j+\cdots+p_{jk-1}(x_{k-1}),\cdots)$$

$$\leq\max\{\nu_i(x_i),\nu_i(p_{ii+1}(x_{i+1})),\cdots,\nu_i(p_{ij-1}(x_{j-1}))\}\vee\max\{\nu_j(x_j),\cdots,\nu_j(p_{jk-1}(x_{k-1}))\}\wedge\cdots$$

$$\leq\max\{\nu_i(x_i),\nu_{i+1}(x_{i+1}),\cdots,\nu_{j-1}(x_{j-1})\}\vee\max\{\nu_j(x_j),\nu_{j+1}(x_{j+1}),\cdots,\nu_{k-1}(x_{k-1})\}\wedge\cdots$$

$$=\bigwedge_{m\in M}\nu_n(x_n)$$

$$\leq\bigwedge_{n\in N}\nu_n(x_n).$$

Thus, $\overline{g}_0,\overline{g}_1:\left(\bigwedge_{n\in N}\varsigma_n,\bigwedge_{n\in N}\varpi_n,\bigvee_{n\in N}\nu_n\right)_{\prod_{n\in N}F_n}\to\left(\bigwedge_{n\in N}\varsigma_n,\bigwedge_{n\in N}\varpi_n,\bigvee_{n\in N}\nu_n\right)_{\prod_{n\in N}F_n}$ are homomorphisms of neutrosophic modules. Define now $D:\prod_{n\in N}F_n\to\prod_{n\in N}F_n$ as

$$D(x_1,x_2,x_3,\cdots)=\big(x_1+p_{12}(x_2)+\cdots+p_{1i-1}(x_{i-1}),x_2+p_{23}(x_3)+\cdots+p_{2i-1}(x_{i-1}),\cdots,$$
$$x_{i-1},0,x_{i+1}+p_{i+1i+2}(x_{i+2})+\cdots+p_{i+1j-1}(x_{j-1}),x_{i+2}+\cdots+p_{i+2j-1}(x_{j-1}),0,\cdots\big)$$

which is a module homomorphism. Hence, we have

$$\left(\bigwedge_{n\in N}\varsigma_n\right)(x_1+p_{12}(x_2)+\cdots+p_{1i-1}(x_{i-1}),x_2+p_{23}(x_3)+\cdots+p_{2i-1}(x_{i-1}),\cdots,x_{i-1},0,\cdots)$$

$$\begin{aligned}
&=\varsigma_1(x_1+p_{12}(x_2)+\cdots+p_{1i-1}(x_{i-1}))\wedge\varsigma_2(x_2)+p_{23}(x_3)+\cdots+p_{2i-1}(x_{i-1}))\wedge\cdots\\
&\wedge\varsigma_{i-1}(x_{i-1})\wedge\varsigma_i(0)\wedge\varsigma_{i+1}(x_{i+1}+p_{i+1i+2}(x_i+2)+\cdots+p_{i+1j}-1(x_{j-1}))\wedge\cdots\\
&\geq\min\{\varsigma_1(x_1),\varsigma_1(p_{12}(x_2)),\cdots,\varsigma_1(p_{1i-1}(x_{i-1}))\}\wedge\\
&\min\{\varsigma_2(x_2),\varsigma_2(p_{23}(x_3)),\cdots,\varsigma_2(p_{2i-1}(x_i-1))\}\wedge\varsigma_{i-1}(x_{i-1})\wedge\\
&\min\{\varsigma_{i+1}(x_{i+1}),\varsigma_{i+1}(p_{i+1i+2}(x_{i+2})),\cdots,\varsigma_{i+1}(p_{i+1j}-1(x_{j-1}))\}\wedge\cdots\\
&\geq\min\{\varsigma_1(x_1),\varsigma_2(x_2),\cdots,\varsigma_{i-1}(x_{i-1})\}\wedge\min\{\varsigma_2(x_2),\varsigma_3(x_3),\cdots,\varsigma_{i-1}(x_{i-1})\}\wedge\\
&\varsigma_{i-1}(x_{i-1})\wedge\varsigma_{i+1}(x_{i+1})\wedge\cdots\\
&=\bigwedge_{k=1}^{i-1}\varsigma_k(x_k)\wedge\bigwedge_{k=2}^{i-1}\mu_k(x_k)\wedge\cdots\\
&=\bigwedge_{n\in N}\varsigma_n(x_n),
\end{aligned}$$

$$\left(\bigwedge_{n\in N}\xi_n\right)(x_1+p_{12}(x_2)+\cdots+p_{1i-1}(x_{i-1}),x_2+p_{23}(x_3)+\cdots+p_{2i-1}(x_{i-1}),\cdots,x_{i-1},0,\cdots)$$

$$\begin{aligned}
&=\varpi_1(x_1+p_{12}(x_2)+\cdots+p_{1i-1}(x_{i-1}))\wedge\varpi_2(x_2)+p_{23}(x_3)+\cdots+p_{2i-1}(x_{i-1}))\wedge\cdots\\
&\wedge\varpi_{i-1}(x_{i-1})\wedge\varpi_i(0)\wedge\varpi_{i+1}(x_{i+1}+p_{i+1i+2}(x_i+2)+\cdots+p_{i+1j}-1(x_{j-1}))\wedge\cdots\\
&\geq\min\{\varpi_1(x_1),\varpi_1(p_{12}(x_2)),\cdots,\varpi_1(p_{1i-1}(x_{i-1}))\}\wedge\\
&\min\{\varpi_2(x_2),\varpi_2(p_{23}(x_3)),\cdots,\xi_2(p_{2i-1}(x_i-1))\}\wedge\varpi_{i-1}(x_{i-1})\wedge\\
&\min\{\varpi_{i+1}(x_{i+1}),\varpi_{i+1}(p_{i+1i+2}(x_{i+2})),\cdots,\varpi_{i+1}(p_{i+1j}-1(x_{j-1}))\}\wedge\cdots\\
&\geq\min\{\varpi_1(x_1),\xi_2(x_2),\cdots,\varpi_{i-1}(x_{i-1})\}\wedge\min\{\varpi_2(x_2),\xi_3(x_3),\cdots,\varpi_{i-1}(x_{i-1})\}\wedge\\
&\varpi_{i-1}(x_{i-1})\wedge\varpi_{i+1}(x_{i+1})\wedge\cdots\\
&=\bigwedge_{k=1}^{i-1}\varpi_k(x_k)\wedge\bigwedge_{k=2}^{i-1}\varpi_k(x_k)\wedge\cdots\\
&=\bigwedge_{n\in N}\varpi_n(x_n),
\end{aligned}$$

$$\left(\bigvee_{n\in N}\nu_n\right)(x_1+p_{12}(x_2)+\cdots+p_{1i-1}(x_{i-1}),x_2+p_{23}(x_3)+\cdots+p_{2i-1}(x_{i-1}),\cdots,x_{i-1},0,\cdots)$$

$$=\nu_1(x_1+p_{12}(x_2)+\cdots+p_{1i-1}(x_{i-1}))\vee\nu_2(x_2)+p_{23}(x_3)+\cdots+p_{2i-1}(x_{i-1}))$$

$$\vee\cdots\vee\nu_{i-1}(x_{i-1})\vee\nu_i(0)\vee\nu_{i+1}(x_{i+1}+p_{i+1i+2}(x_i+2)+\cdots+p_{i+1j}-1(x_{j-1}))\vee\cdots$$

$$\geq\max\{\nu_1(x_1),\nu_1(p_{12}(x_2)),\cdots,\nu_1(p_{1i-1}(x_{i-1}))\}\vee$$

$$\max\{\nu_2(x_2),\nu_2(p_{23}(x_3)),\cdots,\nu_2(p_{2i-1}(x_i-1))\}\vee\nu_{i-1}(x_{i-1})\vee$$

$$\max\{\nu_{i+1}(x_{i+1}),\nu_{i+1}(p_{i+1i+2}(x_{i+2})),\cdots,\nu_{i+1}(p_{i+1j}-1(x_{j-1}))\}\vee\cdots$$

$$\geq\max\{\nu_1(x_1),\nu_2(x_2),\cdots,\nu_{i-1}(x_{i-1})\}\vee$$

$$\max\{\nu_2(x_2),\nu_3(x_3),\cdots,\nu_{i-1}(x_{i-1})\}\vee\nu_{i-1}(x_{i-1})\vee\nu_{i+1}(x_{i+1})\vee\cdots$$

$$=\bigvee_{k=1}^{i-1}\nu_k(x_k)\vee\bigvee_{k=2}^{i-1}\nu_k(x_k)\vee\cdots$$

$$=\bigvee_{n\in N}\nu_n(x_n).$$

Therefore, $\overline{D}:\left(\bigwedge_{n\in N}\varsigma_n,\bigwedge_{n\in N}\varpi_n,\bigvee_{n\in N}\nu_n\right)_{\prod_{n\in N}F_n}\to\left(\bigwedge_{n\in N}\varsigma_n,\bigwedge_{n\in N}\varpi_n,\bigvee_{n\in N}\nu_n\right)_{\prod_{n\in N}F_n}$ is a homomorphism of neutrosophic modules. By some calculation, One can show that $\overline{D}$ is a neutrosophic chain homotopy between $\check{h_0}\circ\overline{g_0}$ and $\check{h_1}\circ\overline{g_1}$ homomorphisms. Then, the following cohomology modules of neutrosophic cochain complexes are quasi isomorphic (see [1]).

$$\overline{0}\to\left(\bigwedge_{n\in N}\varsigma_n,\bigwedge_{n\in N}\varpi_n,\bigvee_{n\in N}\nu_n\right)_{\prod_{n\in N}F_n}\overset{\overline{d}}{\to}\left(\bigwedge_{n\in N}\varsigma_n,\bigwedge_{n\in N}\varpi_n,\bigvee_{n\in N}\nu_n\right)_{\prod_{n\in N}F_n}\to\overline{0}$$

$$\overline{0}\to\left(\bigwedge_{s\in S}\varsigma_s,\bigwedge_{s\in S}\varpi_s,\bigvee_{s\in S}\nu_s\right)_{\prod_{s\in S}F_s}\overset{\overline{d}}{\to}\left(\bigwedge_{s\in S}\varsigma_s,\bigwedge_{s\in S}\varpi_s,\bigvee_{s\in S}\nu_s\right)_{\prod_{s\in S}F_s}\to\overline{0}$$

By the fact that $\varprojlim^{(1)}$ is the first cohomology module, we are done. □

Remark 3.18. Since $\varprojlim(\varsigma_n,\varpi_n,\nu_n)_{F_n}=\operatorname{Ker}\overline{d}$ and $p_{n+1n}(x_{n+1})=x_n$ are satisfied for each $\{x_n\}\in\varprojlim F_n$, one obtains

$$\varsigma_n(x_n)=\varsigma_n(p_{n+1n}(x_{n+1}))\geq\varsigma_{n+1}(x_{n+1}),$$
$$\varpi_n(x_n)=\varpi_n(p_{n+1n}(x_{n+1}))\geq\varpi_{n+1}(x_{n+1}),$$
$$\nu_n(x_n)=\nu_n(p_{n+1n}(x_{n+1}))\leq\nu_{n+1}(x_{n+1}),$$

that is, for each $\{x_n\}\in\operatorname{Ker}\overline{d}$, $\{\varsigma_n(x_n)\}$ is a decreasing sequence, $\{\varpi_n(x_n)\}$ is a decreasing sequence and $\{\nu_n(x_n)\}$ is an increasing sequence.

Theorem 3.19. *For every* $\{x_n''\} \in \operatorname{Ker}\overline{d}$, *if* $\lim\limits_{n\to\infty} \varsigma_n''(x_n'') = 0$ *or* $\lim\limits_{n\to\infty} \varpi_n''(x_n'') = 0$ *or* $\lim\limits_{n\to\infty} \nu_n''(x_n'') = 1$ *and the following diagram*

$$\begin{array}{ccccccccc} & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & (\varsigma_2', \varpi_2', \nu_2')_{F_2'} & \to & (\varsigma_2, \varpi_2, \nu_2)_{F_2} & \to & (\varsigma_2'', \varpi_2'', \nu_2'')_{F_2''} & \to & 0 \\ 0 & \to & (\varsigma_1', \varpi_1', \nu_1')_{F_1'} & \to & (\varsigma_1, \varpi_1, \nu_1)_{M_1} & \to & (\varsigma_1'', \varpi_1'', \nu_1'')_{F_1''} & \to & 0 \end{array}$$

is a short exact sequence of inverse system of neutrosophic modules, then the sequence

$$\begin{aligned} \overline{0} \to \varprojlim(\varsigma_n', \varpi_n', \nu_n')_{F_n'} \to \varprojlim(\varsigma_n, \varpi_n, \nu_n)_{F_n} \to \varprojlim(\varsigma_n'', \varpi_n'', \nu_n'')_{F_n''} \to \varprojlim(\varsigma_n', \varpi_n', \nu_n')_{M_n} \\ \to \varprojlim(\varsigma_n, \varpi_n, \nu_n)_{F_n} \to \varprojlim(\varsigma_n'', \varpi_n'', \nu_n'')_{F_n''} \to \overline{0}, \end{aligned}$$

is exact.

Proof. For an inverse system of neutrosophic modules $\{(\varsigma_n, \varpi_n, \nu_n)_{F_n}\}_{n\in N}$ one has that

$$C' = \overline{0} \xrightarrow{\overline{0}} (\varsigma_A', \varpi_A', \nu_A')_{\prod\limits_{n\in N} F_n} \xrightarrow{\overline{d}} (\varsigma_A', \varpi_A', \nu_A')_{\prod\limits_{n\in N} F_n} \xrightarrow{\overline{0}} \overline{0} \xrightarrow{\overline{0}} \cdots$$

is a cochain complex of neutrosophic modules.

$$\begin{aligned} H^0(C) &= \varprojlim((\varsigma_n, \varpi_n, \nu_n)_{F_n})_{F_n}, \\ H^1(C) &= \varprojlim{}^{(1)}((\varsigma_n, \varpi_n, \nu_n)_{F_n})_{F_n}, \\ H^k(C) &= 0, k \geq 2 \end{aligned} \tag{5}$$

are neutrosophic cohomology modules of this complex. analogously, for the inverse system of modules $\{(\varsigma_n', \varpi_n', \nu_n')_{M_n'})\}$ and $\{(\varsigma_n'', \varpi_n'', \nu_n'')_{F_n''})\}$, we can establish the following neutrosophic cochain complexes

$$C' = \overline{0} \xrightarrow{\overline{0}} (\varsigma_A', \varpi_A', \nu_A')_{\prod\limits_{n\in N} F_n'} \xrightarrow{\overline{d}} (\varsigma_A', \varpi_A', \nu_A')_{\prod\limits_{n\in N} F_n'} \xrightarrow{\overline{0}} \overline{0} \xrightarrow{\overline{0}} \cdots$$

$$C'' = \overline{0} \xrightarrow{\overline{0}} (\varsigma_A'', \varpi_A'', \nu_A'')_{\prod\limits_{n\in N} F_n''} \xrightarrow{\overline{d}} (\varsigma_A'', \varpi_A'', \nu_A'')_{\prod\limits_{n\in N} F_n''} \xrightarrow{\overline{0}} \overline{0} \xrightarrow{\overline{0}} \cdots$$

Observe that, cohomology modules of this complexes have the form of (5). By the hypothesis of this theorem, the following sequence

$$\overline{0} \to C' \to C \to C'' \to \overline{0}$$

is a short exact sequence of cochain complexes of neutrosophic modules. Mind that in general, the following sequence of cohomology modules of this sequence

$$0 \to H^0(C') \to H^0(C) \to H^0(C'') \xrightarrow{\overline{\partial}} H^1(C) \to H^1(C'') \to H^2(C'') \to \cdots$$

is not exact. This is so, since $\overline{\partial}$ is often not a homomorphism of neutrosophic modules. By the fact that $H^0(C'') = \operatorname{Ker} d''$ and $\lim\limits_{n\to\infty} \varsigma_n''(x_n'') = 0$ ($\lim\limits_{n\to\infty} \varpi_n''(x_n'') = 0$, $\lim\limits_{n\to\infty} \nu_n''(x_n'') = 1$), the grade function $\varsigma''(\varpi'', \nu'')$ of a neutrosophic module $(\varsigma'', \varpi'', \nu'')_{H^0(c'')}$ is equal to the grade function

$\varsigma''(\varpi'', \nu'')$ is equal to the grade function indicated in (2). Thus, $\overline{\partial}$ is a homomorphism of neutrosophic modules. This means that the sequence of neutrosophic homology modules

$$0 \to H^0(C') \to H^0(C) \to H^0(C'') \xrightarrow{\overline{\partial}} H^1(C) \to H^1(C'') \to H^2(C') \to \cdots$$

is exact. By (5), we obtain the following exact sequence of neutrosophic modules $\overline{0} \to \varprojlim(\varsigma'_n, \varpi'_n, \nu'_n)_{F'_n} \to \varprojlim(\varsigma_n, \varpi_n, \nu_n)_{F_n} \to \varprojlim(\varsigma''_n, \varpi''_n, \nu''_n)_{F''_n} \to \varprojlim(\varsigma'_n, \varpi'_n, \nu'_n)_{F_n} \to \varprojlim(\varsigma_n, \varpi_n, \nu_n)_{F_n} \to \varprojlim(\varsigma''_n, \varpi''_n, \nu''_n)_{F''_n} \to \overline{0}$. □

Let us now investigate the necessary conditions for which the derivative functor $\varprojlim^{(1)}$ is equal to zero.

Theorem 3.20. *Let the following be an inverse systems of neutrosophic modules*

$$(\varsigma_1, \varpi_1, \nu_1)_{F_1} \xleftarrow{\overline{\varphi_1}} (\varsigma_2, \varpi_2, \nu_2)_{F_2} \xleftarrow{\overline{\varphi_2}} \cdots . \tag{6}$$

If every homomorphisms $\overline{\varphi_n}$ is a neutrosophic epimorphism, then $\varprojlim^{(1)}(\varsigma_n, \varpi_n, \nu_n)_{F_n} = 0$.

Proof. The proof follows from the fact that

$$\overline{d} : \prod_{n=1}^{\infty}(\varsigma_n, \varpi_n, \nu_n)_{F_n} \to \prod_{n=1}^{\infty}(\varsigma_n, \varpi_n, \nu_n)_{F_n}$$

is a neutrosophic epimorphism. □

Definition 3.21. Consider the inverse system of neutrosophic modules (6). If there exists $m \geq n$, for every integer n, such that for all $i \geq m$

$$\mathrm{Im}((\varsigma_i, \varpi_i, \nu_i)_{M_i} \to (\varsigma_n, \varpi_n, \nu_n)_{F_n}) = \mathrm{Im}((\varsigma_m, \varpi_m, \nu_m)_{F_m} \to (\varsigma_n, \varpi_n, \nu_n)_{F_n})$$

then it is said that the inverse system (6) satisfies the *Mittag-Leffler condition.*

Theorem 3.22. *If the inverse system in* (6) *satisfies the Mittag-Leffler condition, then $\varprojlim^{(1)}(\varsigma_n, \varpi_n, \nu_n)_{F_n} = 0$.*

Proof. Let us denote $F'_n = \mathrm{Im}\,\varphi_n^i$, for large i. By the hypothesis of the theorem, the homomorphism $\varphi_{n|F'_{n+1}}$ carries the module F'_{n+1} to F'_n. Then, $\varphi_{n|F'_{n+1}}$ is an epimorphism. Thus, for large i, the homomorphisms

$$\overline{\varphi_n} : (\varsigma_n|_{F'_{n+1}}, \varpi_n|_{F'_{n+1}}, \nu_n|_{F'_{n+1}})_{F'_{n+1}} \to (\varsigma_n|_{F'_n}, \varpi_n|_{F'_n}, \nu_n|_{F'_n})_{F'_n}$$

are epimorphisms. Therefore, by Theorem 3.20, we have $\varprojlim^{(1)}(\varsigma'_n, \varpi'_n, \nu'_n)_{F'_n} = 0$. Now, $\varsigma'_n = \varsigma_n|_{F'_n}, \varpi'_n = \varpi_n|_{F'_n}, \nu'_n = \nu_n|_{F'_n}$. Observe the following sequence of the inverse system of neutrosophic quotient modules

$$(\widetilde{\varsigma_1}, \widetilde{\varpi_1}, \widetilde{\nu_1})_{F_1/F'_1} \leftarrow (\widetilde{\varsigma_2}, \widetilde{\varpi_2}, \widetilde{\nu_2})_{F_2/F'_2} \leftarrow \cdots . \tag{7}$$

For every n, there exists $m > n$ such that the homomorphism $F_m/F'_m \to F_n/F'_n$ is a zero homomorphism. Then, $\varprojlim^{(1)}(\varsigma_n, \varpi_n, \nu_n)_{F_n/F'_n} = 0$. This implies that the limit of inverse system in (7) is equal to 0. Hence, $\varprojlim^{(1)}(\varsigma_n, \varpi_n, \nu_n)_{F_n/F'_n} = 0$. Now, let us see on the following short exact sequence of inverse systems in the category of N-Mod

$$\overline{0} \to \{(\varsigma'_n, \varpi'_n, \nu'_n)_{F'_n}\} \to \{(\varsigma_n, \varpi_n, \nu_n)_{F_n}\} \to \{(\varsigma_n, \varpi_n, \nu_n)_{F_n/F'_n}\} \to \overline{0}. \tag{8}$$

Granting that $(F_n/F'_n) = 0$. By utilizing Theorem 3.20 to the sequence (8), we obtain the following exact sequence

$$\begin{aligned} \overline{0} \to \varprojlim{}^{(1)}(\varsigma'_n, \varpi'_n, \nu'_n)_{F'_n} \to \varprojlim{}^{(1)}(\varsigma_n, \varpi_n, \nu_n)_{F_n} \to \varprojlim{}^{(1)}(\varsigma_n, \varpi_n, \nu_n)_{F_n/F'_n} \to \\ \varprojlim{}^{(1)}(\varsigma'_n, \varpi'_n, \nu'_n)_{F'_n} \to \varprojlim{}^{(1)}(\varsigma_n, \varpi_n, \nu_n)_{F_n} \to \varprojlim{}^{(1)}(\varsigma_n, \varpi_n, \nu_n)_{F_n/F'_n} \to \overline{0}. \end{aligned} \tag{9}$$

Since $\varprojlim^{(1)}(\varsigma'_n, \varpi'_n, \nu'_n)_{F'_n} = \overline{0}$, $\varprojlim^{(1)}(\varsigma_n, \varpi_n, \nu_n)_{F_n/F'_n} = \overline{0}$ and $\varprojlim^{(1)}(\varsigma_n, \varpi_n, \nu_n)_{F_n/F'_n} = \overline{0}$, respectively. Sequence (9) becomes

$$\overline{0} \to \varprojlim{}^{(1)}(\varsigma'_n, \varpi'_n, \nu'_n)_{F'_n} \to \varprojlim{}^{(1)}(\varsigma_n, \varpi_n, \nu_n)_{F_n} \to \overline{0} \to \overline{0} \to \varprojlim{}^{(1)}(\varsigma_n, \varpi_n, \nu_n)_{F_n} \to \overline{0} \to \overline{0}.$$

This proves that $\varprojlim^{(1)}(\varsigma_n, \varpi_n, \nu_n)_{F_n} = \overline{0}$. □

4. **Direct system of neutrosophic modules**

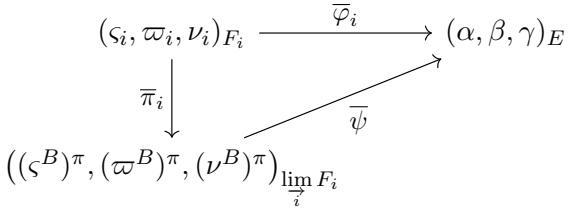
In this section we present some basic properties of direct systems of neutrosophic modules. Let

$$(\overline{\varsigma}, \overline{\varpi}, \overline{\nu})_{\overline{F}} = \left\{(\varsigma_i, \varpi_i, \nu_i)_{F_i}, \overline{p}^{i'i}\right\}_{i \in \Delta} \tag{10}$$

be a direct system of neutrosophic modules, where $(\varsigma^B, \varpi^B, \nu^B)_{\bigoplus\limits_i F_i}$ is a neutrosophic module and $\pi : \bigoplus\limits_i F_i \to \varinjlim\limits_i F_i$ a canonical epimorphism. Moreover, one can consider the neutrosophic module $\left((\varsigma^B)^\pi, (\varpi^B)^\pi, (\nu^B)^\pi\right)_{\varinjlim\limits_i F_i}$.

Theorem 4.1. *Every direct system in the representation* (10) *has a limit in the category of* N-Mod *which is equal to the neutrosophic module* $\left((\varsigma^B)^\pi, (\varpi^B)^\pi, (\nu^B)^\pi\right)_{\varinjlim\limits_i F_i}$.

Proof. It suffices to demonstrate that, there exists a unique homomorphism of neutrosophic modules $\overline{\psi} : \left((\varsigma^B)^\pi, (\varpi^B)^\pi, (\nu^B)^\pi\right)_{\varinjlim\limits_i F_i} \to (\alpha, \beta, \gamma)_N$, making the diagram



commutative, where $\overline{\varphi} = \{\overline{\varphi}_i : (\varsigma_i, \varpi_i, \nu_i)_{F_i} \to (\alpha, \beta, \gamma)_E\}_{i\in\Delta}$ is a family of homomorphisms of neutrosophic modules providing the commutativity of the diagram

$$\begin{array}{ccc} (\varsigma_i, \varpi_i, \nu_i)_{F_i} & \xrightarrow{\overline{\varphi}_i} & (\alpha, \beta, \gamma)_E \\ \Big\downarrow{\scriptstyle \overline{p}^{i,i'}} & \nearrow & \\ (\varsigma_{i'}, \varpi_{i'}, \nu_{i'})_{F_{i'}} & & \end{array}$$

$\overline{l_i} : (\varsigma_i, \varpi_i, \nu_i)_{F_i} \to (\varsigma^B, \varpi^B, \nu^B)_{\underset{i}{\oplus} F_i}$ are usual injections and $\pi_i = \pi \circ l_i$. For every $x \in \varinjlim_i F_i$, there exists $x_i \in F_i$ such that $\pi_i(x_i) = x$. If $\pi_{i'}(x_{i'}) = x$ for each $x_{i'} \in F_{i'}$, then $\varphi_{i'}(x_{i'})$ is equal to $\varphi_i(x_i)$. We define the homomorphism $\psi : \varinjlim_i F_i \to E$ by $\psi(x) = \varphi_i(x_i)$. We want to show that $\overline{\psi}$ is a homomorphism of neutrosophic modules. For each $x \in \varinjlim_i F_i$, let $\pi \circ l_i(x_i) = x$. Here,

$$\begin{aligned} (\varsigma^B)^\pi(x) &= \sup(\underset{i}{\vee} \varsigma_i)(x) = \sup\{\underset{i}{\vee} \varsigma_i(x) : \pi_i(x_i) = x\}, \\ (\varpi^B)^\pi(x) &= \sup(\underset{i}{\vee} \varpi_i)(x) = \sup\{\underset{i}{\vee} \varpi_i(x) : \pi_i(x_i) = x\}, \\ (\nu^B)^\pi(x) &= \inf(\underset{\alpha}{\wedge} \nu_i)(x) = \inf\{\underset{i}{\wedge} \nu_i(x) : \pi_i(x_i) = x\}. \end{aligned}$$

Therefore,

$$\alpha(\psi(x)) = i(\varphi_i(x_i)) \geq \varsigma_i(x_i), \quad \beta(\psi(x)) = \beta(\varphi_i(x_i)) \geq \varpi_i(x_i), \quad \gamma(\psi(x)) = \alpha(\varphi_i(x_i)) \leq \nu_i(x_i).$$

Since this inequality is satisfied for each x_i such that $\pi_i(x_i) = x$, we have $\alpha(\psi(x)) \geq (\varsigma^B)^\pi(x)$, $\beta(\psi(x)) \geq (\varpi^B)^\pi(x)$, $\gamma(\psi(x)) \geq (\nu^B)^\pi(x)$. From the definition of $\overline{\psi}$, it is obvious that the above diagram is commutative. We can easily show that $\varinjlim$ is a functor from the category of direct systems of neutrosophic modules to the category of neutrosophic modules. □

We can now focus on the problem of the exact direct limit of exact sequences of direct systems of neutrosophic modules. Let

$$\begin{aligned} \overline{F} &= \left\{(\varsigma_i, \varpi_i, \nu_i)_{F_i}, \overline{p}^{i'i}\right\}_{i\in\Delta}, \\ \overline{F}' &= \left\{(\varsigma'_i, \varpi'_i, \nu'_i)_{F'_i}, \overline{p}^{i'i}\right\}_{i\in\Delta}, \\ \overline{F}'' &= \left\{(\varsigma''_i, \varpi''_i, \nu''_i)_{F''_i}, \overline{p}^{i'i}\right\}_{i\in\Delta}, \end{aligned}$$

be direct systems of neutrosophic modules, and let

$$\overline{F}' \xrightarrow{\check{h}} \overline{F} \xrightarrow{\overline{g}} \overline{F}'' \tag{11}$$

be an exact sequence.

The following holds.

Theorem 4.2. *Direct limit of the exact sequence* (11),

$$\varinjlim_i(\varsigma_i', \varpi_i', \nu_i')_{F_i'} \to \varinjlim_i(\varsigma_i, \varpi_i, \nu_i)_{F_i} \to \varinjlim_i(\varsigma_i'', \varpi_i'', \nu_i'')_{F_i''},$$

is exact.

Proof. Since the sequence (11) is exact, then the ordinary sequence of R-modules $F_i' \to F_i \to F_i''$ is exact for every $i \in \Delta$. Hence, the sequence $\{F_i'\}_i \xrightarrow{\{h_i\}} \{F_i\}_i \xrightarrow{\{g_i\}} \{F_i''\}_i$ is an exact sequence of direct system of ordinary modules. Then, taking the limit one obtains the sequence

$$\varinjlim_i F_\alpha' \xrightarrow{\varinjlim_i h_i} \varinjlim_\alpha F_i \xrightarrow{\varinjlim_\alpha g_i} \varinjlim_i F_i'', \tag{12}$$

which is also exact. Therefore, the following sequence of neutrosophic modules

$$\left((\varsigma'^B)^\pi, (\varpi'^B)^\pi, (\nu'^B)^\pi\right)_{\varinjlim_i F_i'} \xrightarrow{\overline{\varinjlim_i h_i}} \left((\varsigma^B)^\pi, (\varpi^B)^\pi, (\nu^B)^\pi\right)_{\varinjlim_i F_i} \xrightarrow{\overline{\varinjlim_i g_i}} \left((\varsigma''^B)^\pi, (\varpi''^B)^\pi, (\nu''^B)^\pi\right)_{\varinjlim_i F_i''}$$

fulfills the relations

$$(\varsigma^B)^\pi|_{\operatorname{Im}\varinjlim_i h_i} = (\varsigma^B)^\pi|_{\operatorname{Ker}\varinjlim_i g_i},$$
$$(\varpi^B)^\pi|_{\operatorname{Im}\varinjlim_i h_i} = (\varpi^B)^\pi|_{\operatorname{Ker}\varinjlim_i g_i},$$
$$(\nu^B)^\pi|_{\operatorname{Im}\varinjlim_i h_i} = (\nu^B)^\pi|_{\operatorname{Ker}\varinjlim_i g_i},$$

which are true, since sequence (12) is exact. □

Corollary 4.3. *The direct limit functor preserves monomorphism and epimorphism in the category of neutrosophic modules.*

Let us see on the direct system of chain complexes. Let I be a directed set, for every $i \in I$ suppose

$$C(i) = \left\{(\varsigma_n^{(i)}, \varpi_n^{(i)}, \nu_n^{(i)})_{F_n^{(i)}}, \overline{\partial}_n : (\varsigma_n(i), \varpi_n(i), \nu_n(i))_{F_n(i)} \to (\varsigma_{n-1}(i), \varpi_{n-1}(i), \nu_{n-1}(i))_{F_{n-1}(i)}\right\}_n$$

is a chain complex of neutrosophic modules and for every $i < j$, let $\check{h}_{ij} : C(i) \to C(j)$ be a morphism of chain complexes and $\{C(i), \check{h}_{ij}\}$ be a direct system of these chain complexes.

Theorem 4.4. *The limit of homology modules of direct system of chain complexes of neutrosophic modules is quasi isomorphic to the homology modules of the limit of this direct system, i.e.,* $H_n\left(\varinjlim_i C(i)\right) \simeq \varinjlim_i H_n(C(i))$.

Proof. It is proved by using Corollary 4.3. Therefore, we have

$$\begin{aligned}\varinjlim H_n(C(i)) &= \varinjlim \left(\widetilde{\varsigma_n}(i), \widetilde{\varpi_n}(i), \widetilde{\nu_n}(i)\right)_{\operatorname{Ker}\overline{\partial}_n(i)|_{\operatorname{Im}\overline{\partial}_{n+1}(i)}} \\ &\approx \varinjlim \left(\varsigma_n|_{\operatorname{Ker}\overline{\partial}_n(i)}, \varpi_n|_{\operatorname{Ker}\overline{\partial}_n(i)}, \nu_n|_{\operatorname{Ker}\overline{\partial}_n(i)}\right)_{\operatorname{Ker}\overline{\partial}_n(i)}\Big|_{\varinjlim\left(\varsigma_n|_{\operatorname{Ker}\overline{\partial}_n(i)}, \varpi_n|_{\operatorname{Ker}\overline{\partial}_n(i)}, \nu_n|_{\operatorname{Ker}\overline{\partial}_n(i)}\right)_{\operatorname{Ker}\overline{\partial}_n(i)}} \\ &\approx \operatorname{Ker}\varinjlim \overline{\partial}_n(i)|\varinjlim \overline{\partial}_n(i) \\ &= H_n\left(\varinjlim C(i)\right).\end{aligned}$$

□

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The integral of 2- refined rational neutrosophic functions

Iqbal Ahmed Musa[1] and Yaser Ahmad Alhasan[2,*]

[1]Deanship the Preparatory Year, Prince Sattam bin Abdulaziz University, Alkharj, Saudi Arabia.; i.abdulah@psau.edu.sa
[2]Deanship of the Preparatory Year, Prince Sattam bin Abdulaziz University, Alkharj, Saudi Arabia.; y.alhasan@psau.edu.sa

[*]Corresponding author: y.alhasan@psau.edu.sa

Abstract: The main objective of this paper is to study the integral of 2- refined rational neutrosophic functions, by providing direct rules that make it easy for us to find the integral of 2- refined rational neutrosophic functions, where the division of refined neutrosophic number was used to find the integral of 2- refined rational neutrosophic functions. In addition to proving these rules and providing a set of examples that illustrate these ideas.

Keywords: refined rational neutrosophic functions; 2- refined rational; integrals.

1. Introduction and Preliminaries

As an alternative to the existing logics, Smarandache proposed the Neutrosophic Logic to represent a mathematical model of uncertainty, vagueness, ambiguity, imprecision, undefined, unknown, incompleteness, inconsistency, redundancy, contradiction, where Smarandache made refined neutrosophic numbers available in the following form: $(a, b_1I_1, b_2I_2, \dots, b_nI_n)$ where $a, b_1, b_2, \dots, b_n \in R \text{ or } C$ [1]. Agboola introduced the concept of refined neutrosophic algebraic structures [2]. In addition, the refined neutrosophic rings I was studied in paper [3], where it assumed that I splits into two indeterminacies I_1 [contradiction (true (T) and false (F))] and I_2 [ignorance (true (T) or false (F))]. Abobala presented the papers on some special substructures of refined neutrosophic rings and a study of ah-substructures in n-refined neutrosophic vector spaces [6-7]. Alhasan.Y and Abdulfatah. R also presented a study on the division of refined neutrosophic number [8]:

$$\frac{\dot{a_1} + \dot{b_1}I_1 + \dot{c_1}I_2}{\dot{a_2} + \dot{b_2}I_1 + \dot{c_2}I_2} \equiv \frac{\dot{a_1}}{\dot{a_2}} + \left[\frac{\dot{a_2}\dot{b_1} + \dot{b_1}\dot{c_2} - \dot{a_1}\dot{b_2} - \dot{b_2}\dot{c_1}}{(\dot{a_2} + \dot{c_2})(\dot{a_2} + \dot{b_2} + \dot{c_2})}\right] I_1 + \left[\frac{\dot{a_2}\dot{c_1} - \dot{a_1}\dot{c_2}}{\dot{a_2}(\dot{a_2} + \dot{c_2})}\right] I_2$$

Where: $\dot{a_2} \neq 0$, $\dot{a_2} \neq -\dot{c_2}$ and $\dot{a_2} \neq -\dot{b_2} - \dot{c_2}$

In addition to studying the integral calculus according to the logic of neutrosophic by presenting a set of papers on that [9-10]. The AH-Isometry was extended to n-Refined AH-Isometry by Smarandache & Abobala in 2024 [11]

Finally, there are papers presented in n-Valued Refined Neutrosophic Logic and Its Applications in Physics, Neutrosophic Rings I [4-5].

2. The integral of 2- refined rational neutrosophic functions

2.1 integral of standard 2- refined rational neutrosophic functions

- **Standard 2- refined rational neutrosophic integral I:**

$$\int \frac{a_1 + b_1 I_1 + c_1 I_2}{x^2 - (a_2 + b_2 I_1 + c_2 I_2)^2} dx$$

$$= \frac{1}{2}\left(\frac{a_1}{a_2} + \left[\frac{a_2 b_1 + b_1 c_2 - a_1 b_2 - b_2 c_1}{(a_2 + c_2)(a_2 + b_2 + c_2)}\right] I_1 + \left[\frac{a_2 c_1 - a_1 c_2}{a_2(a_2 + c_2)}\right] I_2\right) ln\left|\frac{x - a_2 - b_2 I_1 - c_2 I_2}{x + a_2 + b_2 I_1 + c_2 I_2}\right| + C$$

Whereas $a_2 \neq 0,\ a_2 \neq -c_2,\ a_2 \neq -b_2 - c_2, C = a_0 + b_0 I_1 + c_0 I_2\ and\ a_0,\ b_0,\ c_0$ are real numbers.

Proof:

$$\frac{a_1 + b_1 I_1 + c_1 I_2}{x^2 - (a_2 + b_2 I_1 + c_2 I_2)^2} = \frac{a_1 + b_1 I_1 + c_1 I_2}{(x - a_2 - b_2 I_1 - c_2 I_2)(x + a_2 + b_2 I_1 + c_2 I_2)}$$

$$= \frac{a_1 + b_1 I_1 + c_1 I_2}{2(a_2 + b_2 I_1 + c_2 I_2)}\left[\frac{2(a_2 + b_2 I_1 + c_2 I_2)}{(x - a_2 - b_2 I_1 - c_2 I_2)(x + a_2 + b_2 I_1 + c_2 I_2)}\right]$$

$$= \frac{a_1 + b_1 I_1 + c_1 I_2}{2(a_2 + b_2 I_1 + c_2 I_2)}\left[\frac{x + a_2 + b_2 I_1 + c_2 I_2 - x + a_2 + b_2 I_1 + c_2 I_2}{(x - a_2 - b_2 I_1 - c_2 I_2)(x + a_2 + b_2 I_1 + c_2 I_2)}\right]$$

$$= \frac{a_1 + b_1 I_1 + c_1 I_2}{2(a_2 + b_2 I_1 + c_2 I_2)}\left[\frac{x + a_2 + b_2 I_1 + c_2 I_2 - (x - a_2 - b_2 I_1 - c_2 I_2)}{(x - a_2 - b_2 I_1 - c_2 I_2)(x + a_2 + b_2 I_1 + c_2 I_2)}\right]$$

$$= \frac{a_1 + b_1 I_1 + c_1 I_2}{2(a_2 + b_2 I_1 + c_2 I_2)}\left[\frac{1}{x - a_2 - b_2 I_1 - c_2 I_2} - \frac{1}{x + a_2 + b_2 I_1 + c_2 I_2}\right]$$

$$\Longrightarrow \int \frac{a_1 + b_1 I_1 + c_1 I_2}{x^2 - (a_2 + b_2 I_1 + c_2 I_2)^2} dx$$

$$= \int \frac{a_1 + b_1 I_1 + c_1 I_2}{2(a_2 + b_2 I_1 + c_2 I_2)}\left[\frac{1}{x - a_2 - b_2 I_1 - c_2 I_2} - \frac{1}{x + a_2 + b_2 I_1 + c_2 I_2}\right] dx$$

$$= \frac{a_1 + b_1 I_1 + c_1 I_2}{2(a_2 + b_2 I_1 + c_2 I_2)}\left(\int \frac{1}{x - a_2 - b_2 I_1 - c_2 I_2} dx - \int \frac{1}{x + a_2 + b_2 I_1 + c_2 I_2} dx\right)$$

$$= \frac{a_1 + b_1 I_1 + c_1 I_2}{2(a_2 + b_2 I_1 + c_2 I_2)}(ln|x - a_2 - b_2 I_1 - c_2 I_2| - ln|x + a_2 + b_2 I_1 + c_2 I_2|) + C$$

$$= \frac{1}{2}\left(\frac{a_1}{a_2} + \left[\frac{a_2 b_1 + b_1 c_2 - a_1 b_2 - b_2 c_1}{(a_2 + c_2)(a_2 + b_2 + c_2)}\right] I_1 + \left[\frac{a_2 c_1 - a_1 c_2}{a_2(a_2 + c_2)}\right] I_2\right) ln\left|\frac{x - a_2 - b_2 I_1 - c_2 I_2}{x + a_2 + b_2 I_1 + c_2 I_2}\right| + C$$

Example 1

Evaluate:

$$\int \frac{4 - 2I_1 + I_2}{x^2 - 64 - 19I_1 - 17I_2} dx$$

Solution:

$$x^2 - 64 - 19I_1 - 17I_2 = x^2 - \left(\sqrt{64 + 19I_1 + 17I_2}\right)^2$$

$$= x^2 - (8 + I_1 + I_2)^2$$

Whereas: $\sqrt{64 + 19I_1 + 17I_2} = \sqrt{64} + \left[\sqrt{64 + 19 + 17} - \sqrt{64 + 17}\right]I_1 + \left[\sqrt{64 + 17} - \sqrt{64}\right]I_2$

$\Longrightarrow \quad \sqrt{64 + 19I_1 + 17I_2} = 8 + I_1 + I_2$

$$\int \frac{4 - 2I_1 + I_2}{x^2 - 64 - 19I_1 - 17I_2}\,dx = \frac{1}{2}\left(\frac{4 - 2I_1 + I_2}{8 + I_1 + I_2}\right) ln\left|\frac{x - 8 - I_1 - I_2}{x + 8 + I_1 + I_2}\right| + C$$

$$= \left(\frac{1}{4} - \frac{23}{180}I_1 + \frac{1}{36}I_2\right) ln\left|\frac{x - 8 - I_1 - I_2}{x + 8 + I_1 + I_2}\right| + C$$

Whereas $C = a_0 + b_0I_1 + c_0I_2\ and\ a_0,\ b_0,\ c_0$ are real numbers.

Let's check the answer:

$$\frac{d}{dx}\left[\left(\frac{1}{4} - \frac{23}{180}I_1 + \frac{1}{36}I_2\right) ln\left|\frac{x - 8 - I_1 - I_2}{x + 8 + I_1 + I_2}\right| + C\right]$$

$$= \left(\frac{1}{4} - \frac{23}{180}I_1 + \frac{1}{36}I_2\right)(ln|x - 8 - I_1 - I_2| - ln|x + 8 + I_1 + I_2|)$$

$$= \left(\frac{1}{4} - \frac{23}{180}I_1 + \frac{1}{36}I_2\right)\left(\frac{1}{x - 8 - I_1 - I_2} - \frac{1}{x + 8 + I_1 + I_2}\right)$$

$$= \left(\frac{1}{4} - \frac{23}{180}I_1 + \frac{1}{36}I_2\right)\left(\frac{x + 8 + I_1 + I_2 - (x - 8 - I_1 - I_2)}{(x - 8 - I_1 - I_2)(x + 8 + I_1 + I_2)}\right)$$

$$= \left(\frac{1}{4} - \frac{23}{180}I_1 + \frac{1}{36}I_2\right)\left(\frac{2(8 + I_1 + I_2)}{x - (64 + 19I_1 + 17I_2)}\right)$$

$$= \left(\frac{1}{2} - \frac{23}{90}I_1 + \frac{1}{18}I_2\right)\left(\frac{8 + I_1 + I_2}{x - 64 - 19I_1 - 17I_2}\right)$$

$$= \frac{4 - 2I_1 + I_2}{x - 64 - 19I_1 - 17I_2}\ (The\ same\ integral\ function)$$

- **Standard 2- refined rational neutrosophic integral II:**

$$\int \frac{a_1 + b_1I_1 + c_1I_2}{(a_2 + b_2I_1 + c_2I_2)^2 - x^2}\,dx$$

$$= \frac{1}{2}\left(\frac{a_1}{a_2} + \left[\frac{a_2b_1 + b_1c_2 - a_1b_2 - b_2c_1}{(a_2 + c_2)(a_2 + b_2 + c_2)}\right]I_1 + \left[\frac{a_2c_1 - a_1c_2}{a_2(a_2 + c_2)}\right]I_2\right) ln\left|\frac{a_2 + b_2I_1 + c_2I_2 + x}{a_2 + b_2I_1 + c_2I_2 - x}\right| + C$$

Whereas $a_2 \neq 0,\ a_2 \neq -c_2,\ a_2 \neq -b_2 - c_2, C = a_0 + b_0I_1 + c_0I_2\ and\ a_0,\ b_0,\ c_0$ are real numbers.

Proof:

$$\frac{a_1 + b_1I_1 + c_1I_2}{(a_2 + b_2I_1 + c_2I_2)^2 - x^2} = \frac{a_1 + b_1I_1 + c_1I_2}{(a_2 + b_2I_1 + c_2I_2 - x)(a_2 + b_2I_1 + c_2I_2 + x)}$$

$$= \frac{a_1 + b_1I_1 + c_1I_2}{2(a_2 + b_2I_1 + c_2I_2)}\left[\frac{2(a_2 + b_2I_1 + c_2I_2)}{(a_2 + b_2I_1 + c_2I_2 + x)(a_2 + b_2I_1 + c_2I_2 - x)}\right]$$

$$= \frac{a_1 + b_1I_1 + c_1I_2}{2(a_2 + b_2I_1 + c_2I_2)}\left[\frac{a_2 + b_2I_1 + c_2I_2 + x + a_2 + b_2I_1 + c_2I_2 - x}{(a_2 + b_2I_1 + c_2I_2 + x)(a_2 + b_2I_1 + c_2I_2 - x)}\right]$$

$$= \frac{a_1 + b_1I_1 + c_1I_2}{2(a_2 + b_2I_1 + c_2I_2)}\left[\frac{x + a_2 + b_2I_1 + c_2I_2 + (a_2 + b_2I_1 + c_2I_2 - x)}{(a_2 + b_2I_1 + c_2I_2 + x)(a_2 + b_2I_1 + c_2I_2 - x)}\right]$$

$$= \frac{a_1 + b_1I_1 + c_1I_2}{2(a_2 + b_2I_1 + c_2I_2)}\left[\frac{1}{a_1 + b_1I_1 + c_1I_2 + x} + \frac{1}{a_2 + b_2I_1 + c_2I_2 - x}\right]$$

$$\Rightarrow \int \frac{a_1 + b_1I_1 + c_1I_2}{x^2 - (a_2 + b_2I_1 + c_2I_2)^2}dx$$

$$= \int \frac{a_1 + b_1I_1 + c_1I_2}{2(a_2 + b_2I_1 + c_2I_2)}\left[\frac{1}{a_1 + b_1I_1 + c_1I_2 + x} + \frac{1}{a_2 + b_2I_1 + c_2I_2 - x}\right]dx$$

$$= \frac{a_1 + b_1I_1 + c_1I_2}{2(a_2 + b_2I_1 + c_2I_2)}\left(\int \frac{1}{a_1 + b_1I_1 + c_1I_2 + x}dx - \int \frac{-1}{a_2 + b_2I_1 + c_2I_2 - x}dx\right)$$

$$= \frac{a_1 + b_1I_1 + c_1I_2}{2(a_2 + b_2I_1 + c_2I_2)}(ln|a_1 + b_1I_1 + c_1I_2 + x| - ln|a_2 + b_2I_1 + c_2I_2 - x|) + C$$

$$= \frac{1}{2}\left(\frac{a_1}{a_2} + \left[\frac{a_2b_1 + b_1c_2 - a_1b_2 - b_2c_1}{(a_2 + c_2)(a_2 + b_2 + c_2)}\right]I_1 + \left[\frac{a_2c_1 - a_1c_2}{a_2(a_2 + c_2)}\right]I_2\right) ln\left|\frac{a_2 + b_2I_1 + c_2I_2 + x}{a_2 + b_2I_1 + c_2I_2 - x}\right| + C$$

Example 2

Evaluate:

$$\int \frac{3I_1 + 5I_2}{1 + 72I_1 + 8I_2 - x^2}dx$$

Solution:

$$1 + 72I_1 + 8I_2 - x^2 = \left(\sqrt{1 + 72I_1 + 8I_2}\right)^2 - x^2$$

$$= \left(\sqrt{1 + 72I_1 + 8I_2}\right)^2 - x^2$$

$$= (1 + 6I_1 + 2I_2)^2 - x^2$$

Whereas: $\sqrt{1 + 72I_1 + 8I_2} = \sqrt{1} + \left[\sqrt{1 + 72 + 8} - \sqrt{1 + 8}\right]I_1 + \left[\sqrt{1 + 8} - \sqrt{1}\right]I_2$

$$\Rightarrow \sqrt{1 + 72I_1 + 8I_2} = 1 + 6I_1 + 2I_2$$

$$\int \frac{3I_1 + 5I_2}{1 + 72I_1 + 8I_2 - x^2}dx = \frac{1}{2}\left(\frac{3I_1 + 5I_2}{1 + 6I_1 + 2I_2}\right) ln\left|\frac{1 + 6I_1 + 2I_2 + x}{1 + 6I_1 + 2I_2 - x}\right| + C$$

$$= \left(-\frac{7}{18}I_1 + \frac{5}{6}I_2\right) ln\left|\frac{1 + 6I_1 + 2I_2 + x}{1 + 6I_1 + 2I_2 - x}\right| + C$$

Whereas $C = a_0 + b_0I_1 + c_0I_2$ *and* $a_0,\ b_0,\ c_0$ are real numbers.

Let's check the answer:

$$\frac{d}{dx}\left[\left(-\frac{7}{18}I_1+\frac{5}{6}I_2\right)ln\left|\frac{1+6I_1+2I_2+x}{1+6I_1+2I_2-x}\right|+C\right]$$

$$=\left(-\frac{7}{18}I_1+\frac{5}{6}I_2\right)(ln|1+6I_1+2I_2+x|-ln|1+6I_1+2I_2-x|)$$

$$=\left(-\frac{7}{18}I_1+\frac{5}{6}I_2\right)\left(\frac{1}{1+6I_1+2I_2+x}-\frac{-1}{1+6I_1+2I_2-x}\right)$$

$$=\left(-\frac{7}{18}I_1+\frac{5}{6}I_2\right)\left(\frac{1+6I_1+2I_2-x+1+6I_1+2I_2+x}{(1+6I_1+2I_2+x)(1+6I_1+2I_2-x)}\right)$$

$$=\left(-\frac{7}{18}I_1+\frac{5}{6}I_2\right)\left(\frac{2(1+6I_1+2I_2)}{(1+6I_1+2I_2)^2-x^2}\right)$$

$$=\left(-\frac{7}{9}I_1+\frac{5}{3}I_2\right)\left(\frac{1+6I_1+2I_2}{1+72I_1+8I_2-x^2}\right)$$

$$=\frac{3I_1+5I_2}{1+72I_1+8I_2-x^2}\quad (The\ same\ integral\ function)$$

- **Standard 2- refined rational neutrosophic integral III:**

$$\int\frac{a_1+b_1I_1+c_1I_2}{x^2+(a_2+b_2I_1+c_2I_2)^2}dx$$

$$=\frac{1}{2}\Big(\frac{a_1}{a_2}+\left[\frac{a_2b_1+b_1c_2-a_1b_2-b_2c_1}{(a_2+c_2)(a_2+b_2+c_2)}\right]I_1+\left[\frac{a_2c_1-a_1c_2}{a_2(a_2+c_2)}\right]I_2\Big)tan^{-1}\left(\left(\frac{1}{a_2}+\left[\frac{-b_2}{(a_2+c_2)(a_2+b_2+c_2)}\right]I_1-\left[\frac{c_2}{a_2(a_2+c_2)}\right]I_2\right)x\right)+C$$

Whereas $a_2\neq 0,\ a_2\neq -c_2,\ a_2\neq -b_2-c_2, C=a_0+b_0I_1+c_0I_2\ and\ a_0,\ b_0,\ c_0$ are real numbers.

Proof:
Let's put: $x=(a_2+b_2I_1+c_2I_2)\,tan\,\vartheta \implies dx=(a_2+b_2I_1+c_2I_2)sec^2\vartheta\ d\vartheta$

Then:

$$x^2+(a_2+b_2I_1+c_2I_2)^2=(a_2+b_2I_1+c_2I_2)^2tan^2\vartheta+(a_2+b_2I_1+c_2I_2)^2$$

$$=(a_2+b_2I_1+c_2I_2)^2(tan^2\vartheta+1)$$

$$=(a_2+b_2I_1+c_2I_2)^2sec^2\vartheta$$

$$\implies \int\frac{a_1+b_1I_1+c_1I_2}{x^2+(a_2+b_2I_1+c_2I_2)^2}dx=\int\frac{a_1+b_1I_1+c_1I_2}{(a_2+b_2I_1+c_2I_2)^2sec^2\vartheta}(a_2+b_2I_1+c_2I_2)sec^2\vartheta\ d\vartheta$$

$$=\int\frac{a_1+b_1I_1+c_1I_2}{a_2+b_2I_1+c_2I_2}d\vartheta=\frac{a_1+b_1I_1+c_1I_2}{a_2+b_2I_1+c_2I_2}\int d\vartheta$$

$$=\frac{a_1+b_1I_1+c_1I_2}{a_2+b_2I_1+c_2I_2}\vartheta=\frac{a_1+b_1I_1+c_1I_2}{a_2+b_2I_1+c_2I_2}tan^{-1}\left(\frac{x}{a_2+b_2I_1+c_2I_2}\right)+C$$

Where:

$$\vartheta = tan^{-1}\left(\frac{x}{a_2 + b_2 I_1 + c_2 I_2}\right)$$

Hence:

$$\int \frac{a_1 + b_1 I_1 + c_1 I_2}{x^2 + (a_2 + b_2 I_1 + c_2 I_2)^2} dx$$

$$= \left(\frac{a_1}{a_2} + \left[\frac{a_2 b_1 + b_1 c_2 - a_1 b_2 - b_2 c_1}{(a_2 + c_2)(a_2 + b_2 + c_2)}\right] I_1 + \left[\frac{a_2 c_1 - a_1 c_2}{a_2(a_2 + c_2)}\right] I_2\right) tan^{-1}\left(\left(\frac{1}{a_2} + \left[\frac{-b_2}{(a_2 + c_2)(a_2 + b_2 + c_2)}\right] I_1 - \left[\frac{c_2}{a_2(a_2 + c_2)}\right] I_2\right) x\right) + C$$

Whereas $a_2 \neq 0,\ a_2 \neq -c_2,\ a_2 \neq -b_2 - c_2,\ C = a_0 + b_0 I_1 + c_0 I_2\ and\ a_0,\ b_0,\ c_0$ are real numbers.

Example 3

Evaluate:

$$\int \frac{-6 + 7I_1 + 8I_2}{x^2 + 25 + 80I_1 + 39I_2} dx$$

Solution:

$$x^2 + 25 + 80I_1 + 39I_2 = x^2 + \left(\sqrt{25 + 80I_1 + 39I_2}\right)^2$$

$$x^2 + 25 + 80I_1 + 39I_2 = x^2 + (5 + 4I_1 + 3I_2)^2$$

Whereas: $\sqrt{25 + 80I_1 + 39I_2} = \sqrt{25} + \left[\sqrt{25 + 80 + 39} - \sqrt{25 + 39}\right] I_1 + \left[\sqrt{25 + 39} - \sqrt{25}\right] I_2$

$$\Longrightarrow \sqrt{1 + 72I_1 + 8I_2} = 5 + 4I_1 + 3I_2$$

$$\int \frac{-6 + 7I_1 + 8I_2}{x^2 + 25 + 80I_1 + 39I_2} dx = \int \frac{-6 + 7I_1 + 8I_2}{x^2 + (5 + 4I_1 + 3I_2)^2} dx$$

$$= \left(\frac{-6 + 7I_1 + 8I_2}{5 + 4I_1 + 3I_2}\right) tan^{-1}\left(\frac{x}{5 + 4I_1 + 3I_2}\right) + C$$

$$= \left(\frac{-6}{5} + \frac{1}{2} I_1 + \frac{29}{20} I_2\right) tan^{-1}\left(\left(\frac{1}{5} - \frac{1}{24} I_1 - \frac{3}{40} I_2\right) x\right) + C$$

Let's check the answer:

$$\frac{d}{dx}\left[\left(\frac{-6}{5} + \frac{1}{2} I_1 + \frac{29}{20} I_2\right) tan^{-1}\left(\left(\frac{1}{5} - \frac{1}{24} I_1 - \frac{3}{40} I_2\right) x\right) + C\right]$$

$$= \left(\frac{-6}{5} + \frac{1}{2} I_1 + \frac{29}{20} I_2\right) \frac{\frac{1}{5} - \frac{1}{24} I_1 - \frac{3}{40} I_2}{\left(\frac{1}{5} - \frac{1}{24} I_1 - \frac{3}{40} I_2\right)^2 x^2 + 1}$$

$$= \left(\frac{-6}{5} + \frac{1}{2}I_1 + \frac{29}{20}I_2\right) \frac{\frac{1}{5} - \frac{1}{24}I_1 - \frac{3}{40}I_2}{\left(\frac{1}{5} - \frac{1}{24}I_1 - \frac{3}{40}I_2\right)^2 \left(x^2 + \frac{1}{\left(\frac{1}{5} - \frac{1}{24}I_1 - \frac{3}{40}I_2\right)^2}\right)}$$

$$= \left(\frac{-6}{5} + \frac{1}{2}I_1 + \frac{29}{20}I_2\right) \frac{1}{\left(\frac{1}{5} - \frac{1}{24}I_1 - \frac{3}{40}I_2\right)\left(x^2 + \frac{1}{\frac{1}{25} - \frac{5}{576}I_1 - \frac{39}{1600}I_2}\right)}$$

$$= \left(\frac{\frac{-6}{5} + \frac{1}{2}I_1 + \frac{29}{20}I_2}{\frac{1}{5} - \frac{1}{24}I_1 - \frac{3}{40}I_2}\right) \frac{1}{\left(x^2 + \frac{1}{\frac{1}{25} - \frac{5}{576}I_1 - \frac{39}{1600}I_2}\right)}$$

$$= (-6 + 7I_1 + 8I_2) \frac{1}{(x^2 + 25 + 80I_1 + 39I_2)}$$

$$= \frac{-6 + 7I_1 + 8I_2}{x^2 + 25 + 80I_1 + 39I_2} \quad (The\ same\ integral\ function)$$

3. Conclusions

In this paper, we were able to discuss three rules that made it easy for us to solve and find the integral of 2- refined rational neutrosophic functions directly without following long solutions and methods to find these integrals. In addition, to verify the validity of these rules, we provided a proof for each rule. We were also able to obtain results that were verified through a set of examples.

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Stock Price Predictions with LSTM-ARIMA Hybrid Model under Neutrosophic Treesoft sets with MCDM interaction

G. Dhanalakshmi [1] Dr. S. Sandhiya [2*] Florentin Smarandache [3]

[1]Research Scholar, Department of Mathematics,

E-mail: dlklal.govardhan@gmail.com

[2]Assistant Professor, Department of Mathematics,

Vels Institute of Science, Technology and Advanced Studies, Chennai, Tamilnadu.

* Corresponding Author: E-mail: sandhyasundarr@gmail.com

[3]Department of Mathematics, University of New Mexico, Gallup, NM 87301, USA.
E-mail: smarand@unm.edu

Abstract The stock market is regarded as volatile, complex, tumultuous, and dynamic. Forecasting stock performance has proven to be a challenging endeavour due to its increasing need for investment and growth prospects. At the forefront of machine learning, deep learning models facilitate the straightforward and efficient exploration and identification of optimal stocks, the hybrid forecasting models (LSTM and ARIMA) are used in the prediction of stock increase. This paper incorporated the MCDM technique to determine the optimal stocks for investment. The Analytic Hierarchy Process (AHP) is used to assign weights to various financial factors. These weights are then used by the Technique for Order Preference by Similarity to Ideal solution (TOPSIS) technique, which is a component Multi Criteria Decision Making method (MCDM), to compute and rank the optimal stocks for investment. Stock analysis involves considering numerous criteria and sub-criteria, which might lead to an unsuitable answer. To address this uncertainty, we utilize Neutrosophic Treesoft sets, which primarily handle numerous criteria, sub-criteria, and an increased number of sub-sub-criteria. Given a larger number of criteria, we will be capable of providing a precise solution to the problem. Furthermore, the definitions of fuzzy treesoft sets and neutrosophic treesoft sets have been presented for the first time. A plotly graph is generated to compare the real and projected stock prices for all the equities. All these are implemented using the program language python, which seems to be simple and easily understandable when compared to the other programming languages like Julia, MATLAB and so on. This hybrid methodology facilitates the forecast of stock prices, the ranking of stocks based on several financial and non-financial factors using AHP and TOPSIS, and the visualization of the outcomes.

Keywords: MCDM, TOPSIS, AHP, Treesoft sets, deep learning, LSTM, ARIMA, Stocks, equities

Abbreviations:

MCDM- Multi Criteria Decision Making

TOPSIS- Technique for Order Preference by Similarity to Ideal Solution

AHP- Analytic Hierarchy process

LSTM- Long Short Term Memory

ARIMA- Autoregressive Integrated Moving Average

1. Introduction: Ownership in a business is represented by stocks, which are often called shares or equity. Investors get a stake in a company when they buy stock; their ownership stake grows in direct proportion to their shareholdings. To finance operations, growth, and expansion, businesses issue stocks to the public. Investment and economic progress on a global scale are propelled by stocks, which play a pivotal role in the financial markets. An equity exchange is a platform where investors participate in the trading of equities. As an intermediary, it facilitates the transfer of capital from investors to enterprises. Among the most prominent stock exchanges are: New York Stock Exchange (NYSE), NASDAQ, Tokyo Stock Exchange Corporation, London Stock Exchange. The purpose of these exchanges is to facilitate stock trading and guarantee that transactions are carried out in a transparent and regulated setting [13].

For a comprehensive understanding of the stock market, it is crucial to grasp the many elements that influence stock prices and investor choices. The aforementioned include:
Supply and demand, Earnings Reports, Market Sentiment, Interest rates and inflation, Company news and industry trends. Equity instruments are essential in the financial markets, providing investors with chances to amass money and experience capital appreciation. A comprehensive grasp of the intricacies of stock markets, encompassing valuation techniques, market dynamics, and risk elements, is crucial for making sound investing choices [15].

Given the on-going evolution of stock markets due to the incorporation of new technology and worldwide trends, investors need to stay well-informed and flexible in order to successfully navigate this extremely volatile environment. The Python programming language is an accessible instrument for software development that has experienced significant growth in recent years. This is widely used in artificial intelligence, deep learning, neural networks, and machine learning technologies. [21]

Deep learning is a comprehensive designation for machine learning methodologies that incorporate deep neural networks consisting of several layers. Deep learning is one of the new technologies which give the investors a clear idea about investing in the best stocks to avail large profit. It also provides the investors with a precise understanding of how to invest in the most profitable equities for significant financial gains. [23]

Deep learning model particularly Long Short-Term Memory (LSTM) models are components of deep learning architectures that excel in capturing intricate patterns in extensive dataset, often surpassing conventional models in situations when the data is abundant and intricate. It is form of recurrent neural network (RNN) often employed to forecast future values using past data. A linear statistical model known as ARIMA is commonly employed to analyse

and predict stationary time series data. The temporal correlation (linear dependencies) is captured by employing autoregressive (AR), differencing (I), and moving average (MA) components. The ARIMA model is valuable for making short-term forecasts when the data demonstrates consistency or can be rendered stationary by differencing. Frequently, the combination of ARIMA with LSTM results in hybrid models that harness the individual strengths of both approaches. Due to the strong efficacy of ARIMA in capturing linear trends and the competence of LSTMs in detecting non-linear correlations, the combination of these two approaches leads to more precise predictions. Linear trends and non-linear fluctuations are frequently seen features in the prediction of stock market dynamics. The integration of both forms of interdependence in hybrid models has the potential to enhance the precision of stock price prediction [12].

Multi-criteria Decision Making (MCDM) refers to a set of strategies and methodologies used to evaluate, prioritize, and select from different alternatives based on many conflicting criteria. This methodology is widely employed in various fields including business, engineering, environmental management, and finance, where key decision-makers must meticulously evaluate several factors before arriving at a decision [11]. TOPSIS is a sophisticated Multi Criteria Decision Making (MCDM) technique that assists decision-makers in selecting the most optimal solution across several domains. The TOPSIS algorithm arranges choices based on their approximate distance to an optimum solution (the most desirable outcome) and a negative ideal solution (the least desirable outcome). The superiority of an alternative is established by its closeness to the optimal solution and its separation from the negative ideal [8].

To efficiently handle and address uncertainty and imprecision, the mathematical framework known as Treesoft sets is extensively used in the fields of multi-criteria decision making (MCDM) and fuzzy logic. Although traditional crisp sets are characterised by binary membership, where an element is either included in a set or not, treesoft sets provide a method to handle data with partial or uncertain membership. Treesoft sets use a tree-like structure to represent different degrees of membership or preferences, therefore facilitating a more complex assessment. In complicated procedures for decision-making including the evaluation of several factors with varying degrees of importance, this structure is particularly well-suited.

Treesoft sets are a versatile and effective tool for managing uncertainty by expanding the conventional set theory to suit complex decision-making dilemmas. They are especially advantageous when there are contradictory criteria, since they can help in effectively representing both qualitative and quantitative aspects. Here we implement a parallel hybridization strategy, in which the LSTM and ARIMA models independently generate forecasts based on historical stock price data. The final prediction for the stocks is obtained by aggregating the predicted values from both models. LSTM is employed to learn intricate patterns and long-term dependencies, while ARIMA is employed to capture linear trends. This method capitalizes on the strengths of both models. The output acquired from the hybrid LSTM-ARIMA model with MCDM techniques applied to Treesoft sets using TOPSIS

provides a precise answer for the most prominent real-world problem, which is the stock increase prediction, when all of these factors are combined.

Motivation and Novelty: The need for a more thorough and accurate framework for evaluating stocks that incorporates sustainability considerations, market sentiment, and financial measures in order to make well-informed investment decisions is what motivates our effort. The intricacy of real-world uncertainty is frequently overlooked by traditional approaches, where Neutrosophic Treesoft sets can be useful. This method seeks to more accurately model both quantitative and qualitative characteristics by fusing the flexibility of Neutrosophic Treesoft set theory with a variety of financial metrics. The integration of environmental and sentiment data is further motivated by the increased emphasis on sustainable and responsible investing, which gives it relevance to contemporary decision-making scenarios. This work's key innovation is the application of Neutrosophic Treesoft sets in stock appraisal, a currently unexamined area. This method facilitates the sophisticated management of unclear, imprecise, and inconsistent information that conventional models find challenging. The integration of LSTM and ARIMA models for forecasting stock price fluctuations, coupled with an extensive AHP and TOPSIS assessment technique, signifies a notable advancement. This complex model is both theoretically distinctive and practically beneficial due to its improved accuracy and relevance in financial analysis.

Research gap: The inability of current stock evaluation techniques to handle uncertainties in financial and qualitative data, such as sentiment and environmental factors, highlights the need for strategies like Neutrosophic Treesoft sets that efficiently handle ambiguous and inconsistent data. When paired with sophisticated uncertainty models, Multi-criteria Decision-making (MCDM) techniques, such as AHP and TOPSIS, are also underutilized, which restricts their capacity to handle both quantitative and qualitative criteria.

2. Preliminaries:

2.1 Linguistic Variables [1]: If the universe of discourse X is a continuous space (the real line R or its subset), we typically divide it into multiple fuzzy sets whose membership function's cover X in a more or less uniform manner. Linguistic values or linguistic labels are the terms used to describe these imprecise sets, which are typically named after adjectives that are commonly used in our daily language, such as "large," "medium," or "small". Consequently, the linguistic variable is frequently used to refer to the universe of discourse X.

Table 1 Linguistic variable

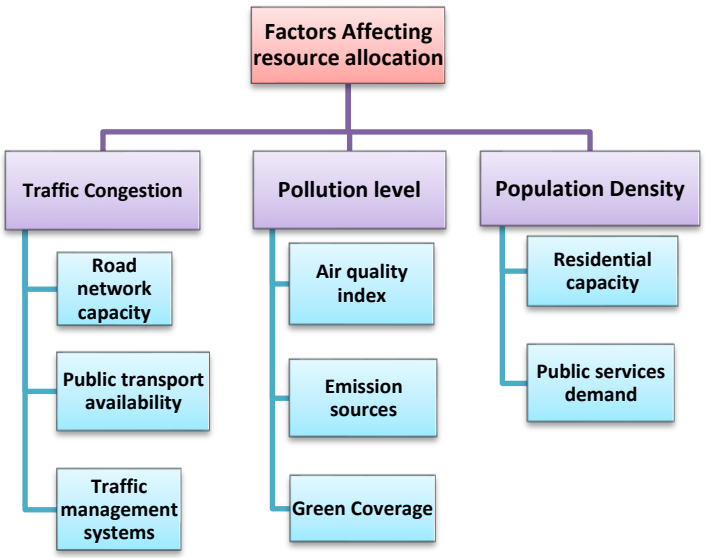
Linguistic Terms	Numerical range
Very Small	[0.0-0.225]
Small	[0.225-0.415]
Medium	[0.415-0.635]
Large	[0.635-0.825]
Very Large	[0.825-1.0]

2.2 Fuzzy set [24]: Denote the generic element of the universe $Q\ as\ q$. A fuzzy set ϑ defined on the universe Q can be characterized as a function $\vartheta: Q \to [0,1]$. The notation μ_{ϑ} is commonly employed to denote the membership function associated with the fuzzy set ϑ. This function maps elements from the universe of discourse Q to the interval $[0,1]$, assigning to each element q in Q a corresponding real number $\mu_{\vartheta(q)}$ that lies within this range.

2.3 Soft sets [2, 14]: Soft sets on Q are characterized by a pair (ω, Y), where Y comprises all the properties that define the members of the universe of discourse, and ω represents a function $\omega: Y \to P(Q)$. According to the findings presented by (Molodstov), a soft set on Q can be characterized as a parameterized collection of subsets of Q, with the set of parameters denoted as Y.

2.4 Hypersoft sets [4, 16, and 19]: The distinct attributes $\mathcal{g}_1, \mathcal{g}_2, \mathcal{g}_3, \dots, \mathcal{g}_n\ for\ n \geq 1\ be\ n$ well defined attributes whose attribute values are contained in the sets $\mathcal{G}_1, \mathcal{G}_2, \mathcal{G}_3, \dots, \mathcal{G}_n$ here $\mathcal{G}_i \cap \mathcal{G}_j = \emptyset$ for $i \neq j$. A hypersoft set over the universal set Q is a pair $(\omega, \mathcal{G}_1 \times \mathcal{G}_2 \times \mathcal{G}_3 \times \dots \times \mathcal{G}_n)$, where ω is the mapping expressed as, $\omega: \mathcal{G}_1 \times \mathcal{G}_2 \times \mathcal{G}_3 \times \dots \times \mathcal{G}_n \to P(Q)$.

Example 1: Following the efficient allocation of resources in a smart city by tackling the obstacles presented by various parameters and sub-parameters across distinct zones where, $\mathcal{Z} = \{z_1, z_2, z_3, z_4, z_5\}$ represents the designated zones for consideration, the optimal zone is identified within this smart city.



Let $\breve{\mathcal{A}} = \{\mathcal{G}_1 \times \mathcal{G}_2 \times \mathcal{G}_3\}$ be a set of parameters

$$\breve{\mathcal{A}} = \{\mathcal{G}_1 \times \mathcal{G}_2 \times \mathcal{G}_3\} = \{a_{11}, a_{12}, a_{13}\} \times \{a_{21}, a_{22}, a_{23}\} \times \{a_{31}, a_{32}\}$$

Where, $\breve{\mathcal{A}} = \{\breve{a_1}, \breve{a_2}, \breve{a_3}, \breve{a_4}, \breve{a_5}, \breve{a_6}, \breve{a_7}, \breve{a_8}, \breve{a_9}, \breve{a_{10}}, \breve{a_{11}}, \breve{a_{12}}, \breve{a_{13}}, \breve{a_{14}}, \breve{a_{15}}, \breve{a_{16}}, \breve{a_{17}}, \breve{a_{18}}\}$ then the Hypersoft sets is given by:

$$(\omega, \breve{\mathcal{A}})$$
$$= \left\{ \begin{array}{c} (\widetilde{a_1}(z_1, z_2)), (\widetilde{a_2}(z_3, z_5)), (\widetilde{a_3}(z_4, z_3)), (\widetilde{a_4}(z_1)), (\widetilde{a_5}(z_2, z_4, z_5)), (\widetilde{a_6}(z_3)), \\ (\widetilde{a_7}(z_1, z_3)), (\widetilde{a_8}(z_1, z_3, z_5)), (\widetilde{a_9}(z_2, z_4)), (\widetilde{a_{10}}(z_1)), (\widetilde{a_{11}}(z_3)), (\widetilde{a_{12}}(z_1, z_2)), \\ (\widetilde{a_{13}}(z_1, z_4, z_5)), (\widetilde{a_{14}}(z_2, z_3, z_4)), \left(\widetilde{a_{16}}(z_1, z_3, z_4, z_5)\right), (\widetilde{a_{17}}(z_1, z_2, z_3, z_4, z_5)), \\ (\widetilde{a_{18}}(z_1, z_2, z_3)) \end{array} \right\}$$

2.5 Treesoft sets [18]: Let Q be the universal set and M a non-empty subset of Q, with $P(M)$ the power set of M. Let $\mathcal{G}$ be the set of attributes $\{\mathcal{G}_1, \mathcal{G}_2, \mathcal{G}_3, \dots, \mathcal{G}_n\}$, for integer $n \geq 1$, where $\mathcal{G}_1, \mathcal{G}_2, \dots, \mathcal{G}_n$ are attributes of first level (since they have one-digit indexes). Each attribute $\mathcal{G}_i$, $1 \leq i \leq n$, is formed by sub-attributes:

$$\mathcal{G}_1 = \{\mathcal{G}_{1,1}, \mathcal{G}_{1,2}, \dots\}$$

$$\mathcal{G}_2 = \{\mathcal{G}_{2,1}, \mathcal{G}_{2,2}, \dots\}$$

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$$\mathcal{G}_n = \{\mathcal{G}_{n,1}, \mathcal{G}_{n,2}, \dots\}$$

where $\mathcal{G}_{i,j}$ are sub-attributes (or attributes of second level) (since they have two-digit indexes). Again, each sub-attribute $\mathcal{G}_{i,j}$ is formed by sub-sub-attributes(or attributes of third level): $\mathcal{G}_{i,j,k}$ and so on, as much refinement as needed into each application, up to sub-sub-…-sub-attributes or attributes of $m - level$ (or having m digits into indexes):$\mathcal{G}_{i_1,i_2,\dots,i_m}$. Therefore, a graph is formed, that we denote as Tree$(\mathcal{G})$, whose root is $\mathcal{G}$ (considered as level zero), then nodes of $level1, level2, \dots, levelm$, we call it as leaves of the graph-tree, all terminal nodes (nodes that have no descendants). Then the treesoft set is:

$$\mathcal{F}: P\big(Tree(\mathcal{G})\big) \to P(M)$$

Tree$(\mathcal{G})$ is the set of all nodes and leaves ($from\ level\ 1\ to\ levelm$) of the graph-tree, and $P\big(Tree(\mathcal{G})\big)$ is the power set of the Tree$(\mathcal{G})$. All node sets of the Treesoft set of $level\ m$ are:

$$Tree(\mathcal{G}) = \{\mathcal{G}_{i_1} | i_1 = 1,2, \dots\} \cup \{\mathcal{G}_{i_1,i_2} | i_1, i_2 = 1,2, \dots\} \cup \dots \cup$$

$\{\mathcal{G}_{i_1,i_2,\dots,i_m} | i_1, i_2, \dots, i_m = 1,2, \dots\}$.The first set is formed by the nodes of $level\ 1$, second set by the nodes of $level\ 2$, third set by the nodes of $level\ 3$, and so on; the last set is formed by the nodes of $level\ m$.

Remark 1: Consider

$$Tree(\mathcal{G}) = \mathcal{D}$$
$$\mathcal{D} = \{\mathcal{G}_{i_1} | i_1 = 1,2, \dots\} \cup \{\mathcal{G}_{i_1,i_2} | i_1, i_2 = 1,2, \dots\} \cup \dots \cup \{\mathcal{G}_{i_1,i_2,\dots,i_m} | i_1, i_2, \dots, i_m = 1,2, \dots\}$$

Where $\mathcal{K}_1 = \{\mathcal{G}_{i_1} | i_1 = 1,2, \dots\}, \mathcal{K}_2 = \{\mathcal{G}_{i_1,i_2} | i_1, i_2 = 1,2, \dots\} \dots \quad \mathcal{K}_m = \{\mathcal{G}_{i_1,i_2,\dots,i_m} | i_1, i_2, \dots, i_m\}$

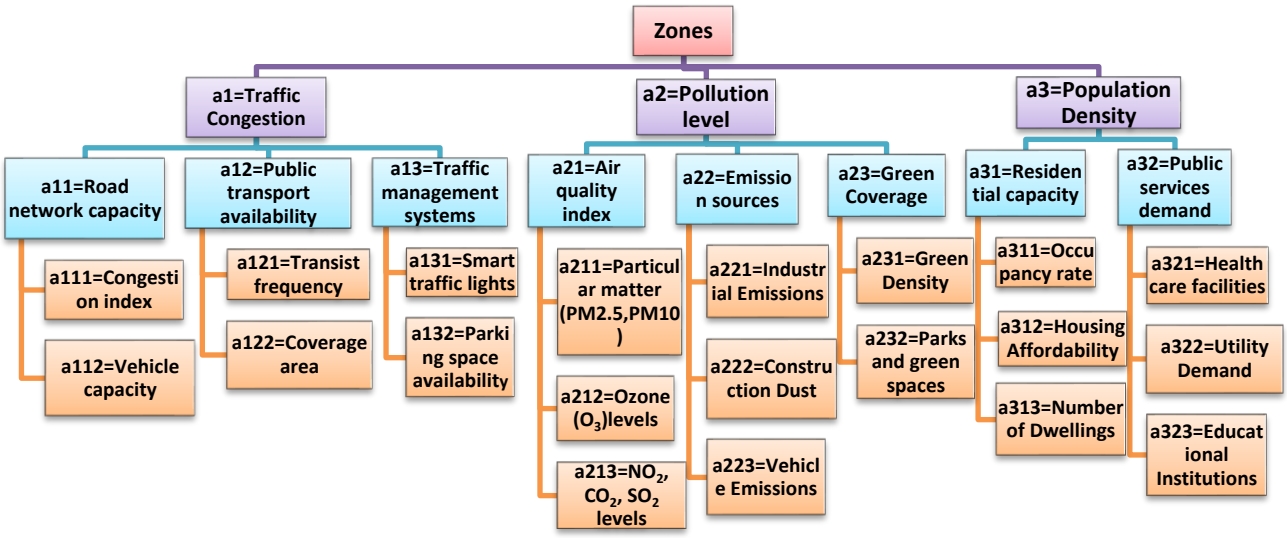
Therefore $Tree(\mathcal{G})$ can also be written as follows:

$$Tree(\mathcal{G}) = \mathcal{K}_1 \cup \mathcal{K}_2 \cup, \dots, \cup \mathcal{K}_m$$

2.6 Fuzzy Treesoft sets: Q the universal set and $FP(Q)$ the be a family of all fuzzy power set Q. $\mathcal{G}$ set of attributes, $\mathcal{G} = \{\mathcal{G}_1, \mathcal{G}_2, \mathcal{G}_3, \dots, \mathcal{G}_n\}$, for integer $n \geq 1$, where $\mathcal{G}_1, \mathcal{G}_2, \dots, \mathcal{G}_n$ are attributes of first level. Each attribute $\mathcal{G}_i, 1 \leq i \leq n$, is formed by the sub-attributes $\mathcal{G}_{i,j}$ attributes of second level. Again, each sub-attribute $\mathcal{G}_{i,j}$ is formed by sub-sub-attributes of third level $\mathcal{G}_{i,j,k}$ and so on, as much refinement needed into each application, upto sub-sub-…-sub-attributes or attributes of m level. A fuzzy treesoft set defined as the pair $\big(\mathcal{F}, P(Tree(\mathcal{G}))\big)$ where $\mathcal{F}: P\big(Tree(\mathcal{G})\big) \to FP(Q)$

Here $Tree(\mathcal{G})$ is the set of all nodes from level 1 *to* level m. $P\big(Tree(\mathcal{G})\big)$ is the powerset of $Tree(\mathcal{G})$, therefore we have $\mathcal{F}: P(\mathcal{K}_1 \cup \mathcal{K}_2 \cup \dots \cup \mathcal{K}_m) \to FP(Q)$ (from the above Remark 1)

Example 2 [6]: Consider the set $\mathcal{Z} = \{z_1, z_2, z_3, z_4, z_5\}$ are the different zones in a smart city, and $\mathcal{P}(\mathcal{Z})$ the power set $\mathcal{Z}$ with the equivalent attributes $\mathcal{T} = \{t_1, t_2, t_3\}$



$$a_1 = \mathcal{Z}_1 = \{a_{11} = Road\ capacity, a_{12} = Public\ transport\ availablity, a_{13} = Traffic\ management\ systems\};$$

$$a_2 = \mathcal{Z}_2 = \{a_{21} = Air\ quality\ index, a_{22} = Emission\ sources, a_{23} = Green\ coverage\};$$

$$a_3 = \mathcal{Z}_3 = \{a_{31} = Residential\ capacity, a_{32} = Public\ services\ demand\}$$

Below is the example of Treesoft sets, the function $\mathcal{F}$ takes the values:

$$\mathcal{F}(Road\ capacity, air\ quality\ index, construction\ dust) = \{z_1, z_2, z_3\}$$

$$\mathcal{F}(Road\ capacity, air\ quality\ index, utility\ demand) = \{z_2, z_4\}$$

$$\mathcal{F}(Road\ capacity, air\ quality\ index, construction\ dust) \cup \mathcal{F}(Traffic\ congestion, air\ quality\ index, utility\ demand) = \{z_1, z_2, z_3, z_4\}$$

Fuzzy treesoft sets is given by:

Consider the function $\mathcal{F}: \mathcal{Z}_1 \cup \mathcal{Z}_2 \cup \mathcal{Z}_3 \to \mathcal{P}(\mathcal{Z})$

Assume the relation

$$\mathcal{F}(\mathcal{Z}_1 \cup \mathcal{Z}_2 \cup \mathcal{Z}_3)$$

$$= \mathcal{F}\big(Road\ capacity(a_{11}), construction\ dust(a_{222}), utility\ demand(a_{322})\big)$$

$$= \mathcal{F}\left(\frac{z_1}{a_{11}(0.7), a_{222}(0.8), a_{322}(0.6)}, \frac{z_2}{a_{11}(0.5), a_{222}(0.2), a_{322}(0.7)}, \frac{z_3}{a_{11}(0.4), a_{222}(0.7), a_{322}(0.6)}, \frac{z_4}{a_{11}(0.8), a_{222}(0.4), a_{322}(0.7)}, \frac{z_5}{a_{11}(0.5), a_{222}(0.6), a_{322}(0.7)}\right)$$

2.7 Neutrosophic sets [10, 20]: A neutrosophic set $\mathcal{N}$ on the universe Q is defined as

$$\mathcal{N} = \{\langle q, \mathcal{T}_{\mathcal{N}}(q), \mathcal{I}_{\mathcal{N}}(q), \mathcal{F}_{\mathcal{N}}(q)\rangle : q \in Q\}$$

where $\mathcal{T}_{\mathcal{N}}, \mathcal{I}_{\mathcal{N}}, \mathcal{F}_{\mathcal{N}}: Q \to]0^-, 1^+[$ and $0^- \leq \mathcal{T}_{\mathcal{N}} + \mathcal{I}_{\mathcal{N}} + \mathcal{F}_{\mathcal{N}} \leq 3^+$.The neutrosophic set derives its value from genuine standard or non-standard subsets of $]0^-, 1^+[$ from an intellectual point of view. However, it is challenging to employ a neutrosophic set with a value from an actual standard or non-standard subset of $]0^-, 1^+[$ in real-world scientific and engineering applications. Therefore, we examine the neutrosophic set, which derives its value from the subset of [0,1].

2.8 Neutrosophic Hypersoft sets [17, 18, and 20]: Take Q to be the universal set, and $P(Q)$ be the power set of Q. Considering the well-defined attributes $\mathcal{g}_1, \mathcal{g}_2, \dots, \mathcal{g}_n$and the associated attribute values are the sets $\mathcal{G}_1, \mathcal{G}_2, \mathcal{G}_3, \dots, \mathcal{G}_n$.Suppose that $\mathcal{G}_i \cap \mathcal{G}_j = \emptyset$ for $i \neq j$ and $i, j \in \{1,2,3, \dots, n\}$, the relation $\mathcal{G}_1 \times \mathcal{G}_2 \times \mathcal{G}_3 \times \dots \times \mathcal{G}_n = \mathcal{S}$ then the pair $(\omega, \mathcal{S})$ is referred to as neutrosophic hypersoft set over Q, where $\omega: \mathcal{G}_1 \times \mathcal{G}_2 \times \mathcal{G}_3 \times \dots \times \mathcal{G}_n \to P(Q)$,

$\omega(\mathcal{G}_1 \times \mathcal{G}_2 \times \mathcal{G}_3 \times \dots \times \mathcal{G}_3) = \{\langle q, \mathcal{T}(\omega(\mathcal{S})), \mathcal{I}(\omega(\mathcal{S})), \mathcal{F}(\omega(\mathcal{S}))\rangle\}$, where $\mathcal{T}$ the membership value of truth, $\mathcal{I}$ the membership value of indeterminacy and $\mathcal{F}$ the membership value of falsity. These values are defined as $\mathcal{T}, \mathcal{I}, \mathcal{F}: Q \to [0,1]$. In addition

$$0 \leq \mathcal{T}(\omega(\mathcal{S})) + \mathcal{I}(\omega(\mathcal{S})) + \mathcal{F}(\omega(\mathcal{S})) \leq 3$$

2.9 Neutrosophic Treesoft sets: Q the universal set and $P(Q)$ be the power set of Q. $\mathcal{G}$ a set of attributes, $\mathcal{G} = \{\mathcal{G}_1, \mathcal{G}_2, \mathcal{G}_3, \dots, \mathcal{G}_n\}$, for integer $n \geq 1$, where $\mathcal{G}_1, \mathcal{G}_2, \dots, \mathcal{G}_n$ are attributes of first level. Each attribute $\mathcal{G}_i$, $1 \leq i \leq n$, is formed by the sub-attributes $\mathcal{G}_{i,j}$ attributes of second level. Again, each sub-attribute $\mathcal{G}_{i,j}$ is formed by sub-sub-attributes of third level $\mathcal{G}_{i,j,k}$ and so on, as much refinement needed into each application, upto sub-sub-,…-sub-attributes or attributes of m level.The relation $P(\mathcal{K}_1 \cup \mathcal{K}_2 \cup \dots \cup \mathcal{K}_m) = \mathcal{D}$ then the neutrosophic treesoft set defined as the pair $(\omega, \mathcal{D})$ where ω: $P(\mathcal{K}_1 \cup \mathcal{K}_2 \cup \dots \cup \mathcal{K}_m) \to P(Q)$ (using Remark 1),

$\omega\big(P(\mathcal{K}_1 \cup \mathcal{K}_2 \cup \dots \cup \mathcal{K}_m)\big) = \big\{\langle q, \mathcal{T}\big(\omega(\mathcal{D})\big), \mathcal{I}\big(\omega(\mathcal{D})\big), \mathcal{F}\big(\omega(\mathcal{D})\big)\rangle\big\}$, where $\mathcal{T}, \mathcal{I}, \mathcal{F} \to [0,1]$. Additionally $0 \leq \mathcal{T}\big(\omega(\mathcal{D})\big) + \mathcal{I}\big(\omega(\mathcal{D})\big) + \mathcal{F}\big(\omega(\mathcal{D})\big) \leq 3$

Example 3: Considering the same example used for Hypersoft sets and, by applying the conditions of neutrosophic hypersoft sets we get the following:

$$(\omega, \check{\mathcal{A}})$$

$$= \left\{ \begin{array}{l} \left(\widetilde{a_1}\left(\frac{z_1}{(0.2,0.4,0.6)}, \frac{z_2}{(0.3,0.5,0.7)}, \frac{z_3}{(0.8,0.5,0.8)}, \frac{z_4}{(0.1,0.8,0.4)}, \frac{z_5}{(0.4,0.1,0.7)}\right)\right), \\ \left(\widetilde{a_2}\left(\frac{z_1}{(0.5,0.6,0.7)}, \frac{z_2}{(0.2,0.3,0.7)}, \frac{z_3}{(0.7,0.4,0.5)}, \frac{z_4}{(0.2,0.4,0.6)}, \frac{z_5}{(0.1,0.7,0.1)}\right)\right), \\ \left(\widetilde{a_3}\left(\frac{z_1}{(0.4,0.5,0.6,)}, \frac{z_2}{(0.1,0.2,0.3)}, \frac{z_3}{(0.8,0.5,0.2)}, \frac{z_4}{(0.5,0.2,0.6)}, \frac{z_5}{(0.8,0.2,0.5)}\right)\right), \\ \left(\widetilde{a_4}\left(\frac{z_1}{(0.8,0.2,0.3)}, \frac{z_2}{(0.5,0.2,0.4)}, \frac{z_3}{(0.8,0.5,0.6)}, \frac{z_4}{(0.4,0.1,0.6)}, \frac{z_5}{(0.8,0.5,0.3)}\right)\right), \\ \left(\widetilde{a_5}\left(\frac{z_1}{(0.6,0.2,0.3)}, \frac{z_2}{(0.5,0.4,0.6)}, \frac{z_3}{(0.4,0.2,0.6)}, \frac{z_4}{(0.7,0.9,0.1)}, \frac{z_5}{(0.8,0.5,0.7)}\right)\right), \\ \left(\widetilde{a_6}\left(\frac{z_1}{(0.2,0.5,0.8)}, \frac{z_2}{(0.3,0.6,0.9)}, \frac{z_3}{(0.1,0.4,0.7)}, \frac{z_4}{(0.1,0.5,0.9)}, \frac{z_5}{(0.3,0.5,0.7)}\right)\right), \\ \left(\widetilde{a_7}\left(\frac{z_1}{(0.2,0.4,0.7)}, \frac{z_2}{(0.4,0.9,0.3)}, \frac{z_3}{(0.7,0.9,0.2)}, \frac{z_4}{(0.1,0.6,0.7)}, \frac{z_5}{(0.8,0.4,0.6)}\right)\right), \\ \left(\widetilde{a_8}\left(\frac{z_1}{(0.4,0.2,0.1)}, \frac{z_2}{(0.8,0.3,0.1)}, \frac{z_3}{(0.4,0.5,0.1)}, \frac{z_4}{(0.3,0.5,0.7)}, \frac{z_5}{(0.1,0.5,0.2)}\right)\right), \\ \left(\widetilde{a_9}\left(\frac{z_1}{(0.4,0.2,0.8)}, \frac{z_2}{(0.7,0.1,0.3)}, \frac{z_3}{(0.4,0.3,0.9)}, \frac{z_4}{(0.1,0.2,0.3)}, \frac{z_5}{(0.3,0.5,0.7)}\right)\right), \\ \left(\widetilde{a_{10}}\left(\frac{z_1}{(0.3,0.4,0.9)}, \frac{z_2}{(0.2,0.5,0.7)}, \frac{z_3}{(0.9,0.2,0.4)}, \frac{z_4}{(0.4,0.7,0.3)}, \frac{z_5}{(0.7,0.8,0.3)}\right)\right), \\ \left(\widetilde{a_{11}}\left(\frac{z_1}{(0.4,0.8,0.3)}, \frac{z_2}{(0.4,0.8,0.1)}, \frac{z_3}{(0.7,0.3,0.4)}, \frac{z_4}{(0.1,0.5,0.3)}, \frac{z_5}{(0.7.0.5,0.9)}\right)\right), \\ \left(\widetilde{a_{12}}\left(\frac{z_1}{(0.1,0.5,0.7)}, \frac{z_2}{(0.3,0.5,0.9)}, \frac{z_3}{(0.7,0.5,0.9)}, \frac{z_4}{(0.2,0.4,0.6)}, \frac{z_5}{(0.8,0.6,0.2)}\right)\right), \\ \left(\widetilde{a_{13}}\left(\frac{z_1}{(0.5,0.8,0.2)}, \frac{z_2}{(0.5,0.2,0.3)}, \frac{z_3}{(0.9,0.2,0.4)}, \frac{z_4}{(0.3,0.4,0.1)}, \frac{z_5}{(0.7,0.2,0.1)}\right)\right), \\ \left(\widetilde{a_{14}}\left(\frac{z_1}{(0.6,0.7,0.2)}, \frac{z_2}{(0.4,0.8,0.2)}, \frac{z_3}{(0.3,0.4,0.9)}, \frac{z_4}{(0.2,0.5,0.4)}, \frac{z_5}{(0.3,0.6,0.9)}\right)\right), \\ \left(\widetilde{a_{15}}\left(\frac{z_1}{(0.2,0.5,0.8)}, \frac{z_2}{(0.7,0.4,0.1)}, \frac{z_3}{(0.1,0.5,0.9)}, \frac{z_4}{(0.7,0.3,0.5)}, \frac{z_5}{(0.2,0.5,0.3)}\right)\right), \\ \left(\widetilde{a_{16}}\left(\frac{z_1}{(0.6,0.5,0.4)}, \frac{z_2}{(0.8,0.9,0.7)}, \frac{z_3}{(0.7,0.5,0.3)}, \frac{z_4}{(0.4,0.9,0.2)}, \frac{z_5}{(0.3,0.1,0.2)}\right)\right), \\ \left(\widetilde{a_{17}}\left(\frac{z_1}{(0.7,0.6,0.4)}, \frac{z_2}{(0.4,0.6,0.5)}, \frac{z_3}{(0.3,0.5,0.9)}, \frac{z_4}{(0.4,0.2,0.1)}, \frac{z_5}{(0.2,0.4,0.5)}\right)\right), \\ \left(\widetilde{a_{18}}\left(\frac{z_1}{(0.4,0.5,0.6)}, \frac{z_2}{(0.7,0.8,0.9)}, \frac{z_3}{(0.5,0.8,0.4)}, \frac{z_4}{(0.8,0.2,0.4)}, \frac{z_{50}}{(0.6,0.4,0.1)}\right)\right) \end{array} \right\}$$

Example 4: Take the example utilized for treesoft sets and by implementing the specifications of neutrosophic treesoft sets, we derive the outcome:

Neutrosophic treesoft sets is given by

$$\mathcal{F}(\mathcal{Z}_1 \cup \mathcal{Z}_2 \cup \mathcal{Z}_3)$$

$$= \mathcal{F}\big(Road\ capacity(a_{11}), Construction\ dust(a_{222}), Utility\ demand(a_{322})\big)$$

$$= \left\{ \begin{array}{l} \left(\dfrac{\mathcal{Z}_1}{a_{11}(0.25,0.71,0.63), a_{222}(0.68,0.47,0.31), a_{322}(0.47,0.51,0.28)}\right); \\ \left(\dfrac{\mathcal{Z}_2}{a_{11}(0.52,0.84,0.61), a_{222}(0.82,0.56,0.71), a_{322}(0.59,0.62,0.48)}\right); \\ \left(\dfrac{\mathcal{Z}_3}{a_{11}(0.84,0.67,0.45), a_{222}(0.74,0.73,0.75), a_{322}(0.74,0.61,0.81)}\right); \\ \left(\dfrac{\mathcal{Z}_4}{a_{11}(0.94,0.21,0.52), a_{222}(0.24,0.84,0.86), a_{322}(0.75,0.84,0.56)}\right); \\ \left(\dfrac{\mathcal{Z}_5}{a_{11}(0.24,0.45,0.55), a_{222}(0.91,0.54,0.64), a_{322}(0.72,0.61,0.53)}\right) \end{array} \right\}$$

3. Multi Criteria Decision Making Technique (MCDM):

3.1 Analytical Hierarchy Process (AHP) [5]: This is the simplest weight calculation method, here is the algorithm for the method:

Step 1: Define the Decision Problem

Step 2: Identify the criteria and sub-criteria

Step 3: Establish the pairwise comparison matrix

Step 4: Normalize pairwise comparison matrix

Step 5: Calculate criteria weights

Step 6: Check consistency of pairwise matrix

Step 7: Calculate final scores and make the decisions

3.2 Technique for Order of Preference by Similarity to Ideal Solution (TOPSIS) [3]:

Step 1: Create an evaluation matrix

Step 2: The evaluation matrix is normalized using the below formula

$$r_{ij} = \frac{x_{ij}}{\sqrt{\sum_{k-1}^{m} {x_{kj}}^2}}, i = 1,2,\dots,m; j = 1,2,\dots,n$$

Step 3: Calculate the weighted normalized decision matrix

$$t_{ij} = r_{ij}.\mathcal{W}_j, i = 1,2,\dots,m; j = 1,2,\dots,n$$

where $\mathcal{W}_j = \frac{w_j}{\sum_{k=1}^{n} w_k}, j = 1,2,\dots,n$

Step 4: Establish the worst alternative ($\mathcal{A}_w$) and best alternative ($\mathcal{A}_b$);

$$\mathcal{A}_w = \{\langle max(t_{ij}|i = 1,2,\dots,m)|j \in \mathcal{J}_-\rangle, \langle min(t_{ij}|i = 1,2,\dots,m)|j \in \mathcal{J}_+\rangle\} \equiv \{t_{wj}|j = 1,2,\dots,n\}$$

$\mathcal{A}_b = \{\langle min(t_{ij}|i = 1,2,\dots,m)j \in \mathcal{J}_-\rangle, \langle max(t_{ij}|i = 1,2,\dots,m)|j \in \mathcal{J}_+\rangle\} \equiv \{t_{bj}|j = 1,2,\dots,n\}$

Step 5: Compute the $\mathcal{L}^2$ the distance among the target alternative i & worst condition $\mathcal{A}_w$

$$d_{iw} = \sqrt{\sum_{j=1}^{n}(t_{ij} - t_{wj})^2}, i = 1,2,\dots,m$$

and distance among the alternative i & best condition $\mathcal{A}_b$

$$d_{ib} = \sqrt{\sum_{j=1}^{n}(t_{ij} - t_{bj})^2}, i = 1,2,\dots,m$$

where d_{iw} & d_{ib} are $\mathcal{L}^2$- norm distances from a target alternative to the worst & best conditions, correspondingly.

Step 6: Determine how similar it is to the worst situation.

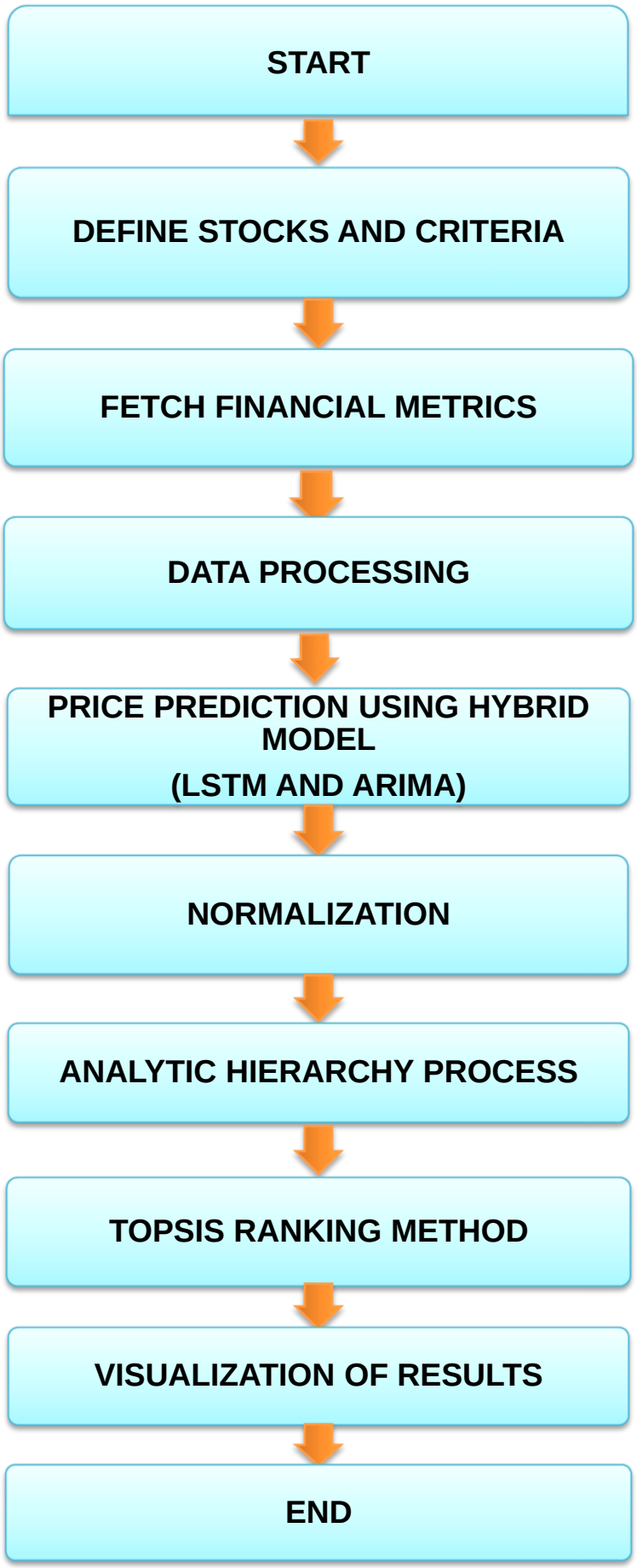
$$s_{iw} = \frac{d_{iw}}{(d_{iw} + d_{ib})}, 0 \leq s_{iw} \leq 1, i = 1,2,\dots,m.$$

$s_{iw} = 0$ if and only if the alternative solution has the worst condition and

$s_{iw} = 1$ if and only if the alternative solution has the best condition

Step 7: Rank the alternative using $s_{iw}(i = 1,2,\dots,m)$.

Flow Chart for Stock analysis and Prediction process: The below flowchart gives a brief outline of the working of the proposed model.


START
DEFINE STOCKS AND CRITERIA
FETCH FINANCIAL METRICS
DATA PROCESSING
PRICE PREDICTION USING HYBRID MODEL
(LSTM AND ARIMA)
NORMALIZATION
ANALYTIC HIERARCHY PROCESS
TOPSIS RANKING METHOD
VISUALIZATION OF RESULTS
END

Formulation of the problem: Investors are confronted with the challenge of selecting the most advantageous equities from a competitive and diverse market place in the contemporary financial landscape. Furthermore, the increasing significance of market sentiment, risk management, growth potential, and sustainability (ESG) considerations requires a more comprehensive evaluation approach. The objective of the decision-maker is to address the challenge of comprehensive stock evaluation by creating a multi-criteria decision-making (MCDM) framework that will evaluate seven influential stocks: Apple Inc. (AAPL), Microsoft Corp. (MSFT), Amazon. Com Inc. (AMZN), Alpabet Inc. (GOOGL), Tesla Inc. (TSLA), NVIDIA Corp. (NVDA), and Johnson & Johnson (JNJ). The assessment will be predicated on seven primary criteria, each of which is further divided into sub-criteria and sub-sub-criteria as listed below:

The criteria's include

$$\mathcal{d} = Finanical\ Performance;\ \mathcal{e} = Market\ Sentiment;\ \mathcal{f} = Risk\ Profile;$$

$$\mathcal{g} = Growth\ Potential;\ \mathcal{h} = Valuation;\ \mathcal{i} = Liquidity;$$

$$\mathcal{j} = Sustainability\ and\ ESG\ factors$$

Each criteria is subdivided into sub-criteria's as

$$d_1 = revenue\ growth;\ d_2 = Profitability;\ d_3 = return\ on\ equity;$$

$$e_1 = social\ media\ sentiment;\ e_2 = news\ sentiment;\ e_3 = analyst\ recommendations;$$

$$\mathcal{f}_1 = volatility;\ \mathcal{f}_2 = financial\ leverage;\ \mathcal{f}_3 = liquidity\ risk;$$

$$\mathcal{g}_1 = predicted\ price\ increase;\ \mathcal{g}_2 = market\ share\ growth;\ \mathcal{g}_3 = innovation;$$

$$\mathcal{h}_1 = price - to - eaarnings;\ \mathcal{h}_2 = price - to - book\ ratio;\ \mathcal{h}_3 = dividend\ yield;$$

$$\mathcal{i}_1 = trading\ volume;\ \mathcal{i}_2 = cash\ flow;\ \mathcal{i}_3 = short - term\ solvency;$$

$$\mathcal{j}_1 = environmental\ score;\ \mathcal{j}_2 = social\ responsibility;\ \mathcal{j}_3 = governance;$$

Following are the sub-sub-criteria's of each sub-criteria's

$$d_{11} = quarterly;\ d_{12} = annual\ ;\ d_{13} = 5 - year\ CAGR;$$

$$d_{21} = net\ profit\ margin;\ d_{22} = operating\ margin;\ d_{23} = gross\ margin;$$

$$d_{31} = TTM;\ d_{32} = 3 - year\ average;\ d_{33} = 5 - year\ average;$$

$$e_{11} = twitter;\ e_{12} = reddit;\ e_{13} = stock\ twits;$$

$$e_{21} = positive\ ;\ e_{22} = negative\ ;\ e_{23} = neutral\ news\ ratio;$$

$$e_{31} = buy; e_{32} = hold; e_{33} = sell\ ratings;$$

$$\mathbb{f}_{11} = 1 - year; \mathbb{f}_{12} = 5 - year; \mathbb{f}_{13} = beta;,$$

$$\mathbb{f}_{21} = debt - to - equity; \mathbb{f}_{22} = interest\ coverage;\ \mathbb{f}_{31} = current\ ration;\ \mathbb{f}_{32} = quick\ ratio; \mathbb{f}_{33} = cash\ ratio;$$

$$\mathbb{g}_{11} = 1 - month; \mathbb{g}_{12} = 3 - month; \mathbb{g}_{13} = 6 - month;$$

$$\mathbb{g}_{21} = industry\ comparison;$$

$$\mathbb{g}_{31} = R\&D\ investments; \mathbb{g}_{32} = patents; \mathbb{g}_{33} = new\ product\ launches;$$

$$\mathbb{h}_{11} = forward\frac{P}{E}; \mathbb{h}_{12} = trailing\frac{P}{E}; \mathbb{h}_{13} = industry\ comparison;$$

$$\mathbb{h}_{21} = book\ value\ per\ share; \mathbb{h}_{22} = market\ price\ per\ share;$$

$$\mathbb{h}_{31} = payout\ ratio; \mathbb{h}_{32} = growth\ rate; \mathbb{h}_{33} = historical\ yield;$$

$$\mathbb{i}_{11} = average\ daily\ volume; \mathbb{i}_{12} = relative\ volume;$$

$$\mathbb{i}_{21} = operating\ cash\ flow; \mathbb{i}_{22} = free\ cash\ flow, \mathbb{i}_{31} = current\ ratio; \mathbb{i}_{32} = quick\ ratio; \mathbb{i}_{33} = cash\ ratio;$$

$$\mathbb{j}_{11} = carbon\ footprint; \mathbb{j}_{12} = energy\ efficiency; \mathbb{j}_{13} = waste\ management;$$

$$\mathbb{j}_{21} = employee\ diversity; \mathbb{j}_{22} = community\ engagement;$$

$$\mathbb{j}_{31} = board\ diversity; \mathbb{j}_{32} = transparency; \mathbb{j}_{33} = anti - corruption;$$

The below table shows the alternatives and criteria's taken from all the above mentioned criteria's sub-criteria's, sub-sub-criteria's

Table 2 Alternatives and criteria's

	Alternatives	Criteria's
1.	AAPL	Net profit margin
2.	MSFT	P\E ratio
3.	AMZN	Average daily trading volume
4.	GOOGL	Beta (stock availability)
5.	TSLA	Environmental score
6.	NVDA	Sentiment scores
7.	JNJ	Predicted 3-month increase

The following code illustrates a hybrid approach to stock analysis and prediction that employs financial metrics, sentiment scores, environment scores, and hybrid price forecasting models (LSTM and ARIMA). It incorporates the Analytic Hierarchy Process (AHP) for weight determination and TOPSIS for alternative ranking.

Step 1: Here the necessary libraries to be installed and imported in the code

Step 2: Defining stocks and simulated data.

```python
stocks = ['AAPL', 'MSFT', 'AMZN', 'GOOGL', 'TSLA', 'NVDA', 'JNJ']
```

The simulated data includes:

```python
sentiment_scores = {...}
environmental_scores = {...}
```

Step 3: Fetching financial metrics

```python
def fetch_financial_metrics(stocks):
```

The below mentioned are the financial metrics to be considered

- Net profit margin
- P\E ratio
- Average daily trading volume
- Beta
- Environmental score

Step 4: LSTM and ARIMA model prediction

LSTM uses past stock prices to predict future prices

```python
def get_hybrid_predicted_price_increase(ticker, start_date, end_date, sequence_length=60):
    stock_data = yf.download(ticker, start=start_date, end=end_date)
    stock_data = stock_data[['Close']]
```

LSTM works best with scaled data, hence `MinMaxScaler` is used to normalize the stock prices between 0 and 1.

```python
scaler = MinMaxScaler(feature_range=(0, 1))
scaled_data = scaler.fit_transform(stock_data)
```

In order to train the LSTM model, the data is partitioned into sequences. The stock prices are sequence over a period of 60 days.

```python
def create_sequences(data, sequence_length):
```

The TensorFlow/Keras Sequential API is employed to construct a two-layer LSTM model. The Huber loss is employed to mitigate the effects of outliers. For a period of 20 epochs, the model is trained on 80% of the data.

```python
model = Sequential()
...
model.compile(optimizer='adam', loss=Huber())
model.fit(X_train, y_train, batch_size=32, epochs=20, verbose=1)
```

Upon completion of training, the LSTM model forecasts future prices by utilizing the most recent 60 days of stock price data.

```python
last_sequence = scaled_data[-sequence_length:]
predicted_price_scaled = model.predict(last_sequence)
predicted_price_lstm = scaler.inverse_transform(predicted_price_scaled)
```

The stock data is also fitted with an ARIMA (AutoRegressive Integrated Moving Average) model, which aids in the identification of linear trends. The stock price for the following day is predicted [7].

```python
model_arima = ARIMA(stock_data, order=(5, 1, 0))
model_fit = model_arima.fit()
predicted_price_arima = model_fit.forecast(steps=1)
```

The forecasts from LSTM and ARIMA are averaged to obtain a hybrid anticipated price. The percentage increase from the most recent price is calculated and provided [9].

```python
predicted_price = (predicted_price_lstm[0][0] + predicted_price_arima) / 2
```

Step 5: Financial Metrics and Predicted price increase

```python
predicted_increases = {}
for stock in stocks:
    predicted_increases[stock] = get_hybrid_predicted_price_increase(stock, '2023-01-01',
```

In the forthcoming period, the probable price increases for each stock are recorded in predicted_increases.

Step 6: Normalization of Financial Metrics:

Using MinMaxScaler the financial data, sentiment scores, and predicted price increases are normalized for comparison

```python
def normalize_data(data):
    scaler = MinMaxScaler()
    return pd.DataFrame(scaler.fit_transform(data), columns=data.columns, index=data.index
```

Step 7: AHP Method for Weight Calculation [22]

Based on user preferences (such as the significance of criteria such as Net Profit Margin and P\E ratio), a pairwise comparison matrix is generated. The final weights are determined by normalizing the matrix and computing the priority vector.

```python
def ahp_method():
```

Step 8: TOPSIS for Stock Ranking [24]

According to the criteria weights determined by AHP, the financial data is first normalized and subsequently weighted.

```python
evaluation_matrix = normalized_df.values
...
weighted_df = norm_matrix * weights.values
```

The best and worst alternatives are identified for each criterion. For certain criteria (e.g., Net Profit Margin), elevated values are advantageous, whilst for others (e.g., Beta), diminished values are preferred.

```python
def determine_best_worst(weighted_df, is_benefit):
```

The Euclidean distance of each stock from the ideal best and worst solutions is used to calculate the TOPSIS score. The stocks are arranged in accordance with their scores.

```python
d_best, d_worst = calculate_distances(weighted_df, best, worst)
topsis_score = d_worst / (d_best + d_worst)
```

Step 9: Visualization of Predicted prices:

This tool visualizes the actual against expected pricing utilizing plotly. It superimposes both LSTM and ARIMA forecasts onto the real stock price data.

```python
# Plot all stocks in a single graph and save as image and HTML
def plot_all_stocks_actual_vs_predicted_and_save(stocks, start_date, end_date, image_file_
```

Step 10: Final output

The table below displays the rating of seven stocks, the final output presents the model's projection based on the criteria; the predicted 3-month gain fluctuates according to the market's stock value changes.

Table 3 Final ranking

	Alternatives	TOPSIS scores	Ranking
1.	AAPL	0.135899	7
2.	MSFT	0.200582	5
3.	AMZN	0.595582	1

4.	GOOGL	0.219955	4
5.	TSLA	0.406424	3
6.	NVDA	0.558132	2
7.	JNJ	0.159474	6

Comparative study of the Model:

1. **Comparison of Prediction Models: Hybrid LSTM-ARIMA vs. Other Models**
   - **Hybrid Model (LSTM+ARIMA)**

     Advantages: Integrates the efficacy of LSTM, which excels in capturing sequential relationships in time-series data, with ARIMA, which is proficient in modelling trends and seasonality. It enables the application of linear elements using ARIMA and non-linear patterns using LSTM.

     Disadvantages: It requires significant computational resources, as it necessitates the training of both LSTM and ARIMA models. Neural Networks often lack interpretability due to their perceived "black box" nature. **Accuracy:** Hybrid models often surpass single models in analysing complex stock data, contingent upon the stock's volatility and historical trends.

   - **Alternative Models**
     - **Single ARIMA model:** This method is more straightforward and interpretable, as it relies solely on three factors (p,d,q). It works better for steady time series with strong seasonality, but it's less useful for extremely volatile stocks.
     - **Single LSTM Model:** This method proves to be effective in pinpointing non-linear correlations within time-series data. The system requires more data for the best training, and its performance could decline with fewer datasets.
     - **Prophet Model:** Facebook created this tool, which excels at handling time series data that includes distinct seasonal and trend elements. It is quick and relatively easy to modify, yet it might not adequately react to sudden changes in the market.

2. **Comparison of weighting and Ranking Techniques: AHP-TOPSIS vs. Alternative Decision-Making methods:**
   - **AHP-TOPSIS Approach:**

     Advantages: The Analytic Hierarchy Process (AHP) facilitates the integration of expert judgement and enables subjective weighting according to the significance of criteria. TOPSIS is simple to compute and yields a singular score that rates equities based on their closeness to an optimal solution**.**

     Disadvantages: AHP process may exhibit sensitivity to biases in judgment during weight assignment. TOPSIS ignores trade-offs among criteria and assumes that all criteria have equal significance after weighting.

Suitability: It is ideal for situations where expert preference is present and decision-making involves a well-defined set of criteria.

- **Alternative methods:**
  - **Simple Additive Weighting (SAW):** With direct weighting and summation of criteria, it is simpler to compute. The absence of TOPSIS's ideal-best/worst comparison could potentially impact the clarity of the displayed rankings.
  - **Analytic Network Process:** A modification of AHP that considers the interrelationships among criteria. Increased complexity enhances the representation of realistic relationships among decision factors.
  - **Machine Learning-**Based Ranking (e.g., Gradient Boosting): Models can acquire intricate non-linear associations between variables and results. The model demands labelled data, it presents interpretative challenges, and may be inappropriate for scenarios where interpretability is crucial.

Table 4 Comparative study table

Criteria	Hybrid LSTM-ARIMA	ARIMA	AHP-TOPSIS	SAW	ANP
Complexity	High	Medium	Medium	Low	High
Accuracy	High	Medium-High	High	Medium	High
Interpretability	Medium	High	High	High	Medium
Computational Requirements	High	Low	Medium	Low	High
Optimal Application	Volatile stocks	Stable stocks	Criteria ranking	Simple decision-making	Complex interdependencies

Sensitivity Analysis: Sensitivity analysis in this work entails how variations in the weights or significance of each criterion affect the final TOPSIS rankings for each stock. This assists in determining which criteria exert the most significant impact on the decision-making process and the resilience of the rankings to variations in these criteria.

Case 1: By adjusting the weights of the criterion by a small percentage of $\pm$**10**%

Table 5

Stocks	Baseline score	Baseline rank	Net profit +10%	Rank	Net profit -10%	Rank	Average +10%	Rank	Average -10%	Rank
AAPL	0.136	7	0.138	7	0.134	7	0.138	7	0.134	7
MSFT	0.201	5	0.204	5	0.197	5	0.195	5	0.206	5
AMZN	0.596	1	0.593	1	0.597	1	0.578	1	0.613	1
GOOGL	0.220	4	0.221	4	0.219	4	0.214	4	0.226	4

TSLA	0.406	3	0.405	3	0.407	3	0.400	3	0.412	3
NVDA	0.558	2	0.560	2	0.557	2	0.573	2	0.544	2
JNJ	0.159	6	0.166	6	0.153	6	0.155	6	0.164	6

In the aforementioned case, stocks that continuously get high rankings across diverse situations may be regarded as robust in their weight, exhibiting percentage fluctuations in their weights.

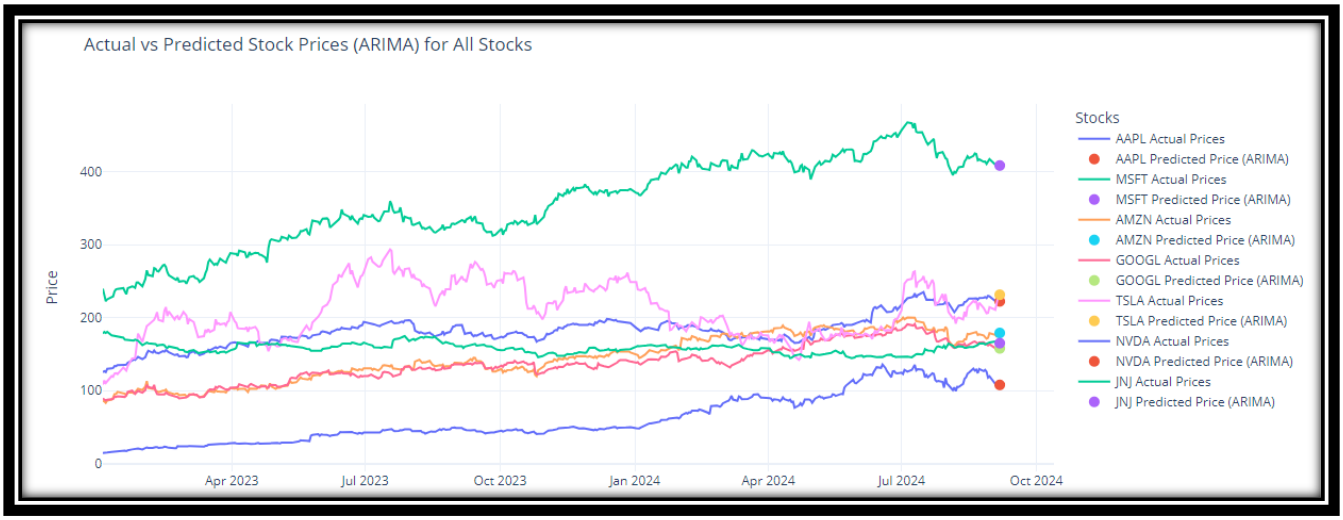
Case 2: Enhancing the weight of a beneficial criteria and lowering the weight of a non-beneficial criteria we get the below table

Table 6

Stocks	Baseline score	Baseline rank	Change in score	Rank
AAPL	0.136	7	0.431	6
MSFT	0.201	5	0.500	3
AMZN	0.596	1	0.445	5
GOOGL	0.220	4	0.478	4
TSLA	0.406	3	0.169	7
NVDA	0.558	2	0.588	1
JNJ	0.159	6	0.558	2

The most significant shifts in ranking occurred when the non-beneficial criterion Beta was elevated and the beneficial criterion environmental score was minimized. These criteria exert the most substantial impact on stock performance as per the model

Description of the graph performance: The graph illustrates a comparison between actual stock prices and forecasted prices for various equities (AAPL, MSFT, AMZN, GOOGL, TSLA, NVDA, JNJ) over a designated time period. Each stock possesses a solitary marker (point) that signifies the anticipated price for the subsequent day following the conclusion of the actual price data (September 7, 2024). The markers are positioned at the terminus of the actual price lines and are typically depicted as larger dots in an alternate colour (eg., red). The marker's location indicates the projected price for the following day, derived from the ARIMA model



Below is the link for the working of a plotly graph based on python coding language

stocks-plot.html

Interpretation of the figure

- **Trend analysis:** The overall trend in the prices of each stock over the specified time frame can be analysed by examining the lines. For example, an upward-sloping price line of a stock suggests that its price is on the rise.
- **Forecasting Accuracy:** The predicted markers serve as a benchmark for comparison with the actual price lines. The forecasting model (ARIMA) has performed has performed well if the predicted price (marker) is in close proximity to the last point of the actual price line.
- **Volatility:** Additionally, it is possible to evaluate the volatility of each stock through assessing the degree of fluctuation in the actual price lines. Larger fluctuations may suggest that the stock in question is more volatile or prone to risk.
- **Investment Decisions:** This graphic aids investors in decision-making by utilizing trends and forecasts. A stock with a robust growth trajectory and an optimistic outlook may be regarded as a favourable investment prospect.

Overall this graphical representation is a potent instrument for the analysis of stock performance and the formulation of well-informed investment decisions based on historical data and predictions.

Limitations and impact: This approach is limited by the complexity arising from the use of Neutrosophic Treesoft sets, which although beneficial for managing uncertainty, may hinder understanding and practical use in real-time trading. The hybrid ARIMA-LSTM model, while excellent for forecasting, can be computationally intensive, presenting difficulties for high-frequency trading. Moreover, the precision depends on the quality of subjective inputs such as sentiment scores, which may be susceptible to bias. Considering these constraints, the

significance of this research it considerable; it improves stock forecast precision by integrating both quantitative and qualitative indicators, facilitating more informed and balanced investment choices. The innovative use of Neutrosophic Treesoft sets may impact future research in uncertainty modelling in finance and other fields where decision-making under ambiguous settings is crucial.

Discussion: A robust methodology for assessing the performance of the selected stocks was established through the integration of MCDM techniques and LSTM/ARIMA predictive models. Although traditional financial metrics are still significant, a more comprehensive and forward-thinking evaluation was achieved by incorporating advanced price forecasting, sentiment analysis, and ESG factors. Stocks such as AAPL and JNJ may be preferred by investors who prioritize sustainable practices and long-term development. Conversely, investors who prioritize high-growth, innovation-driven investments may favour AMZN, NVDA or TSLA, although with a higher risk tolerance.

Conclusion: This paper defines neutrosophic treesoft sets and provides examples of both neutrosophic and fuzzy treesoft sets. Using Neutrosophic treesoft sets along with hybrid predictive models and AHP-TOPSIS ranking methods in this study shows that it is possible to make a flexible, accurate, and advanced framework for financial analysis. The hybrid model for predicting stock price increases combines the long-term dependencies and trend-capturing capabilities of LSTM with the precision of ARIMA in time-series modelling. The use of Neutrosophic treesoft sets allows the framework to systematically capture three essential aspects of information: truth, indeterminacy, and falsity. This structure demonstrates enhanced flexibility and responsiveness to real-world situations marked by ambiguous or conflicting data. Future research may focus on assessing this framework using larger datasets, integrating additional financial indicators, or applying real-time sentiment analysis via NLP to improve its accuracy and adaptability and extend this model to other environments like Intuitionistic, Pythagorean, Bipolar fuzzy sets.

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A Neutrosophic Perspective on Enhancing Quality Virtual Simulation Experimental Education of Integrated Circuits: A New Approach

Lihua Dai[1*], Ben Wang [2], Xuemin Cheng [1], Junyi Xu [3], Hao Wang [1]

1 School of integrated Circuits and Communications, Suzhou Vocational Institute of Industrial Technology, Suzhou, 215104, Jiangsu, China

2 Microsoft (China) Co., LTD. Suzhou Branch, Suzhou, 215000, Jiangsu, China

3 Suzhou Huaxin Microelectronics Co., Ltd., Jiangsu, Suzhou, 21500, Jiangsu, China

*Corresponding Author: Lihua Dai, E-mail: sharon_dlh@163.com

Abstract: With rapid advancements in technology-related industries, integrated circuits come to be the core of modern electronics (IC). This in turn highlights the indispensable need to figure out advanced techniques for delivering high-quality experimental education in integrated circuits (IC). To that end, this work presents a neutrosophic framework to enhance the quality of education based on the selection of simulation-based tools that best improve the experimental skills of students. Given the inherent challenges of uncertainty, ambiguity, and subjectivity of virtual simulation, we propose to incorporate Interval-Valued Neutrosophic Sets (IVNS) to model uncertainty inherent to virtual simulation scenarios and adapt to encode linguistic variables that can be applied to decision-making in the experimental education of ICs. In our framework, A customized CRITIC (Criteria Importance Through Intercriteria Correlation) method is introduced to give weights for the IVNS representation of each criterion of IC simulation tools. Then, an IVNS-adapted ELECTRE-I (Elimination and Choice Expressing Reality) is presented to make a multi-criteria decision based on weighted representations of the IVNS-represented alternative. A public case study is introduced to experiment with the validity of different simulation tools for improving the experimental education of IC. Both quantitative and qualitative results demonstrate that the proposed methodology can offer a new solution to improve the student experience and optimize resource allocation in IC design courses. The implication of these results lies in enabling educators, researchers, and policymakers to easily elevate the quality of experimental education in integrated circuits.

Keywords: Neutrosophic theory, Integrated circuits, Teaching Quality, Decision making

1. Introduction

The study of Integrated circuits (ICs) has been gaining increased interest because of their important role in most modern electronics, which act as a foundational element in the development of technological solutions across different industries [1]. For years and decades, the process of teaching and training students in the field of ICs has conventionally depended on physical laboratories to offer hands-on experience [2]. Nevertheless, these educational mechanisms encounter many limitations including but not limited to constrained access to lab resources, high costs, and difficulty to cope with rapid technological development [3].

As a remedy, virtual simulations emerged as a transformative educational solution that can provide learners with realistic, collaborative environments to experiment with integrated circuits, enabling flexible, cost-effective, and scalable education [4]. This allows students to simulate the design of circuits, test their functionalities, and investigate performance indicators without performing any physical interactions. Despite these advantages, assessing the quality of experimental education delivered through virtual simulations remains a complex challenge due to the multifaceted nature of educational outcomes and the subjectivity involved in evaluating such methods [5].

As an extension to fuzzy logic, Smarandache [6] proposed neutrosophic sets (NSs) as a theoretical framework for modeling uncertainty in quantitative data with three-folded components: truth ($\mathbb{T}$), falsity ($\mathbb{F}$), and indeterminacy ($\mathbb{I}$). This representation leads to informative characterization by concurrently encoding the uncertainty, inarticulacy, and inconsistencies. With application of neutrosophic theory, we can to develop a robust approach to objectively mediate the quality of experimental education in virtual simulation environments. Later, the interval valued neutrosophic sets (IVNS) [7] were developed to extend the concept of traditional NSs to better address practical scenarios (such as virtual simulations) in which, it is hard to determine the exact value of truth, indeterminacy, and falsity. Instead, IVNS allows representing each of these components in the form of an interval within [0,1] which allows a more flexible way to model uncertainty and imprecision. This allows fulfilling the gap between theoretical adaptability and real-world applicability. Inspired by this, IVNS can help IC educators to better comprehend the points of strengths and weaknesses of virtual educational simulations and recognize areas that can be improved [8], [9].

To handle the gaps in the existing literature, this paper proposed a neutrosophic approach to evaluate teaching quality in university chemistry. The contributions of the proposed approaches can be summarized as follows:

- Our framework gets an insightful weighting of each criterion of simulation tools using the CRITIC (Criteria Importance Through Intercriteria Correlation) method.
- Then, we introduced the IVNS-adapted Elimination and Choice Expressing Reality (IVNS-ELECTRE-I) approach to making a correct decision about the best tools that can result in optimal IC education.

- A quantitative case study is presented, on which proof-of-concept analysis is performed to assess and evaluate the results of the proposed approach, comparing its performance against previous approaches.

The remainder of this paper is formulated as follows. First, we provide a detailed discussion of primary concepts and definitions. Section 3 explains the proposed neutrosophic decision-making approach to the best virtual simulation for improving the experimental education of ICs. Section 4 introduces the results of our approach and discusses them. Section 5 concludes the main findings.

2. Fundamentals & Definitions

Definition 1. Given a universal discourse X, the IVNS is configured with three ranged elements: truth ($\mathbb{T}$), falsity ($\mathbb{F}$), and indeterminacy ($\mathbb{I}$) [7]. Each of them has an interval range. $T_N = \left[\mathbb{T}_{N(x)}^{L}, \mathbb{T}_{N(x)}^{U} \subseteq [0,1]\right], I_N(x) = \left[\mathbb{I}_{N(x)}^{L}, \mathbb{I}_{N(x)}^{U} \subseteq [0,1]\right]$, and $F_N(x) = \left[\mathbb{F}_{N(x)}^{L}, \mathbb{F}_{N(x)}^{U} \subseteq [0,1]\right]$. With these ranged components, IVNS can be defined as:

$$N = \left\{\langle x, \left[\mathbb{T}_{N(x)}^{L}, \mathbb{T}_{N(x)}^{U}\right], \left[\mathbb{I}_{N(x)}^{L}, \mathbb{I}_{N(x)}^{U}\right], \left[\mathbb{F}_{N(x)}^{L}, \mathbb{F}_{N(x)}^{U}\right]\rangle \mid x \in X\right\} \quad (1)$$

where

$$0 \leq \mathbb{T}_{N(x)}^{L} + \mathbb{I}_{N(X)}^{L} + \mathbb{F}_{N(x)}^{L} \leq 3.$$

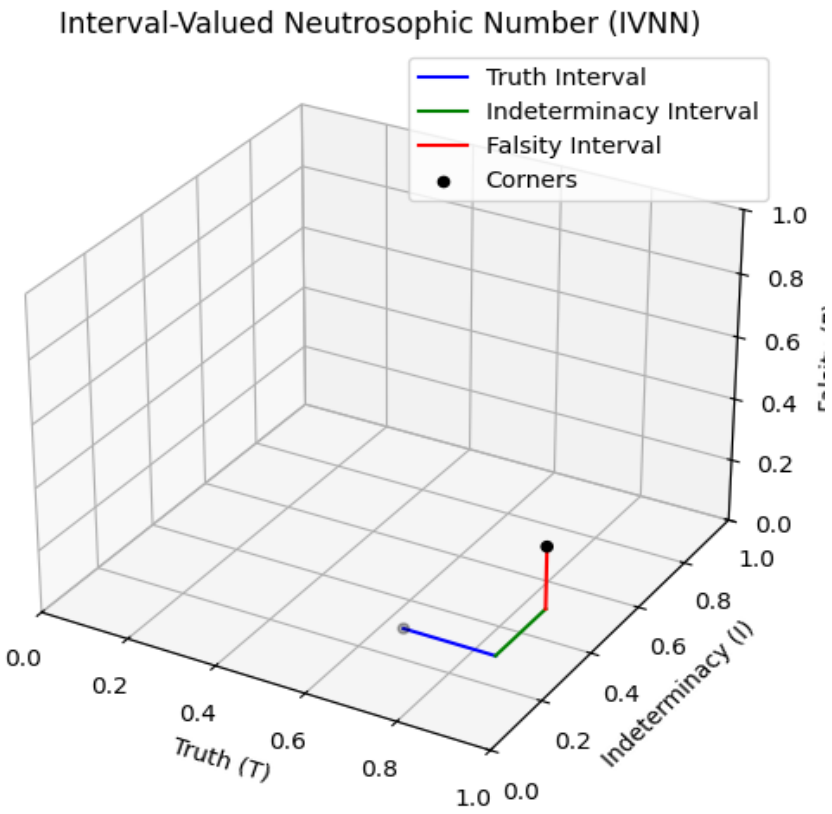

Figure 1. visual representation of IVNS in 3D coordinate system.

Definition 2. Given two IVNs $A = \left[\mathbb{T}_A^L, \mathbb{T}_A^U\right], \left[\mathbb{I}_A^L, \mathbb{I}_A^U\right], \left[\mathbb{F}_A^L, \mathbb{F}_A^U\right]$, and $B = \left[\mathbb{T}_B^L, \mathbb{T}_B^U\right], \left[\mathbb{I}_B^L, \mathbb{I}_B^U\right], \left[\mathbb{F}_B^L, \mathbb{F}_B^U\right]$, there are many mathematical operations that can be applied to them:

- Complement

$$A^c = \left\langle \left[\mathbb{T}_A^L, \mathbb{T}_A^U\right], \left[1 - \mathbb{I}_A^L, 1 - \mathbb{I}_A^U\right], \left[\mathbb{F}_A^L, \mathbb{F}_A^U\right] \right\rangle \tag{2}$$

- Addition

$$A \oplus B = \left\langle \left[\mathbb{T}_A^L + \mathbb{T}_B^L - \mathbb{T}_A^L \mathbb{T}_B^L, \mathbb{T}_A^U + \mathbb{T}_B^U - \mathbb{T}_A^U \mathbb{T}_B^U\right], \left[\mathbb{I}_A^L \mathbb{I}_B^L, \mathbb{I}_A^U \mathbb{I}_B^U\right], \left[\mathbb{F}_A^L \mathbb{F}_B^L, \mathbb{F}_A^U \mathbb{F}_B^U\right] \right\rangle \tag{3}$$

- Multiplication

$$A \otimes B = \left\langle \left[\mathbb{T}_A^L \mathbb{T}_B^L, \mathbb{T}_A^U \mathbb{T}_B^U\right], \left[\mathbb{I}_A^L + \mathbb{I}_B^L - \mathbb{I}_A^L \mathbb{I}_B^L, \mathbb{I}_A^U + \mathbb{I}_B^U - \mathbb{I}_A^U \mathbb{I}_B^U\right], \left[\mathbb{F}_A^L + \mathbb{F}_B^L - \mathbb{F}_A^L \mathbb{F}_B^L, \mathbb{F}_A^U + \mathbb{F}_B^U - \mathbb{F}_A^U \mathbb{F}_B^U\right] \right\rangle \tag{4}$$

Definition 3. Given two IVNS $A = \left[\mathbb{T}_A^L, \mathbb{T}_A^U\right], \left[\mathbb{I}_A^L, \mathbb{I}_A^U\right], \left[\mathbb{F}_A^L, \mathbb{F}_A^U\right]$, and $b = \left[\mathbb{T}_B^L, \mathbb{T}_B^U\right], \left[\mathbb{I}_B^L, \mathbb{I}_B^U\right], \left[\mathbb{F}_B^L, \mathbb{F}_B^U\right]$, the following relations apply as follows:

$$a \subseteq b \text{ if and only if } \mathbb{T}_A^L \leq \mathbb{T}_B^L, \mathbb{T}_A^U \leq \mathbb{T}_B^U; \mathbb{I}_A^L \geq \mathbb{I}_B^L, \mathbb{I}_A^U \geq \mathbb{I}_B^U; \mathbb{F}_A^L \geq \mathbb{F}_B^L, \mathbb{F}_A^U \geq \mathbb{F}_B^U \tag{5}$$

$$A = b \text{ if and only if } A \subseteq b \text{ and } b \subseteq A. \tag{6}$$

Definition 4. The weighted averaging operator for IVNS is defined to aggregate a number of IVNS weighted by the weight vector $Y = (y_1, \ldots, y_j, \ldots, y_n))$ and $\sum_{j=1}^n y_j = 1$.

$$\begin{aligned} &WA_{IVNNs}(x_1, \ldots, x_j, \ldots, x_n) = \sum_{j=1}^n y_j x_j \\ &= \left\langle \left[1 - \prod_{j=1}^n \left(1 - \mathbb{T}_j^L\right)^{y_j}, 1 \right. \right. \\ &- \prod_{j=1}^n \left(1 \right. \\ &\left. \left. - \mathbb{T}_j^U\right)^{y_j}\right], \left[\prod_{j=1}^n \left(\mathbb{I}_j^L\right)^{y_j}, \prod_{j=1}^n \left(\mathbb{I}_j^U\right)^{y_j}\right], \left[\prod_{j=1}^n \left(\mathbb{F}_j^L\right)^{y_j}, \prod_{j=1}^n \left(\mathbb{F}_j^U\right)^{y_j}\right] \rangle \end{aligned} \tag{7}$$

Definition 5. Given an IVNS $A = \left[\mathbb{T}_A^L, \mathbb{T}_A^U\right], \left[\mathbb{I}_A^L, \mathbb{I}_A^U\right], \left[\mathbb{F}_A^L, \mathbb{F}_A^U\right]$, the deneutrosophication can be computed as follows:

$$\mathfrak{D}(A) = \left(\frac{\left(\mathbb{T}_A^L + \mathbb{T}_A^U\right)}{2} + \left(1 - \frac{\left(\mathbb{I}_A^L + \mathbb{I}_A^U\right)}{2}\right)\left(\mathbb{I}_A^U\right) - \left(\frac{\left(\mathbb{F}_A^L + \mathbb{F}_A^U\right)}{2}\right)\left(1 - \mathbb{F}_A^U\right)\right) \tag{8}$$

3. Research Approach

In this section, we provide a detailed description of the IVNS-based decision-making framework that integrates a customized CRITIC method with the IVNS-ELECTRE-I approach. IVNS are intelligently used to model uncertainty, vagueness, and imprecision in IC features. This integration enables achieving robust and elastic decision-making that enables addressing complex, multi-criteria problems in uncertain environments.

3.1. Criteria Importance Through Intercriteria Correlation (CRITIC)

In this section, we start to objectively calculate the weight of each criterion criteria according to two key facets namely contrast intensity (which is estimated based on standard deviation) and the interaction among criteria [10]. By analyzing the versatility among criteria, the CRITIC was designed to allocate high weights for criteria that exhibit less correlation with others, which

ensures that the weights represent the significance as well as the exclusivity of each criterion. We decide to apply this method to avoid subjectivity in the process of determination of weight and is predominantly valuable when the decision matrix encompasses various or contradictory information. This is achieved as follows:

First, we establish the initial decision matrix based on deneutrosophication of IVNS matrix, X.

$$X = \left(\mathfrak{D}_{ij}(\mathcal{M})\right)_{m\times n} \tag{9}$$

Second, a normalization matrix, NN, is computed as follows:

$$NN = \left(NN_{ij}\right)_{m\times n} \tag{10}$$

The building of NN_{ij} is conditioned based on the following expression:

$$NN_{ij} = \begin{cases} \dfrac{X_{ij} - minX_j}{maxX_j - minX_j} & j > 0 \\ \dfrac{maxX_j - X_{ij}}{maxX_j - minX_j} & j < 0 \end{cases} \tag{11}$$

Third, we calculate the standard deviation for each criterion, S_j. This can be expressed as follows:

$$S_j = \sqrt{\frac{\sum_{i=1}^{m}\left(NN_{ij} - \overline{NN}_j\right)^2}{n-1}} \tag{12}$$

Fourth, we drive the correlation coefficient between every possible pair of criteria. This can be expressed as follows:

$$r_{bj} = \frac{\sum_{b,j=1}^{n}(NN_{ib} - \overline{NN}_b)\left(NN_{ij} - \overline{NN}_j\right)}{\sqrt{\sum_{b=1}^{n}(NN_{ib} - \overline{NN}_b)^2 \sum_{j=1}^{n}\left(NN_{ij} - \overline{NN}_j\right)^2}} \tag{13}$$

Fifth, we compute the contradictory distinctive between every possible pair of criteria, A_j.

$$Q_j \leftarrow \sum_{b=1}^{n}\left(1 - r_{bj}\right) \tag{14}$$

Sixth, we compute the amount of information controlled in $j-th$ criterion, C_j.

$$C_j \leftarrow S_j\, Q_j \tag{15}$$

Finally, we compute the weight of the $j-th$ criterion, w_{j}.

$$w_j \leftarrow \frac{C_j}{\sum_{j=1}^{n} C_j} \tag{16}$$

3.2. IVNS-ELECTRE-I

Herein, we discuss the methodological steps for building IVNS-ELECTRE-I to select the best virtual simulation alternatives. Given a group of simulation alternatives $A = \{a_1, a_2, \dots, a_m\}$, and a group of the evaluation criteria, $A = \{a_1, a_2, \dots, a_n\}$, for academic stockholders. The implementation of IVNS-ELECTRE-I follow this sequence of steps:

Step 1, we take the IVNS values of the alternatives w.r.t different criteria as characterized by a weighted decision matrix, in which weights are obtained by the CRITIC method in Equations (16). The weighting procedure can be implemented in many ways.

$$Y = \left(z_{ij}\right) = \left(z_{ij}^1, z_{ij}^2, z_{ij}^3, \dots, z_{ij}^m\right), \quad z_{ij} = w_j \cdot x_{ij}. \tag{17}$$

In step 2, we calculate fuzzy concordance sets as follows:

$$\mathcal{F}_{pq} = \{1 \le j \le s : v_{pj} \ge v_{qj}, p \ne q; p, q \leftarrow 1,2,3 \cdots n\}, \quad v_{ij} = z_{ij}^1 + z_{ij}^2 + z_{ij}^3 + \cdots + z_{ij}^m. \tag{18}$$

Step 3, we calculate concordance indices according to the following:

$$\mathbb{f}_{pq} = \sum\nolimits_{j \in F_{pq}} w_j \; for \; all \; p, and \; q \tag{19}$$

Step 4, we compute the concordance matrix, $\boldsymbol{\mathcal{F}}$, according to the following:

$$\mathcal{F} = \begin{pmatrix} - & \mathbb{f}_{12} & \mathbb{f}_{13} & \cdots & \mathbb{f}_{1n} \\ \mathbb{f}_{21} & - & f_{23} & \cdots & \mathbb{f}_{2n} \\ \mathbb{f}_{31} & \mathbb{f}_{32} & - & \cdots & \mathbb{f}_{3n} \\ \vdots & \vdots & \vdots & \dots & \vdots \\ \mathbb{f}_{n1} & \mathbb{f}_{n2} & \mathbb{f}_{n3} & \cdots & - \end{pmatrix}. \tag{20}$$

Step 5, we calculate discordance indices, $\mathcal{G}_{pq}$, according to the following:

$$\mathcal{g}_{pq} = \frac{max_{j \in \mathcal{G}_{pq}} \sqrt{\frac{1}{m}\left[(z_{pj}^1 - z_{qj}^1)^2 + (z_{pj}^2 - z_{qj}^2)^2 + \cdots + \left(z_{pj}^m - z_{qj}^m\right)^2\right]}}{max_j \sqrt{\frac{1}{m}\left[(z_{pj}^1 - z_{qj}^1)^2 + (z_{pj}^2 - z_{qj}^2)^2 + \cdots + \left(z_{pj}^m - z_{qj}^m\right)^2\right]}} \tag{21}$$

Step 6, we calculate discordance matrix, $\boldsymbol{\mathcal{G}}$, according to the following:

$$\mathcal{G} = \begin{pmatrix} - & \mathcal{g}_{12} & \mathcal{g}_{13} & \cdots & \mathcal{g}_{1n} \\ \mathcal{g}_{21} & - & \mathcal{g}_{23} & \cdots & \mathcal{g}_{2n} \\ \mathcal{g}_{31} & \mathcal{g}_{32} & - & \cdots & \mathcal{g}_{3n} \\ \vdots & \vdots & \vdots & \dots & \vdots \\ \mathcal{g}_{n1} & \mathcal{g}_{n2} & \mathcal{g}_{n3} & \cdots & - \end{pmatrix}. \tag{22}$$

Step 7, we calculate the levels of concordance and discordance according to the following:

$$\overline{\not{f}} = \frac{1}{n(n-1)} \sum_{\substack{p=1 \\ p\neq q}}^{n} \sum_{\substack{q=1 \\ q\neq p}}^{n} \not{f}_{pq}\,, \qquad \overline{\not{g}} = \frac{1}{n(n-1)} \sum_{\substack{p=1 \\ p\neq q}}^{n} \sum_{\substack{q=1 \\ q\neq p}}^{n} \not{g}_{pq}\,. \tag{23}$$

Step 8, we build the dominance matrices based on concordance and discordance levels as formulated below:

$$H = \begin{pmatrix} - & h_{12} & h_{13} & \dots & h_{1n} \\ h_{21} & - & h_{23} & \dots & h_{2n} \\ h_{31} & h_{32} & - & \dots & h_{3n} \\ \vdots & \vdots & \vdots & \dots & \vdots \\ h_{n1} & h_{n2} & h_{n3} & \dots & - \end{pmatrix}, where\ h_{pq} = \begin{cases} 1, & \not{f}_{pq} \geq \overline{\not{f}}, \\ 0, & \not{f}_{pq} < \overline{\not{f}}. \end{cases} \tag{24}$$

$$L = \begin{pmatrix} - & l_{12} & l_{13} & \dots & l_{1n} \\ l_{21} & - & l_{23} & \dots & l_{2n} \\ l_{31} & l_{32} & - & \dots & l_{3n} \\ \vdots & \vdots & \vdots & \dots & \vdots \\ l_{n1} & l_{n2} & l_{n3} & \dots & - \end{pmatrix}, where\ l_{pq} = \begin{cases} 1, & g_{pq} < \overline{\not{g}}, \\ 0, & g_{pq} \geq \overline{\not{g}}. \end{cases} \tag{25}$$

Step 9, based on the multiplication of the above dominance matrices, we drive a new matrix, M, representing the aggregation of dominance.

$$M = \begin{pmatrix} - & m_{12} & m_{13} & \dots & m_{1n} \\ m_{21} & - & m_{23} & \dots & m_{2n} \\ m_{31} & m_{32} & - & \dots & m_{3n} \\ \vdots & \vdots & \vdots & \dots & \vdots \\ m_{n1} & m_{n2} & m_{n3} & \dots & - \end{pmatrix}. \tag{26}$$

Step 10, the alternatives are ranked according to outrank values of matrix M.

4. Results and Discussions

This section introduces a quantitative case study for our work based on a new university trying to implement virtual simulation to improve experimental teaching of IC design and testing. The decision maker reported that there are many qualitative and quantitative criteria to consider to be able to determine the effectiveness of this virtual education. For each criterion, fuzzy linguistic terms (such as "Very Low," "Low," "Moderate," "High," and "Very High") are used to express and characterize subjective decisions for each assessment criterion. These terms can be later mapped into IVNS. In Table 1, we provide different criteria, related descriptions, and linguistic terms.

Table 1. Summary of criteria in terms of name, description, and linguistic terms.

Criterion	ID	Description	Linguistic terms	Type

Ease of Use	C1	Simplicity of user interface and learning curve.	Difficult, Somewhat Difficult, Moderate, Easy, Very Easy	Cost
Realism Simulations	C2	Degree to which the platform replicates real IC behavior.	Poor, Fair, Good, Very Good, Excellent.	Benefit
Educational features	C3	Degree to which the students can benefit from the platform.	Nonexistent, Limited, Moderate, Extensive, Highly Comprehensive.	Benefit
Cost-Efficiency	C4	The value for money.	Very Expensive, Expensive, Moderate, Affordable, Very Affordable.	Cost
Hardware integration support	C5	The ability to integrate with hardware circuits.	Not Supported, Partially Supported, Moderately Supported, Well Supported, Fully Supported.	Benefit

The decision-maker investigation for simulation tools leads to four distinct alternatives as shown in Table 2.

Table 2. Summary of main educational IC simulation platforms

Simulator	ID	Source
LTspice	Alt1	https://www.analog.com/en/resources/design-tools-and-calculators/ltspice-simulator.html
Proteus	Alt2	https://www.labcenter.com/
PSpice	Alt3	https://www.cadence.com/en_US/home/tools/pcb-design-and-analysis/analog-mixed-signal-simulation/pspice.html
Tina-TI	Alt4	https://www.cadence.com/en_US/home/tools/pcb-design-and-analysis/analog-mixed-signal-simulation/pspice.html
NI Multisim	Alt5	https://www.ni.com/en/support/downloads/software-products/download.multisim.html#452133
SPICE	Alt6	https://publichealth.buffalo.edu/cat/kt4tt/best-practices/need-to-knowledge-ntk-model/ntk-commercial-devices/master-list-of-tools/electrical-electronic-engineering/spice--simulation-program-with-integrated-circuit-emphasis--.html

Using the linguistic variables and criteria mentioned above, we build IVNS-based decision matrix as shown in Table 3.

Table

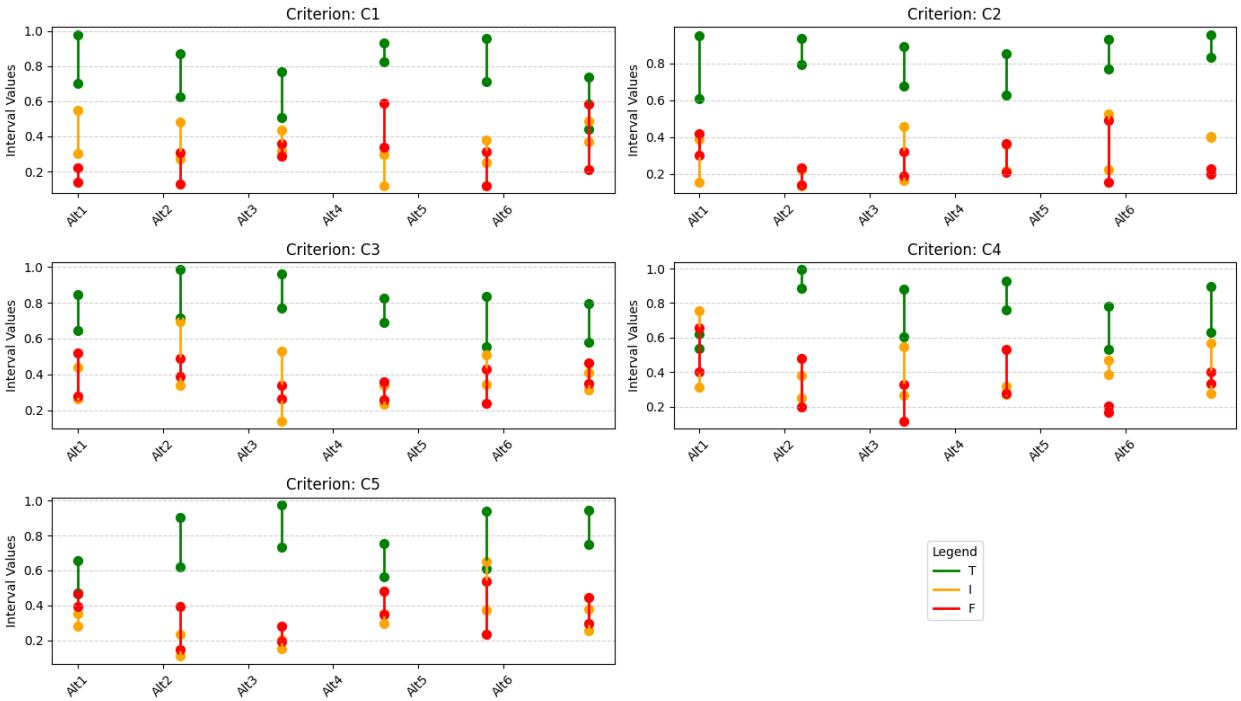

Figure 2. Visualization of IVN components for different alternatives with respect to each criterion.

3. IVNS-based decision matrix.

Criteria / Alternative	C1	C2	C3	C4	C5
Alt1	<[0.7005, 0.9799], [0.302, 0.5471], [0.1409, 0.2194]>	<[0.6068, 0.9488], [0.1558, 0.3894], [0.2996, 0.4197]>	<[0.6471, 0.8459], [0.2634, 0.4383], [0.2821, 0.5221]>	<[0.5396, 0.6196], [0.3112, 0.7575], [0.3997, 0.6581]>	<[0.4651, 0.6566], [0.2805, 0.3523], [0.3932, 0.4719]>
Alt2	<[0.6253, 0.8706], [0.2724, 0.4818], [0.1277, 0.3081]>	<[0.7909, 0.937], [0.1344, 0.2202], [0.1414, 0.2326]>	<[0.7141, 0.983], [0.3384, 0.6968], [0.3902, 0.488]>	<[0.8844, 0.9928], [0.2504, 0.3818], [0.1973, 0.4802]>	<[0.623, 0.9005], [0.1101, 0.2356], [0.1466, 0.3919]>
Alt3	<[0.5065, 0.7682], [0.3155, 0.4341], [0.2863, 0.3568]>	<[0.6761, 0.8892], [0.1656, 0.4542], [0.1896, 0.3209]>	<[0.7679, 0.9618], [0.1415, 0.5318], [0.2672, 0.3406]>	<[0.6061, 0.8784], [0.2668, 0.5466], [0.1172, 0.3312]>	<[0.7311, 0.9745], [0.1545, 0.2031], [0.1933, 0.2822]>
Alt4	<[0.8229, 0.9329], [0.1183, 0.2962], [0.3401, 0.59]>	<[0.6263, 0.8531], [0.2158, 0.3597], [0.2072, 0.3659]>	<[0.6898, 0.8269], [0.237, 0.3403], [0.2587, 0.3605]>	<[0.7617, 0.9264], [0.2722, 0.3198], [0.2791, 0.5303]>	<[0.5632, 0.7563], [0.2973, 0.3534], [0.3465, 0.4842]>
Alt5	<[0.7134, 0.9568], [0.2515, 0.3803], [0.1201, 0.3124]>	<[0.7701, 0.9311], [0.2234, 0.5247], [0.1563, 0.4903]>	<[0.5551, 0.8372], [0.3438, 0.5111], [0.239, 0.4281]>	<[0.5298, 0.7795], [0.3851, 0.4715], [0.1669, 0.2018]>	<[0.6119, 0.9377], [0.3722, 0.6511], [0.2354, 0.54]>
Alt6	<[0.4433, 0.7357], [0.3718, 0.4867], [0.2116, 0.5825]>	<[0.8307, 0.9555], [0.3994, 0.4052], [0.1981, 0.2251]>	<[0.5814, 0.7971], [0.3148, 0.4102], [0.3499, 0.4649]>	<[0.6292, 0.895], [0.2783, 0.5697], [0.3316, 0.4038]>	<[0.749, 0.9417], [0.2582, 0.381], [0.2961, 0.4443]>

Next, we perform deneutrosophication of the decision matrix and apply the CRITIC method to obtain the quantitative weights of each criterion as displayed in Table 4.

Table 4. CRITIC weights for different criteria based on deneutrosophication of IVNS-based decision matrix.

Criteria / Alternative	C1	C2	C3	C4	C5
Alt1	1.014404	0.852345	0.838859	0.751499	0.573253
Alt2	0.897298	0.901605	0.959867	1.023631	0.792896
Alt3	0.701928	0.922753	1.017228	0.916603	0.848929
Alt4	0.922042	0.814195	0.802433	0.879102	0.683934
Alt5	0.94657	1.01425	0.798023	0.777058	0.914423
Alt6	0.701516	0.971319	0.732753	0.871024	0.898862
CRITIC weights	0.1907	0.1443	0.2477	0.2608	0.1562

The concordance matrix, as displayed in Table 5, provides a hot summary of levels of agreement between the alternatives. Each of the tabulated values represents the degree to which one alternative is concordant with another according to the weighted standing of the criteria.

Table 5. Concordance Matrix

	Alt1	Alt2	Alt3	Alt4	Alt5	Alt6
Alt1	0.0000	0.7976	0.8080	0.3740	1.0000	0.9544
Alt2	0.3432	0.0000	0.1681	0.0725	0.3562	0.3106
Alt3	0.9159	0.5726	0.0000	0.6452	0.7171	0.1464
Alt4	0.2707	0.4615	0.6296	0.0000	0.6756	0.6300
Alt5	0.1988	0.7227	0.6425	0.2991	0.0000	0.2754
Alt6	0.9171	0.6657	0.8338	0.6464	0.7183	0.0000

In Table 6, we provide a discordance matrix, representing the degree of divergence between alternatives. This can be regarded as a complementary matrix for the concordance matrix, in which we can identify the areas in which noteworthy differences occur between alternatives.

Table 6. Discordance Matrix

	Alt1	Alt2	Alt3	Alt4	Alt5	Alt6
Alt1	0.0000	1.0000	0.6569	0.3431	1.0000	0.6569
Alt2	0.2941	0.0000	0.2941	0.5392	0.4902	0.4804

Alt3	0.4216	0.3431	0.0000	0.6667	0.6176	0.4706
Alt4	0.4902	0.8039	0.4902	0.0000	0.8039	0.6765
Alt5	0.4118	0.1275	0.0980	0.4118	0.0000	0.4608
Alt6	0.6176	0.4902	0.3922	0.9608	0.9118	0.0000

The dominance matrix, shown in Table 7, integrates the information from the concordance and discordance matrices to determine the dominance relationships among alternatives. Each entry reflects whether one alternative outranks another based on the combined evaluation of concordance and discordance thresholds.

Table 7. Dominance Matrix

	Alt1	Alt2	Alt3	Alt4	Alt5	Alt6
Alt1	0.0	0.0	0.0	0.0	0.0	0.0
Alt2	1.0	0.0	0.0	1.0	0.0	0.0
Alt3	0.0	0.0	0.0	1.0	0.0	0.0
Alt4	0.0	0.0	0.0	0.0	0.0	0.0
Alt5	0.0	0.0	0.0	0.0	0.0	0.0
Alt6	0.0	0.0	0.0	0.0	0.0	0.0

In Figure 3, we provide a graphical representation of the dominance interactions among the alternatives according to the results of the decision-making process. As shown, the comparative strengths of each alternative are represented by directed edges between nodes. Also, sensitivity analysis is performed to figure shows the effect of changing the thresholding parameters $(\overline{\mathfrak{f}}, \overline{\mathfrak{g}})$ on the total dominance. It could be noted that these parameters require careful choice to better determine the final dominance matrix. In Table 8, we provide a comparative analysis results, in which compare the ranking results obtained using the IVNS-ELECTRE-I method and Combinative Distance-Based Assessment(CODAS) reported in [11]. The tabulated results explain the differences in the ranking of different simulation tools for IC design education. When it comes to the IVNS-ELECTRE-I method, Alt2 is ranked the highest, followed by Alt3 and Alt5, while the rankings in CODAS place Alt3 first, followed by Alt5 and Alt6. Notably, the alternative Alt6 in the IVNS-ELECTRE-I method ranks higher than in CODAS, where it is positioned in the lower tiers. On the other hand, Alt1 appears lower in the IVNS-ELECTRE-I method compared to CODAS, reflecting the differing methodologies used to assess the alternatives. Alt4 consistently holds the last rank in both methods.

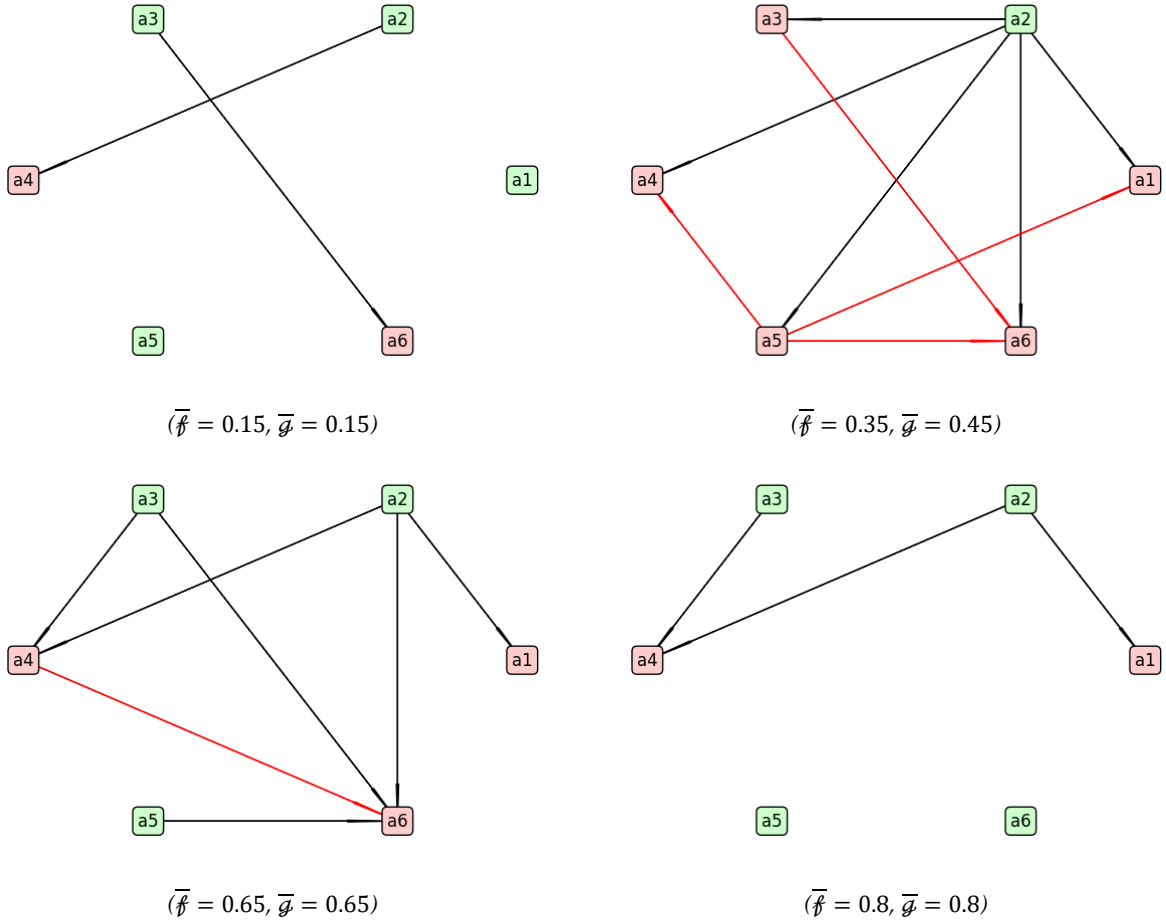

Figure 3. visualization of the dominance relationships between different alternatives

Table 8. Comparative analysis

Rank	IVNS-ELECTRE-I	[11]
1	Alt2	Alt3
2	Alt3	Alt5
3	Alt5	Alt6
4	Alt6	Alt1
5	Alt1	Alt2
6	Alt4	Alt4

5. Conclusions

In this paper, we presented a neutrosophic decision-making framework that integrates IVNS to model uncertainty in the virtual simulation of experimental education of ICs. Our framework

integrates customized CRITIC weighting to get objective insights about the importance of each alternative. Then, we drive an IVNS-ELECTRE-I approach to make multi-criteria decisions about uncertain and imprecise settings of learning tools. Our framework exploited the flexibility of IVNS to model truth, indeterminacy, and falsity degrees, which reflect a richer representation of uncertainty compared to traditional techniques. The results of a numerical case study demonstrate the effectiveness of the framework through detailed computations of concordance, discordance, and dominance matrices.

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Risk Assessment of Steel Structure Engineering Cost using Multi-Criteria Decision-Making Approach under Uncertainty

Bijun Ni*
Department of Construction Engineering and Art Design,
Suzhou Vocational Institute of Industrial Technology (SIIT)，Suzhou, 215000, Jiangsu, China
*Corresponding author, E-mail: 18550464769@163.com

Abstract: Choosing options with complex, ambiguous, and contradicting criteria is a common challenge for decision-makers. To choose the optimal option, they typically use multicriteria decision-making techniques, in which specialists manage the relative importance of conflicting tangible and intangible criteria to identify and evaluate possible courses of action. Making decisions involves one or more people selecting the best option from a range of potential options. This study proposes multi-criteria decision making (MCDM) method to compute the criteria weights are rank the alternatives. The MCDM approach is applied for risk assessment to cost in steel structure engineering. This problem includes many criteria and alternatives. The MCDM method is used under the plithogenic set to deal with uncertainty in the evaluation process. The grey relational analysis (GRA) is an MCDM method used in this study to rank the alternatives. Eight criteria and eight alternatives are gathered to apply the MCDM approach. The sensitivity analysis is conducted in this study.

Keywords: Risk Assessment; Steel Structure; MCDM Approach; Uncertainty; Decision-Making.

1. Introduction

Steel building projects are among the many construction projects that have experienced cost overruns in recent years. Stakeholder satisfaction, project completion, quality and safety, and financial profitability can all be significantly impacted by cost overruns. Depending on the project's size, complexity, location, and nature, the typical percentage of construction project overruns might vary significantly[1], [2].

Project cost overruns can be caused by a variety of factors, but the four primary ones are interface integration, manufacturing management, construction management, and design topology and management. The contractors always make every effort to prevent or mitigate project cost overruns, including having an appropriate project plan, timely project monitoring, and an accurate cost risk estimate, because a cost overrun can cause significant financial losses and profitability for the construction company[3], [4].

Traditional construction contingency research typically uses data analysis methods like statistics and artificial intelligence to methodically sift, categorize, and encode extensive cost databases at

the industry level (macro-level). These results often show the information at the macro level[5], [6]. Each steel building project, however, has unique features that might contribute to cost overruns. It's possible that the project-specific level cannot effectively use the industry-level data. Project managers rarely recognize possible project cost overruns, thus they rely solely on macro-level data to create an efficient customized cost management system[7], [8]. So the risk assessment of cost in steel structure is a MCDM problem with a set of criteria and alternatives[9], [10].

Making judgments when there are several competing criteria is known as multiple criteria decision-making (MCDM). For instance, when a family wishes to purchase a car, they need to think about factors like cost, comfort, safety, fuel efficiency, upkeep, and so forth [11], [12]. There is often a disagreement between these characteristics. Selecting the most affordable, safest, and most comfortable car is a challenging challenge. Buying an automobile is just one of several MCDM issues that are more complex. Making decisions is made much more difficult by the world's growing complexity and unpredictability[13], [14].

A decision maker (DM) must select from a range of potential solutions the one that best meets the assessment criteria in MCDM situations. The DM's experience is crucial in determining how to balance these opposing qualities and get a scientific conclusion. To address this kind of issue, numerous MCDM techniques have been created to date[15], [16].

Fuzzy set theory, which Zadeh initially presented in 1965, is one of the most prominent topics in the previous century for grasping the idea of skepticism and unpredictability difficulties of science. Engineering, architectural modeling, networking science, decision-making processes, and many other practical issues can be resolved with the help of vagueness theory. With the fascinating interpretation of both membership and non-membership functions, Atanassov expanded the concept of fuzzy set theory and presented the significant idea of intuitionistic fuzzy theory. With nourishment, researchers have recently expanded the uncertainty region into new domains and directions, such as triangular, trapezoidal, pentagonal, hexagonal, and heptagonal numbers. Liu and Yuan, respectively, developed the basic idea of a trapezoidal and triangular intuitionistic fuzzy set[17], [18]. Additionally, researchers developed a variety of novel approaches to analytically sketch the concepts and promoted a few enhanced versions of unclear parameters.

Smarandache recently developed the idea that the neutrosophic set perception is composed of three distinct elements: truthiness, indeterminacy, and falseness. The perception of a single-typed neutrosophic set was later proven by Wang et al., and this is a very relevant factor to handle complex problems[19], [20]. As the study went on, Chakraborty et al. developed the robust idea of triangular and trapezoidal neutrosophic numbers to solve a variety of real-world issues.

The plithogenic version of neutrosophic logic, which examines the genesis, character, and extent of neutralities, environments of ambiguity, and their interactions, is employed as a supplement to existing techniques. Plithogeny, this section, promotes the integration and linkages of theories and concepts across a range of scientific domains. Plithogeny is the process by which new entities are created, formed, developed, and evolved from the dynamics and mergers of several

conflicting, neutral, and/or non-contradictory prior entities. Plithogeny promotes the linking and unification of concepts and hypotheses across a range of scientific domains[21], [22].

Motivation of this Study

This study article's primary objective is to support the GRA scheme to promote the gray system and assess the optimal option when dealing with imprecise datasets. We are motivated to carry out our current research effort because the following questions come to mind throughout survey studies to better control and execute the decision-making dilemma.

(1) How plithogenic set can deal with the inaccurate data be included in a realistic model of MCDM?

(2) Which operator in mathematics is suitable for combining the underlying data?

(3) Which method will be most effective in capturing the problem's grey knowledge?

(4) Does our study require an additional method to analyze a debate based on comparative knowledge?

(5) Is the method we used steady and reliable?

Novelty of this study

These days, scientists have demonstrated their interest in developing theories related to the plithogenic domain to support its many applications in many plithogenic fields. Our overarching goal in this study work is to concentrate on MCDM model, which are as follows:

(1) Interpreting the MCDM approach using the plithogenic operators.

(2) To solve our suggested MCDM problem, implement the GRA concept in the plithogenic domain.
(3) Analysis of the ranking results' sensitivity.

(4) A case study of the MCDM is solved using the GRA method to compute the criteria weights and rank the alternatives.

2. Multi-criteria group decision-making problem in plithogenic environment

This section shows the steps of the proposed approach as shown in Figure 1. We evaluate the criteria and alternatives using the plithogenic numbers. Then we apply the plithogenic operators to combine the plithogenic numbers. Then we convert these plithogenic numbers into a crisp value.

2.1 GRA Approach

This part shows the steps of the GRA methodology under the uncertainty framework to rank the alternatives[23], [24]. Various models and techniques have been employed thus far to assess the alternatives. Grey Relational Analysis (GRA) is one of the MCDM models in uncertain situations. This approach, which employs grey numbers, evaluates ambiguity and inadequate information. Interval scale can be used to convert numbers of concepts into numerical values, and grey

numbers represent the decision maker's personal preferences. Even though GRA has been the basis for numerous studies, the GRA technique has limitations and shortcomings. We create a modified, grey-based approach as a MCDM method under inadequate information to get beyond these limitations.

Step 1: Build the decision matrix.

Step 2. Normalize the decision matrix

$$u_{ij} = \frac{x_{ij}}{\sqrt{\sum_{i=1}^{m} x_{ij}^2}}; i = 1, \dots, m; j = 1, \dots, n \quad (1)$$

Step 3. Compute the weighted decision matrix.

$$r_{ij} = w_j u_{ij} \quad (2)$$

Step 4. Compute the grey relational coefficients for positive and negative criteria.

$$q_{ij}^{+} = \frac{m^{+} + dM^{+}}{k_{ij}^{+} + dM^{+}} \quad (3)$$

Where $k_{ij}^{+} = \left| u_{ij} - r_j^{+} \right|$ (4)

$$m^{+} = \min_i \min_j k_{ij}^{+} \quad (5)$$

$$M^{+} = \max_i \max_j k_{ij}^{+} \quad (6)$$

And d refers to the distinguishing coefficient.

$$q_{ij}^{-} = \frac{m^{-} + dM^{-}}{k_{ij}^{-} + dM^{-}} \quad (7)$$

Where $k_{ij}^{-} = \left| u_{ij} - r_j^{-} \right|$ (8)

$$m^{-} = \min_i \min_j k_{ij}^{-} \quad (9)$$

$$M^{-} = \max_i \max_j k_{ij}^{-} \quad (10)$$

Step 5. Compute the grey relational grade.

$$F_i^{+} = \frac{1}{n} \sum_{j=1}^{n} q_{ij}^{+} \quad (11)$$

$$F_i^{-} = \frac{1}{n} \sum_{j=1}^{n} q_{ij}^{-} \quad (12)$$

Step 6. Compute the relative grey relational grade.

$$F_i = \frac{F_i^{+}}{F_i^{+} + F_i^{-}} \quad (13)$$

Step 7. Rank the alternatives.

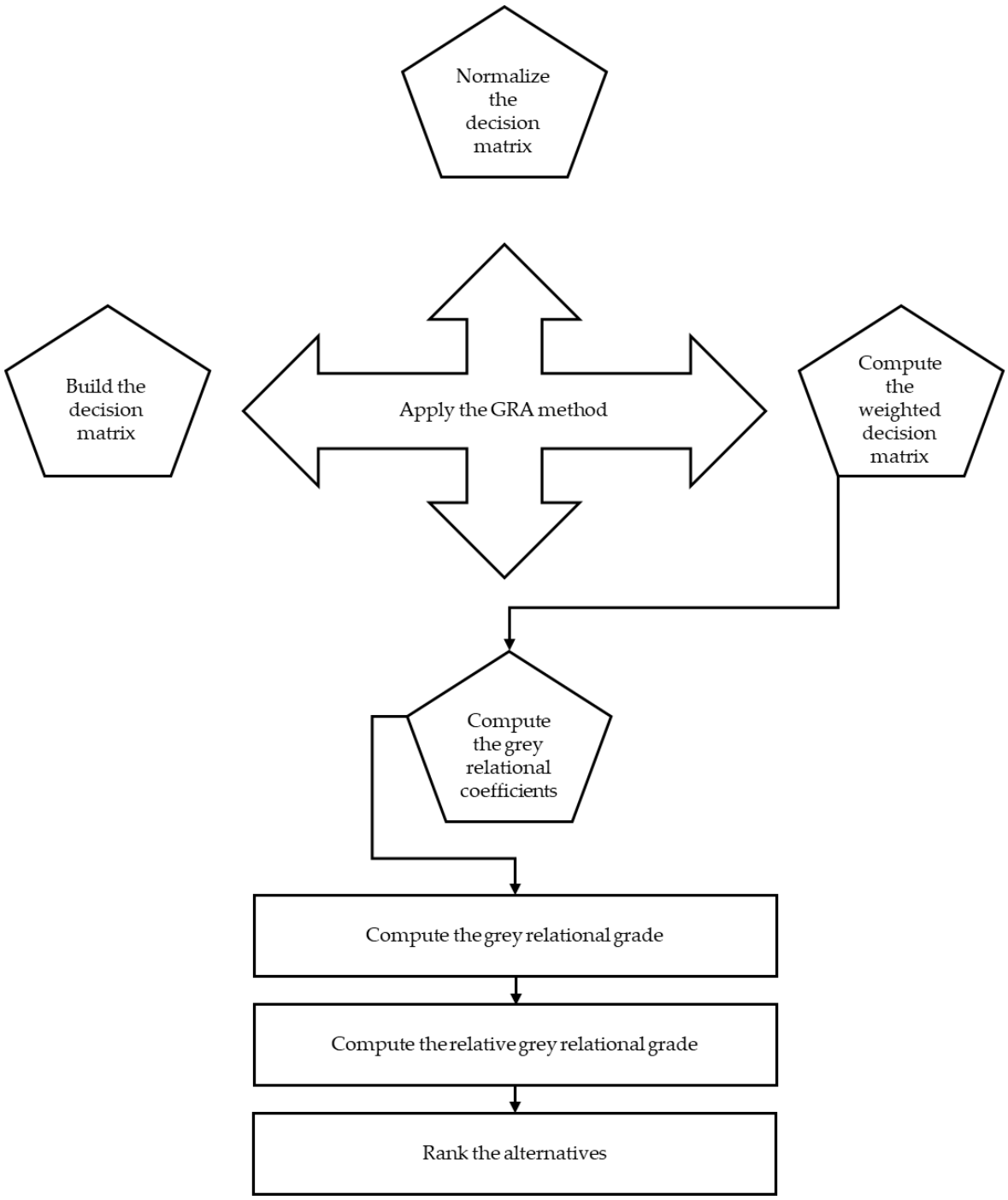
Normalize the decision matrix
Build the decision matrix
Apply the GRA method
Compute the weighted decision matrix
Compute the grey relational coefficients
Compute the grey relational grade
Compute the relative grey relational grade
Rank the alternatives

Figure 1. The details of the proposed approach.

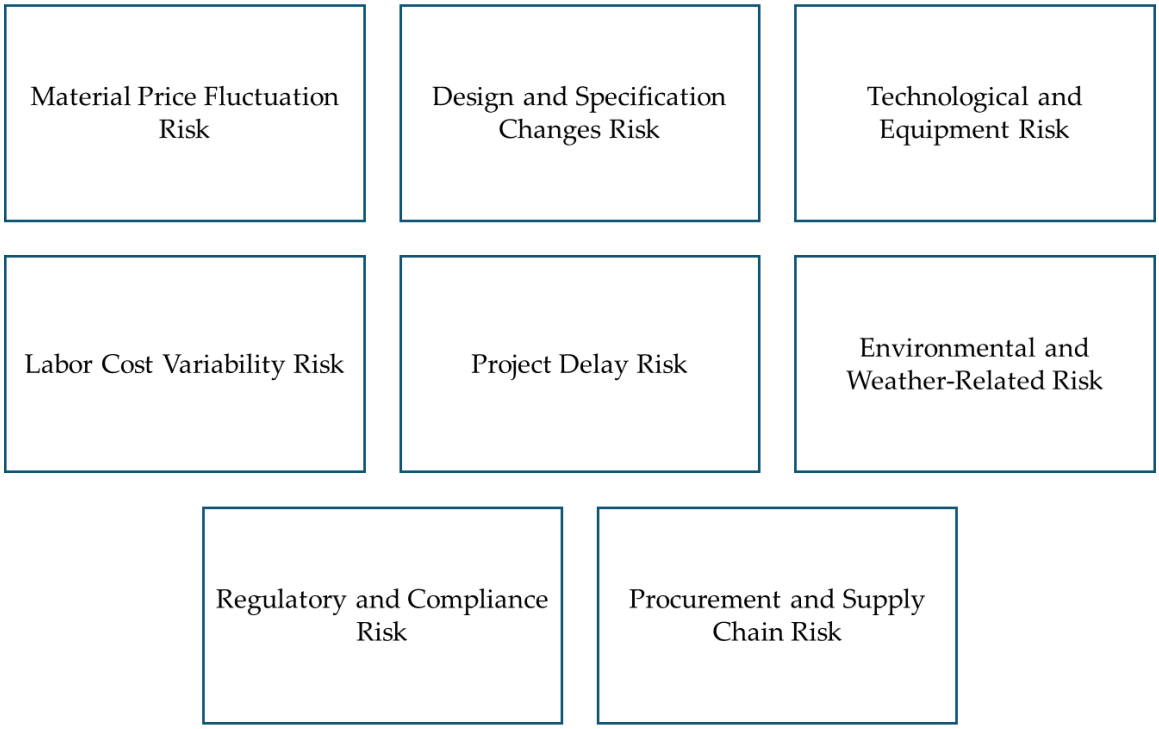
Material Price Fluctuation Risk
Design and Specification Changes Risk
Technological and Equipment Risk
Labor Cost Variability Risk
Project Delay Risk
Environmental and Weather-Related Risk
Regulatory and Compliance Risk
Procurement and Supply Chain Risk

Figure 2. The criteria of this study.

3. Numerical Example

This section shows the steps of the proposed approach. We let three experts evaluate the criteria and alternatives. They gathered eight criteria and eight alternatives as shown in Figure 2. Then we replace their opinions by using the plithogenic numbers as shown in table 1. Then we used plithogenic operators to combine various plithogenic numbers. Then we apply the score function to obtain crisp values. Then we compute the criteria weights by normalizing these values as shown in Table 2 and figure 3.

Table 1. The plithogenic numbers.

	C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8
A_1	(0.25, 0.60, 0.80)	(0.80, 0.10, 0.30)	(0.95, 0.05, 0.05)	(0.80, 0.10, 0.30)	(0.65, 0.30, 0.45)	(0.50, 0.40, 0.60)	(0.25, 0.60, 0.80)	(0.25, 0.60, 0.80)
A_2	(0.80, 0.10, 0.30)	(0.25, 0.60, 0.80)	(0.25, 0.60, 0.80)	(0.65, 0.30, 0.45)	(0.25, 0.60, 0.80)	(0.25, 0.60, 0.80)	(0.80, 0.10, 0.30)	(0.25, 0.60, 0.80)
A_3	(0.80, 0.10, 0.30)	(0.80, 0.10, 0.30)	(0.80, 0.10, 0.30)	(0.25, 0.60, 0.80)	(0.80, 0.10, 0.30)	(0.80, 0.10, 0.30)	(0.80, 0.10, 0.30)	(0.80, 0.10, 0.30)
A_4	(0.95, 0.05, 0.05)	(0.80, 0.10, 0.30)	(0.80, 0.10, 0.30)	(0.80, 0.10, 0.30)	(0.80, 0.10, 0.30)	(0.80, 0.10, 0.30)	(0.95, 0.05, 0.05)	(0.80, 0.10, 0.30)
A_5	(0.95, 0.05, 0.05)	(0.95, 0.05, 0.05)	(0.95, 0.05, 0.05)	(0.80, 0.10, 0.30)	(0.95, 0.05, 0.05)	(0.95, 0.05, 0.05)	(0.95, 0.05, 0.05)	(0.95, 0.05, 0.05)

	A_1	A_2	A_3	A_4	A_5	A_6	A_7	A_8
C_1	(0.95, 0.05, 0.05)	(0.80, 0.10, 0.30)	(0.65, 0.30, 0.45)	(0.50, 0.40, 0.60)	(0.40, 0.70, 0.50)	(0.40, 0.70, 0.50)	(0.80, 0.10, 0.30)	(0.95, 0.05, 0.05)
C_2	(0.65, 0.30, 0.45)	(0.95, 0.05, 0.05)	(0.80, 0.10, 0.30)	(0.65, 0.30, 0.45)	(0.50, 0.40, 0.60)	(0.40, 0.70, 0.50)	(0.40, 0.70, 0.50)	(0.50, 0.40, 0.60)
C_3	(0.95, 0.05, 0.05)	(0.80, 0.10, 0.30)	(0.65, 0.30, 0.45)	(0.50, 0.40, 0.60)	(0.40, 0.70, 0.50)	(0.50, 0.40, 0.60)	(0.40, 0.70, 0.50)	(0.50, 0.40, 0.60)
C_4	(0.80, 0.10, 0.30)	(0.95, 0.05, 0.05)	(0.80, 0.10, 0.30)	(0.65, 0.30, 0.45)	(0.50, 0.40, 0.60)	(0.40, 0.70, 0.50)	(0.40, 0.70, 0.50)	(0.40, 0.70, 0.50)
C_5	(0.95, 0.05, 0.05)	(0.65, 0.30, 0.45)	(0.95, 0.05, 0.05)	(0.80, 0.10, 0.30)	(0.65, 0.30, 0.45)	(0.50, 0.40, 0.60)	(0.40, 0.70, 0.50)	(0.25, 0.60, 0.80)
C_6	(0.95, 0.05, 0.05)	(0.65, 0.30, 0.45)	(0.95, 0.05, 0.05)	(0.80, 0.10, 0.30)	(0.65, 0.30, 0.45)	(0.50, 0.40, 0.60)	(0.40, 0.70, 0.50)	(0.10, 0.75, 0.85)
C_7	(0.95, 0.05, 0.05)	(0.80, 0.10, 0.30)	(0.65, 0.30, 0.45)	(0.50, 0.40, 0.60)	(0.40, 0.70, 0.50)	(0.50, 0.40, 0.60)	(0.40, 0.70, 0.50)	(0.40, 0.70, 0.50)
C_8	(0.95, 0.05, 0.05)	(0.80, 0.10, 0.30)	(0.65, 0.30, 0.45)	(0.50, 0.40, 0.60)	(0.40, 0.70, 0.50)	(0.50, 0.40, 0.60)	(0.40, 0.70, 0.50)	(0.40, 0.70, 0.50)

	A_1	A_2	A_3	A_4	A_5	A_6	A_7	A_8
C_1	(0.10, 0.75, 0.85)	(0.95, 0.05, 0.05)	(0.80, 0.10, 0.30)	(0.65, 0.30, 0.45)	(0.95, 0.05, 0.05)	(0.65, 0.30, 0.45)	(0.80, 0.10, 0.30)	(0.95, 0.05, 0.05)
C_2	(0.80, 0.10, 0.30)	(0.10, 0.75, 0.85)	(0.95, 0.05, 0.05)	(0.80, 0.10, 0.30)	(0.65, 0.30, 0.45)	(0.95, 0.05, 0.05)	(0.80, 0.10, 0.30)	(0.65, 0.30, 0.45)
C_3	(0.95, 0.05, 0.05)	(0.10, 0.75, 0.85)	(0.95, 0.05, 0.05)	(0.80, 0.10, 0.30)	(0.65, 0.30, 0.45)	(0.95, 0.05, 0.05)	(0.40, 0.70, 0.50)	(0.50, 0.40, 0.60)
C_4	(0.80, 0.10, 0.30)	(0.65, 0.30, 0.45)	(0.10, 0.75, 0.85)	(0.95, 0.05, 0.05)	(0.80, 0.10, 0.30)	(0.65, 0.30, 0.45)	(0.95, 0.05, 0.05)	(0.40, 0.70, 0.50)
C_5	(0.65, 0.30, 0.45)	(0.50, 0.40, 0.60)	(0.80, 0.10, 0.30)	(0.10, 0.75, 0.85)	(0.95, 0.05, 0.05)	(0.80, 0.10, 0.30)	(0.65, 0.30, 0.45)	(0.10, 0.75, 0.85)
C_6	(0.50, 0.40, 0.60)	(0.10, 0.75, 0.85)	(0.95, 0.05, 0.05)	(0.80, 0.10, 0.30)	(0.65, 0.30, 0.45)	(0.95, 0.05, 0.05)	(0.80, 0.10, 0.30)	(0.10, 0.75, 0.85)
C_7	(0.40, 0.70, 0.50)	(0.10, 0.75, 0.85)	(0.10, 0.75, 0.85)	(0.95, 0.05, 0.05)	(0.80, 0.10, 0.30)	(0.65, 0.30, 0.45)	(0.95, 0.05, 0.05)	(0.95, 0.05, 0.05)
C_8	(0.25, 0.60, 0.80)	(0.10, 0.75, 0.85)	(0.95, 0.05, 0.05)	(0.80, 0.10, 0.30)	(0.65, 0.30, 0.45)	(0.95, 0.05, 0.05)	(0.80, 0.10, 0.30)	(0.80, 0.10, 0.30)

	A_6	A_7	A_8
C_1	(0.10, 0.75, 0.85)	(0.80, 0.10, 0.30)	(0.95, 0.05, 0.05)
C_2	(0.95, 0.05, 0.05)	(0.10, 0.75, 0.85)	(0.65, 0.30, 0.45)
C_3	(0.95, 0.05, 0.05)	(0.95, 0.05, 0.05)	(0.10, 0.75, 0.85)
C_4	(0.95, 0.05, 0.05)	(0.95, 0.05, 0.05)	(0.95, 0.05, 0.05)
C_5	(0.95, 0.05, 0.05)	(0.95, 0.05, 0.05)	(0.10, 0.75, 0.85)
C_6	(0.95, 0.05, 0.05)	(0.10, 0.75, 0.85)	(0.10, 0.75, 0.85)
C_7	(0.95, 0.05, 0.05)	(0.10, 0.75, 0.85)	(0.95, 0.05, 0.05)
C_8	(0.95, 0.05, 0.05)	(0.65, 0.30, 0.45)	(0.80, 0.10, 0.30)

Table 2. The criteria weights.

	Weights	Rank
C_1	0.130725	8
C_2	0.126733	6
C_3	0.123146	2
C_4	0.128831	7
C_5	0.126123	5
C_6	0.11607	1
C_7	0.124066	3
C_8	0.124305	4

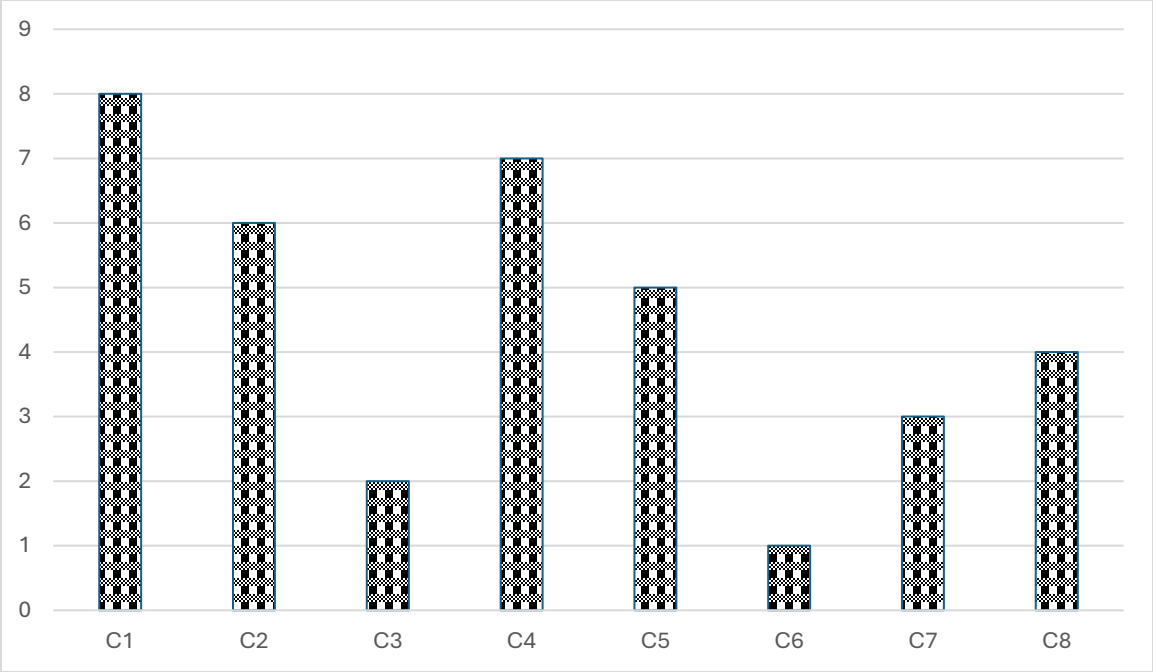

Figure 3. The ranks of criteria.

Then we apply the steps of the plithogenic-GRA methodology to rank the alternatives. We normalize the decision matrix as shown in Table 3 and relations between these values are shown in figure 4. Then we compute the weighted normalized decision matrix.

Table 3. The normalized matrix.

	C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8
A_1	0.266498	0.385242	0.433663	0.397305	0.34256	0.296882	0.199584	0.303221
A_2	0.399655	0.276388	0.243648	0.37616	0.268075	0.214946	0.365123	0.244141
A_3	0.371456	0.414487	0.406735	0.235642	0.41669	0.449818	0.34012	0.407558
A_4	0.356702	0.385242	0.379254	0.39337	0.337122	0.430531	0.414854	0.380021

A_5	0.370854	0.369939	0.356091	0.366792	0.411198	0.407618	0.395419	0.356811
A_6	0.215709	0.384616	0.409281	0.34439	0.391934	0.428936	0.354703	0.410109
A_7	0.389015	0.257172	0.316654	0.378756	0.351577	0.274433	0.291189	0.335486
A_8	0.410662	0.322593	0.220734	0.306249	0.277857	0.229158	0.414854	0.360388

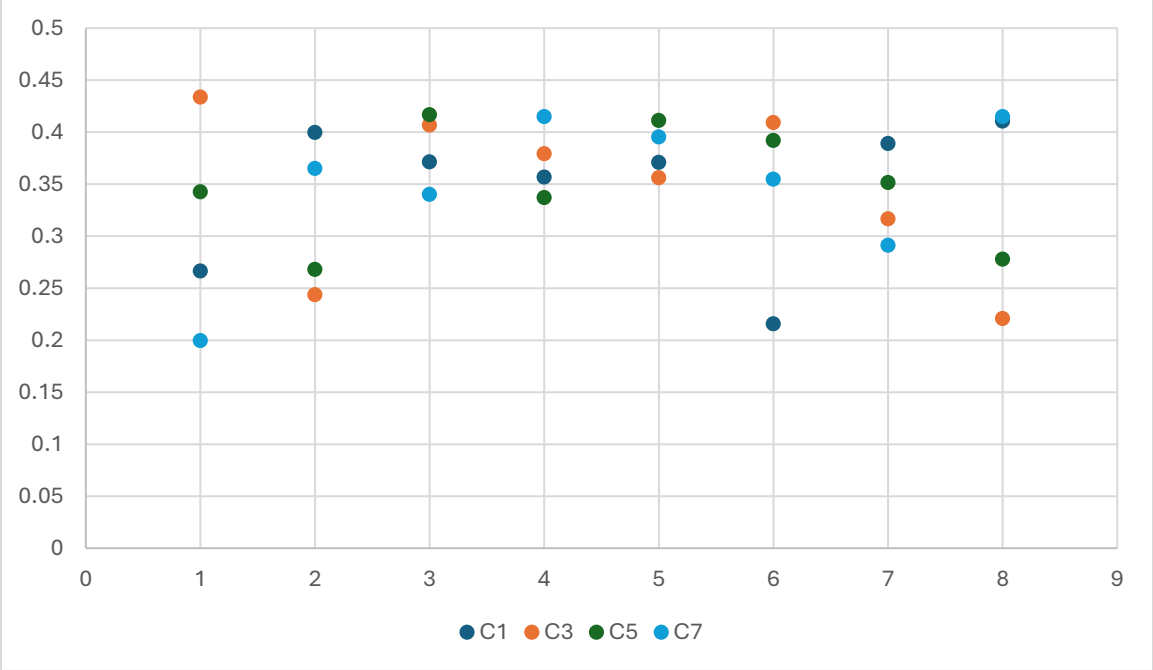

Figure 4. Relations between criteria and alternatives in normalization matrix.

Eq. (3) is used to compute the grey relational coefficient. Eq. (11) is used to compute the grey relational grade. Eq. (13) is used to compute the relative grey relational grade. We rank the alternatives as shown in Table 4 and figure 5.

Table 4. The rank of alternatives.

	F_i^+	F_i^-	F_i	Rank
A_1	0.423298	0.990779	0.299346	5
A_2	0.627571	0.711546	0.468645	8
A_3	0.266815	1.384942	0.161534	1
A_4	0.295717	0.74759	0.283442	4
A_5	0.293804	0.767173	0.276918	3
A_6	0.276851	1.426373	0.162545	2
A_7	0.459617	0.720922	0.389328	7
A_8	0.508939	0.867091	0.36986	6

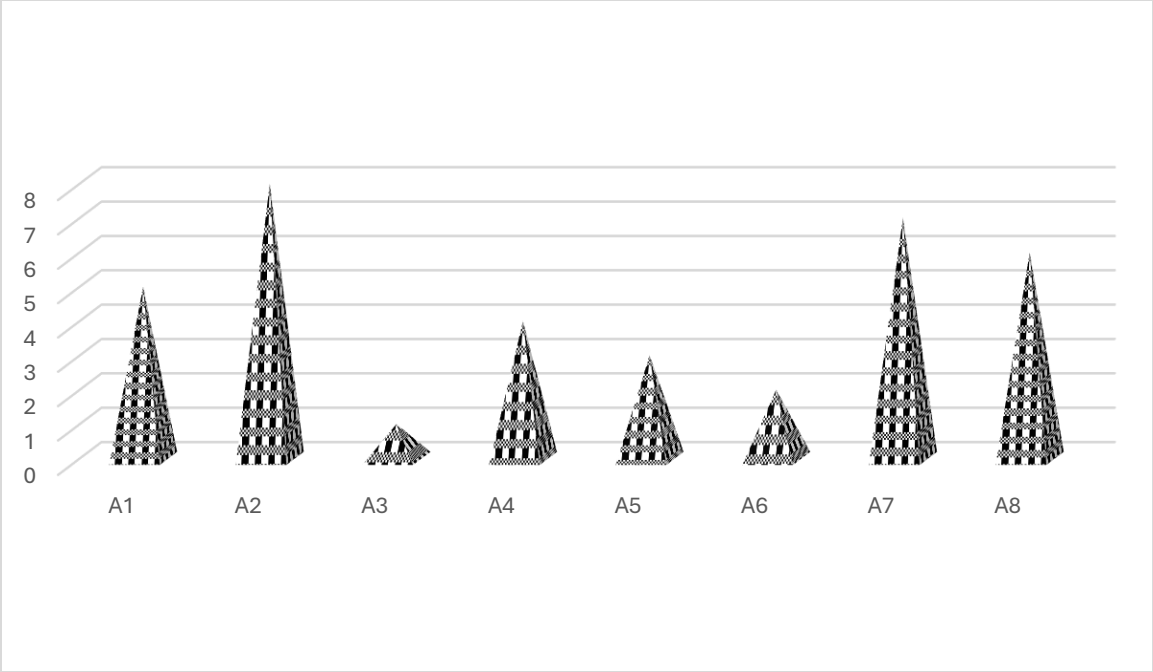

Figure 5. The rank of alternatives.

4. Sensitivity Analysis

This section shows the sensitivity analysis to show the stability of the rank. We change in d parameter value between 0.1 to 0.9 then we apply the GRA approach to rank the alternatives. Then we ranked the alternatives as shown in Table 5 and figure 6. The results show the rank of the alternatives is stable under different values of d parameters. We show alternative 2 is the best and alternative 3 is the worst.

Table 5. The rank of alternatives under sensitivity analysis.

	d=0.1	**d=0.2**	**d=0.3**	**d=0.4**	**d=0.5**	**d=0.6**	**d=0.7**	**d=0.8**	**d=0.9**
A_1	5	5	5	5	5	5	5	5	5
A_2	8	8	8	8	8	8	8	8	8
A_3	1	1	1	1	1	1	1	1	1
A_4	4	4	4	4	4	4	4	4	4
A_5	3	3	3	3	3	3	3	3	3
A_6	2	2	2	2	2	2	2	2	2
A_7	7	7	7	7	7	7	7	7	7
A_8	6	6	6	6	6	6	6	6	6

The results show alternative 2 is the best in all cases, followed by alternative 7, alternative 8, alternative 1, alternative 4, alternative 5, and alternative 6. We observed that alternative 3 is the worst of all alternatives. So, the proposed model is stable and effective and due to its outcome,

alternative 2 is the best under different cases and scenarios. The proposed model can be applied to the different MCDM issues to rank the alternatives due to it stable under different cases.

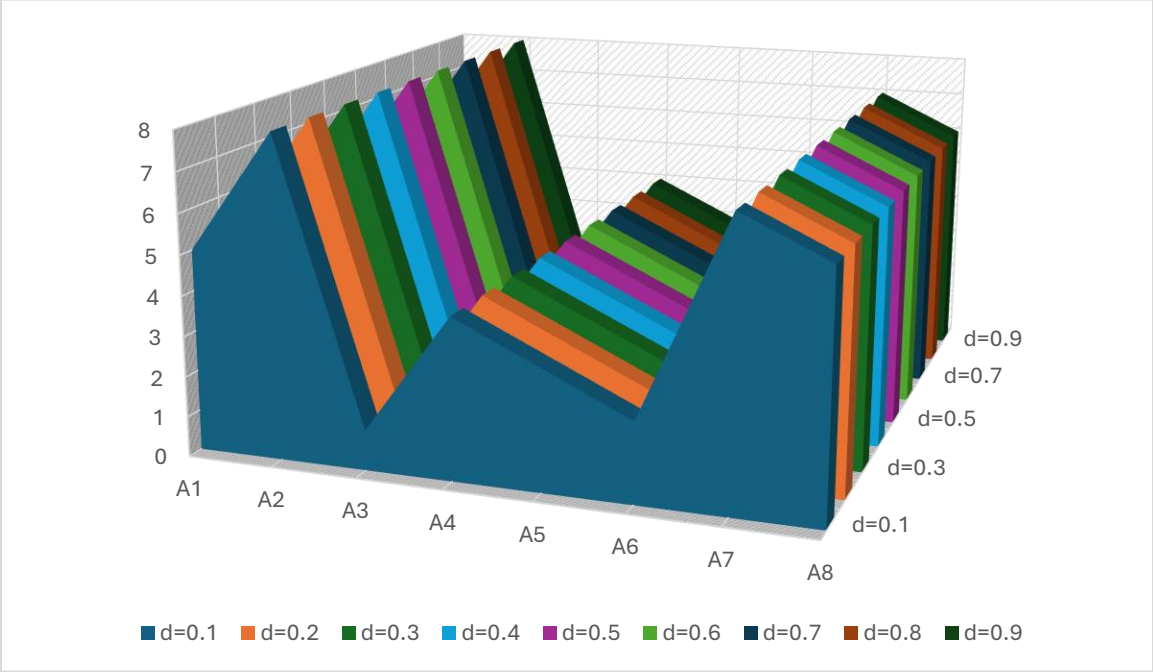

Figure 6. The ranks of alternatives under different cases.

5. Conclusions

This study proposed MCDM approach for risk assessment of cost steel in structure engineering. This study applies the GRA approach under the plithogenic sets to deal with uncertainty information. Three experts have evaluated the criteria and alternative. We combine the plithogenic numbers using the plithogenic operators. The GRA approach is applied to eight criteria and eight alternatives. The results show alternative 2 is the best and alternative 3 is the worst. The sensitivity analysis was conducted to show the different rank of alternatives. There are nine cases in which GRA approach is applied on it to show the rank of alternatives. The results show the ranks of alternatives are stable in different cases.

6. Future works

The study can encompass a variety of specific and hierarchical factors, including market, social, political, and environmental variables. The suggested model may be able to assist with more complicated issues that may or may not involve a greater number of levels of hierarchy. The proposed model can be applied into different MCDM problems to compute the criteria weights and rank the alternatives. Different uncertainties environments can be used to deal with uncertainty and vague data in the evaluation process.

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Single-Valued Neutrosophic Graph with Heptapartitioend Structure

Hanaa fathi[1], M. Myvizhi[2], Ahmed Abdelhafeez[3*,4], Mohamed Refaat Abdellah[5], Mohamed Eassa [3,4], Mohamed S. Sawah[6], Hussam Elbehiery[3]
[1]College of Information Technology, Amman Arab University, Amman, Jordan.
[2]Assistant Professor, KPR Institute of Engineering and Technology, Coimbatore, Tamilnadu, India.
[3]Computer Science Department, Faculty of Information System and Computer Science, October 6 University, Giza, 12585, Egypt
[4]Applied Science Research Center. Applied Science Private University, Amman, Jordan
[5]The Department of Computer Science, College of Information Technology, Misr University for Science and Technology, Cairo, Egypt
[6]Department of Computer Science, Faculty of Information Technology, Ajloun National University P.O.43, Ajloun-26810, JORDAN
*Correspondence: aahafeez.scis@o6u.edu.eg

Abstract
This study applies single-valued neutrosophic sets, which extend the frameworks of fuzzy and intuitionistic fuzzy sets, to graph theory. We introduce a new category of graphs called Single-Valued Heptapartitioned Neutrosophic Graphs (SVHNG) and investigate their characteristics through comprehensive proofs and illustrative examples.

Keywords: Neutrosophic Sets, Heptapartitioend Neutrosophic Set, Neutrosophic Graph, SVHN-Graph.

1. Introduction

Graph theory is a fundamental branch of mathematics that studies the relationships between objects, represented as vertices or nodes, and the connections between them represented as edges (West, 2001; Bondy & Murty, 2008 [1,2]). It provides a powerful tool for modeling and analyzing complex systems, networks, and relationships in various fields, including computer science, physics, biology, and social sciences (Newman, 2010; Barabási, 2016 [3,4]). By representing systems as graphs, researchers and practitioners can gain insights into their structure, behavior, and evolution, and develop efficient algorithms and techniques for solving complex problems.

For every solution, classical MCDM requires exact accurate numbers as decision data. Nevertheless, as decision theory develops and is used more often, we see that: (1) Most choice problems are ambiguous by nature, which prevents decision-makers (DMs) from providing evaluation values in a binary fashion. Examples include ranking universities' overall strength or evaluating urban modernization. (2) Cognitive elements and personal thought patterns frequently cause ambiguity and confusion in practical decision-making. As a result, it gets harder and harder to apply traditional MCDM theory and techniques to real-world decision issues.

In 1965, Zadeh presented FS theory to address this hazy phenomenon. Uncertain membership in sets results from what we try to represent lacking distinct and well-defined bounds, as the FS theory acknowledges. The unit interval [0,1] defines the thesis range, or the domain of study, thereby expressing this association as a fuzzy set. Such phenomena can be examined and dealt with by defining suitable affiliation functions and applying operations and transformations to fuzzy sets. This theory has

wide-ranging applicability in many socioeconomic fields and offers a convincing explanation for the ubiquity of fuzzy occurrences in the objective world.

When assessing things or phenomena, the DMs frequently come across a neutral state distinct from an affirmative or negative position. This condition, the DM's hesitate degree, represents the unpredictability of decision-making processes and may be impacted by several DM-specific characteristics. Several academics have examined and improved traditional FS theory to deal with this ambiguity, suggesting novel FS expansions.

First, the interval fuzzy set, created by researcher Turksen in 1986, expresses the degree of connectivity using interval numbers. The concept of a hesitant fuzzy set was also made by Torra in 2010, which permits the degree of attachment to exist as multiple possible values within a fuzzy set. Interestingly, the degree of reluctance of the decision subject is not directly represented by either interval fuzzy sets or hesitate fuzzy sets. Rather, they statistically characterize uncertainty in decision events by broadening the range of potential values for the degree of attachment.

Second, unlike standard fuzzy sets, the IFSs, developed by scholar Atanassov in 1986, integrate affiliation and unaffiliated degree information. For the first time, they provide a hesitate degree, considering the total affiliation, unaffiliated, and hesitate degrees. IFSs somewhat describe uncertainty in decision events. For example, the intuitionistic fuzzy set can represent a company's vote on whether to invest in a project when a panel of ten DMs is composed of seven in favor, two against, and one abstaining.

It is crucial to remember that the reluctance of decision subjects cannot be fully captured by interval fuzzy sets, IFSs, or hesitate fuzzy sets. Each cannot adequately capture the unpredictability of choice occurrences. One or more of the previously stated classical fuzzy extension sets are the foundation for most of the following fuzzy extension sets. These additional sets improve the capacity to describe fuzziness and uncertainty created by superposing and fusing the original classical fuzzy extension sets. Nevertheless, their internal systems have grown more intricate, which has decreased their usefulness and application.

Neutrosophic sets, introduced by Smarandache in 1998, are a mathematical framework that extends the concept of fuzzy sets to manage indeterminate and inconsistent information (Smarandache, 1998 [5]). Neutrosophic sets are characterized by three membership functions: truth (T), indeterminacy (I), and falsehood (F), which satisfy the condition $T + I + F \leq 3$ (Wang et al., 2010 [6]). This framework provides a powerful tool for modeling and analyzing complex systems, particularly when data is uncertain, imprecise, or incomplete.

Fuzzy graphs are a mathematical framework that combines graph theory with fuzzy set theory, introduced by Zadeh in 1965 [7]. Fuzzy graphs model complex systems where object relationships are uncertain or imprecise (Rosenfeld, 1975 [8]). In a fuzzy graph, each edge is assigned a membership value between 0 and 1, representing the degree of connection or relationship between the vertices (Yeh & Bang, 1975 [9]). This framework provides a powerful tool for modeling and analyzing complex networks, particularly when data is uncertain or incomplete.

Intuitionistic fuzzy graphs are a mathematical framework that combines graph theory with intuitionistic fuzzy set theory, introduced by Atanassov in 1986 [10]. Intuitionistic fuzzy graphs are used to model complex systems where relationships between objects are uncertain, imprecise, or incomplete (Atanassov, 1999 [11]). In an intuitionistic fuzzy graph, each edge is assigned two membership values: a membership value and a non-membership value, which satisfy the condition that the sum of the two values is less than or equal to 1 (Sharma, 2014 [12]). This framework provides a powerful tool for modeling and analyzing complex networks, particularly when data is uncertain or incomplete.

Introduced by Smarandache in 1998, neutrosophic graphs are a mathematical paradigm that blends neutrosophic set theory and graph theory. Complex systems with ambiguous, imperfect, or incomplete interactions between items are modeled using neutrophilic graphs (Smarandache, 2010 [13]). The degree of truth, indeterminacy, and untruth of the link between the vertices is represented by the neutrosophic number allocated to each edge in a neutrosophic network (Broumi et al., 2016 [14]). This paradigm offers a potent tool for modeling and analyzing complicated networks, especially when data is ambiguous or lacking.

Further research on neutrophilic graphs has been done in [15–18]. Interval NGs were first introduced by Broumi et al. [19] and have since been further examined in [20–21]. In a variety of hybrid environments, including neutrosophic soft graphs [22], bipolar SVN graphs [23], rough neutrosophic diagraphs [24], neutrosophic soft, rough graphs [25], and others, NGs have been further investigated. In several settings, recent developments in graph theory have been illustrated in [26].

The concept of heptapartitioned neutrosophic sets was introduced by Florentin Smarandache [27]. A single-valued heptapartitioned neutrosophic set (SVHNS) is a mathematical framework that extends the concept of neutrosophic sets to manage more complex and uncertain information. In an SVHNS, each element is assigned seven membership values: truth (T), indeterminacy (I), falsehood (F), unknown truth (UT), unknown indeterminacy (UI), unknown falsehood (UF), and unknown (U), which satisfy certain conditions (Broumi et al., 2016 [28]). This framework provides a powerful tool for modeling and analyzing complex systems, particularly when data is uncertain, imprecise, or incomplete.

The Single Valued Heptapartitioned Neutrosophic (SVHN) graph is obtained for this study, and its fundamental characteristics are established.

2. Preliminaries

This section offers a few current definitions pertinent to the article's primary findings.

Definition 2.1 [29] Neutrosophic Set A on Y is defined as follows:

$\boldsymbol{B = \{< y, \alpha_B(y), \gamma_B(y), \delta_B(y) >, y \in Y\}}$ where $\boldsymbol{\alpha_B,\ \beta_B,\ \gamma_B : B \to [0,1]}$ and $\boldsymbol{0 \le \alpha_B(x) + \beta_B(y) + \gamma_B(y) \le 3}$.Here, $\boldsymbol{\alpha_B(y)}$ is the degree of membership, $\boldsymbol{\beta_B(y)}$is the degree of indeterminacy and $\boldsymbol{\gamma_B(y)}$ is the degree of non-membership. Here, $\boldsymbol{\alpha_B(x)}$ and $\boldsymbol{\gamma_B(y)}$ are dependent on neutrosophic elements and $\boldsymbol{\beta_B(y)}$ is an independent neutrosophic element.

Definition 2.2 [30] Assume that Y is a universe. A QNS, B on Y with separate neutrosophic components is an example of an object of this sort.

$$\boldsymbol{B = \{< y, \alpha_B(y), C_B(y), U_B(y), \gamma_B(y) >, y \in Y\}}$$

$\boldsymbol{and\ 0 \le \alpha_B(y) + C_B(y) + U_B(y) + \gamma_B(y) \le 4}$ Here, $\boldsymbol{\alpha_B(y)}$ is the truth membership, $\boldsymbol{C_B(y)}$ is contradiction membership, $\boldsymbol{U_B(y)}$ is ignorance membership and $\boldsymbol{\gamma_B(y)}$ is a false membership.

Definition 2.3 [32,33] The set P must not be empty. Every element of P has a PNS over P defined by a truth-membership function $\boldsymbol{\alpha_B(y)}$, a contradiction membership function $\boldsymbol{C_B(y)}$ an ignorance membership function $\boldsymbol{G_B(y)}$ unknown membership function $\boldsymbol{U_B(y)}$ and a falsity membership function $\boldsymbol{\gamma_B(y)}$ such that for each $\boldsymbol{p \in P,\ 0 \le \alpha_B(y) + C_B(y) + G_B(y) + U_B(y) + \gamma_B(y) \le 5}$.

Definition 2.4 [31] Let the universe R be non-empty. In the neutrosophic set Heptapartitioned (HNS) every element has a B over R. Here an absolute truth-membership function. $\boldsymbol{0 \le \alpha_B}$, a relative truth membership function M_B, a contradiction membership function C_A, an ignorance membership function I_A, an unknown membership function U_A, an absolute falsity membership function F_A and a relative falsity membership function K_A such that for each $p \in R,\ T_A, M_A, C_A, I_A, U_A, F_A, K_A \in [0,1]$ and

$$\boldsymbol{B = [\, p, \alpha_B(p), M_B(p), C_B(p), \beta_B(p), U_B(p), \gamma_B(p), K_B(p) : p \in R\,]\ 0 \le \alpha_B(p) + M_B(p) + C_B(p) + I_B(p) + U_B(p) + \gamma_B(p) + K_B(p) \le 7}.$$

3. Single valued Heptapartitioned Neutrosophic Graphs

Definition 3.1. Let $\mathfrak{V} = \{\mathfrak{v}_i, i = 1,2 \ldots . n\}$ Be an unchanging collection of vertices and $\mathcal{E} = \{(\mathfrak{v}_i, \mathfrak{v}_j), i, j = 1,2 \ldots . n\}$ Be the collection of vertices' edges of $\mathfrak{V}$. A Single Valued Heptapartitioned Neutrosophic Graph of $\tilde{\mathcal{G}} = (\mathfrak{V}, \mathcal{E})$ is defined by $\mathcal{G} = (H_1, H_2)$, where (i) $T_{H_1}: \mathfrak{V} \to [0,1], M_{H_1}: \mathfrak{V} \to [0,1], C_{H_1}: \mathfrak{V} \to [0,1], I_{H_1}: \mathfrak{V} \to [0,1], U_{H_1}: \mathfrak{V} \to [0,1], F_{H_1}: \mathfrak{V} \to [0,1],\ K_{H_1}: \mathfrak{V} \to [0,1]$ represents the absolute truth, relative truth, contradiction, ignorance, unknown, absolute falsity, relative falsity membership functions of the vertices $\mathfrak{v}_i \in \mathfrak{V}$ accordingly, such that $0 \leq T_{H_1}(\mathfrak{v}_i) + M_{H_1}(\mathfrak{v}_i) + C_{H_1}(\mathfrak{v}_i) + I_{H_1}(\mathfrak{v}_i) + U_{H_1}(\mathfrak{v}_i) + F_{H_1}(\mathfrak{v}_i) + K_{H_1}(\mathfrak{v}_i) \leq 7$,for every $\mathfrak{v}_i \in \mathfrak{V}(i = 1,2 \ldots . n)$;

(ii) $T_{H_2}: \mathcal{E} \subseteq \mathfrak{V} \times \mathfrak{V} \to [0,1], M_{H_2}: \mathcal{E} \subseteq \mathfrak{V} \times \mathfrak{V} \to [0,1], C_{H_2}: \mathcal{E} \subseteq \mathfrak{V} \times \mathfrak{V} \to [0,1], I_{H_2}: \mathcal{E} \subseteq \mathfrak{V} \times \mathfrak{V} \to [0,1], U_{H_2}: \mathcal{E} \subseteq \mathfrak{V} \times \mathfrak{V} \to [0,1], F_{H_2}: \mathcal{E} \subseteq \mathfrak{V} \times \mathfrak{V} \to [0,1], K_{H_2}: \mathcal{E} \subseteq \mathfrak{V} \times \mathfrak{V} \to [0,1]$ specified by $T_{H_2}(\mathfrak{v}_i, \mathfrak{v}_j) \leq \min\{T_{H_1}(\mathfrak{v}_i), T_{H_1}(\mathfrak{v}_j)\}, M_{H_2}(\mathfrak{v}_i, \mathfrak{v}_j) \leq \min\{M_{H_1}(\mathfrak{v}_i), M_{H_1}(\mathfrak{v}_j)\}, C_{H_2}(\mathfrak{v}_i, \mathfrak{v}_j) \leq \min\{C_{H_1}(\mathfrak{v}_i), C_{H_1}(\mathfrak{v}_j)\}, I_{H_2}(\mathfrak{v}_i, \mathfrak{v}_j) \geq \max\{I_{H_1}(\mathfrak{v}_i), I_{H_1}(\mathfrak{v}_j)\}, U_{H_2}(\mathfrak{v}_i, \mathfrak{v}_j) \geq \max\{U_{H_1}(\mathfrak{v}_i), U_{H_1}(\mathfrak{v}_j)\}, F_{H_2}(\mathfrak{v}_i, \mathfrak{v}_j) \geq \max\{F_{H_1}(\mathfrak{v}_i), F_{H_1}(\mathfrak{v}_j)\}, K_{H_2}(\mathfrak{v}_i, \mathfrak{v}_j) \geq \max\{K_{H_1}(\mathfrak{v}_i), K_{H_1}(\mathfrak{v}_j)\}$.

We know that both H_1 and H_2 is the SVHN set over $\mathfrak{V}$ and $\mathcal{E}$ Accordingly.

Example 3.1. Let $\tilde{\mathcal{G}} = (\mathfrak{V}, \mathcal{E})$ is a graph, where $\mathfrak{V} = \{\mathfrak{v}_1, \mathfrak{v}_2, \mathfrak{v}_3, \mathfrak{v}_4\}$ and $\mathcal{E} = \{(\mathfrak{v}_1, \mathfrak{v}_2), (\mathfrak{v}_2, \mathfrak{v}_3), (\mathfrak{v}_3, \mathfrak{v}_4), (\mathfrak{v}_4, \mathfrak{v}_1)\}$. Here H_1 is an SVHN vertex set of $\mathfrak{V}$ and H_2 is an SVHN edge set of $\mathcal{E}$ specified by the following table values.

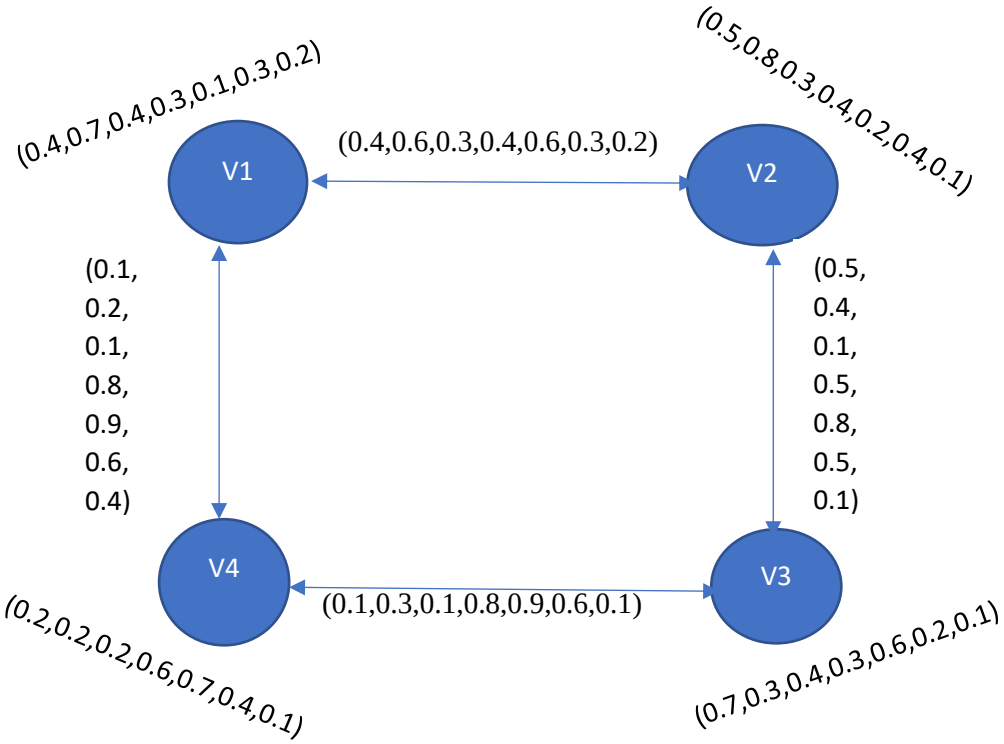
Table 1

	$\mathfrak{v}_1$	$\mathfrak{v}_2$	$\mathfrak{v}_3$	$\mathfrak{v}_4$
T_{H_1}	0.4	0.5	0.7	0.2
M_{H_1}	0.7	0.8	0.3	0.2
C_{H_1}	0.4	0.3	0.4	0.2
I_{H_1}	0.3	0.4	0.3	0.6
U_{H_1}	0.1	0.2	0.6	0.7
F_{H_1}	0.3	0.4	0.2	0.4
K_{H_1}	0.2	0.1	0.1	0.1

Table 2

	$(\mathfrak{v}_1, \mathfrak{v}_2)$	$(\mathfrak{v}_2, \mathfrak{v}_3)$	$(\mathfrak{v}_3, \mathfrak{v}_4)$	$(\mathfrak{v}_4, \mathfrak{v}_1)$
T_{H_2}	0.4	0.5	0.1	0.1
M_{H_2}	0.6	0.4	0.3	0.2
C_{H_2}	0.3	0.1	0.1	0.1
I_{H_2}	0.4	0.5	0.8	0.8
U_{H_2}	0.6	0.8	0.9	0.9
F_{H_2}	0.3	0.5	0.6	0.6
K_{H_2}	0.2	0.1	0.1	0.4

The above table is shown in the following diagram.



Remark 3.1. Let $\mathcal{G} = (H_1, H_2)$ Is an SVHN graph. Afterward, the edge $(\mathcal{v}_i, \mathcal{v}_j)$ is said to be an incident at $\mathcal{v}_i$ and $\mathcal{v}_j$.

Definition 3.2. Let $\mathcal{G} = (H_1, H_2)$ be an SVHN-graph. Subsequently,

(i) $(\mathcal{v}_i, T_{H_1}(\mathcal{v}_i), M_{H_1}(\mathcal{v}_i), C_{H_1}(\mathcal{v}_i), I_{H_1}(\mathcal{v}_i), U_{H_1}(\mathcal{v}_i), F_{H_1}(\mathcal{v}_i), K_{H_1}(\mathcal{v}_i))$ is called a single-valued heptapartitioned neutrosophic (SVHN) vertex.

(ii) $((\mathcal{v}_i, \mathcal{v}_j), T_{H_2}(\mathcal{v}_i, \mathcal{v}_j), M_{H_2}(\mathcal{v}_i, \mathcal{v}_j), C_{H_2}(\mathcal{v}_i, \mathcal{v}_j), I_{H_2}(\mathcal{v}_i, \mathcal{v}_j), U_{H_2}(\mathcal{v}_i, \mathcal{v}_j), F_{H_2}(\mathcal{v}_i, \mathcal{v}_j),$

$K_{H_2}(\mathcal{v}_i, \mathcal{v}_j))$ is called a single-valued heptapartitioned neutrosophic (SVHN) edge.

Definition 3.3. Let $\mathcal{G} = (H_1, H_2)$ be an SVHN-graph. Afterwards, $H = ({H_1}', {H_2}')$ is called an SVHN sub-graph of $\mathcal{G} = (H_1, H_2)$ if $H = ({H_1}', {H_2}')$ is also an SVHN-graph such that:

(i) ${H_1}' \subseteq H_1$ i.e.,${T'}_{H_{1_i}} \leq T_{H_{1_i}}, {M'}_{H_{1_i}} \leq M_{H_{1_i}}, {C'}_{H_{1_i}} \leq C_{H_{1_i}}, {I'}_{H_{1_i}} \geq I_{H_{1_i}}, {U'}_{H_{1_i}} \geq U_{H_{1_i}}, {F'}_{H_{1_i}} \geq F_{H_{1_i}}, {K'}_{H_{1_i}} \geq K_{H_{1_i}}$, for every $\mathcal{v}_i \in \mathfrak{V}$;

(ii) ${H_2}' \subseteq H_2$ i.e.,${T'}_{H_{2_i}} \leq T_{H_{2_i}}, {M'}_{H_{2_i}} \leq M_{H_{2_i}}, {C'}_{H_{2_i}} \leq C_{H_{2_i}}, {I'}_{H_{2_i}} \geq I_{H_{2_i}}, {U'}_{H_{2_i}} \geq U_{H_{2_i}}, {F'}_{H_{2_i}} \geq F_{H_{2_i}}, {K'}_{H_{2_i}} \geq K_{H_{2_i}}$, for every $(\mathcal{v}_i, \mathcal{v}_j) \in \mathcal{E}$.

Example 3.2. Let $\mathcal{G} = (H_1, H_2)$ be an SVHN-graph as shown in the previous example. Afterwards, $H = (H_1', H_2')$ where $\mathfrak{V}' = \{v_1, v_3, v_4\}$; $\mathcal{E}' = \{(v_1, v_3), (v_3, v_4)\}$ specified by the following table values.

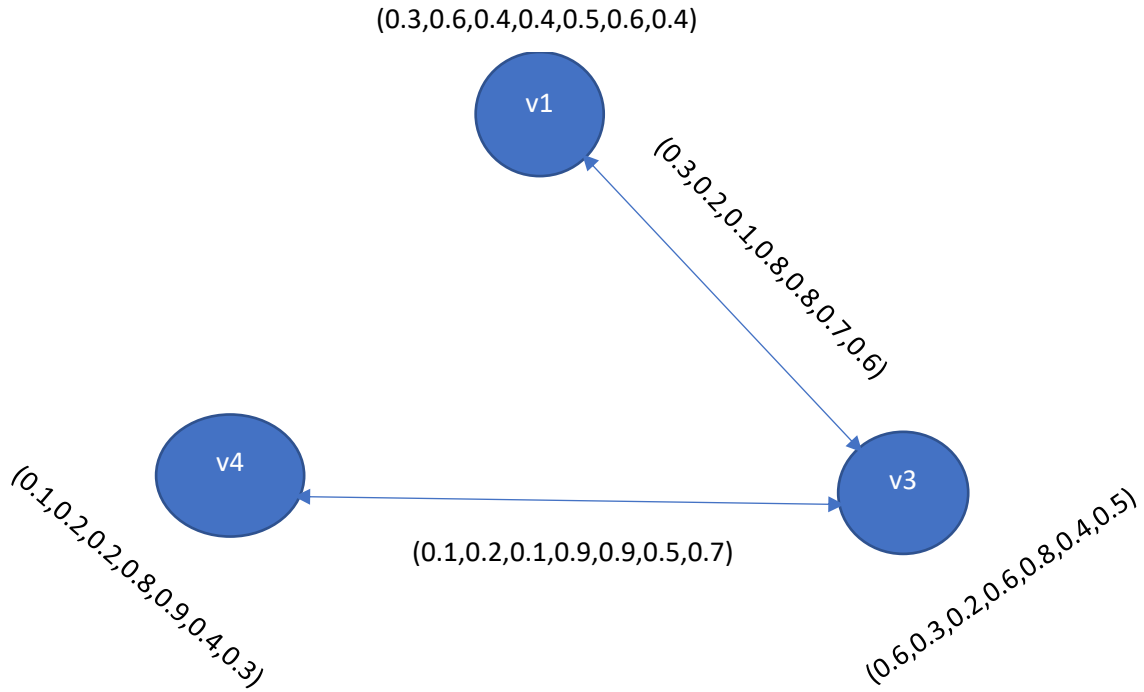
Table 3

	v_1	v_3	v_4
T'_{H_1}	0.3	0.6	0.1
M'_{H_1}	0.6	0.3	0.2
C'_{H_1}	0.4	0.2	0.2
I'_{H_1}	0.4	0.6	0.8
U'_{H_1}	0.5	0.8	0.9
F'_{H_1}	0.6	0.4	0.4
K'_{H_1}	0.4	0.5	0.3

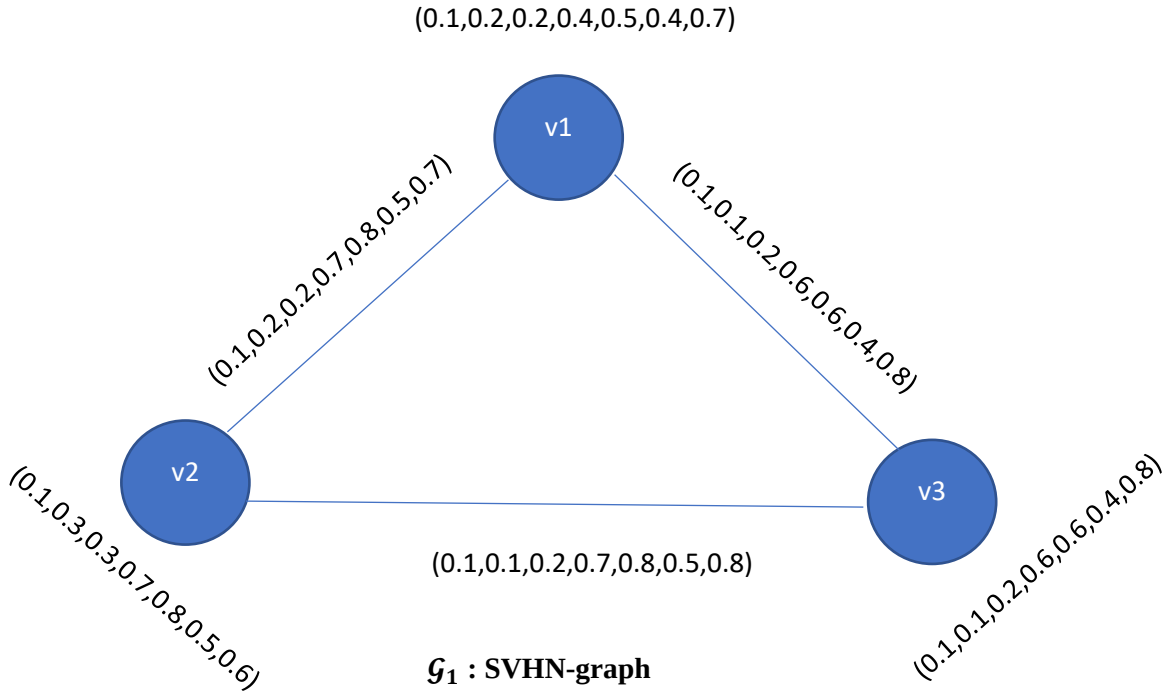
Table 4

	(v_1, v_3)	(v_3, v_4)
T'_{H_2}	0.3	0.1
M'_{H_2}	0.2	0.2
C'_{H_2}	0.1	0.1
I'_{H_2}	0.8	0.9
U'_{H_2}	0.8	0.9
F'_{H_2}	0.7	0.5
K'_{H_2}	0.6	0.7

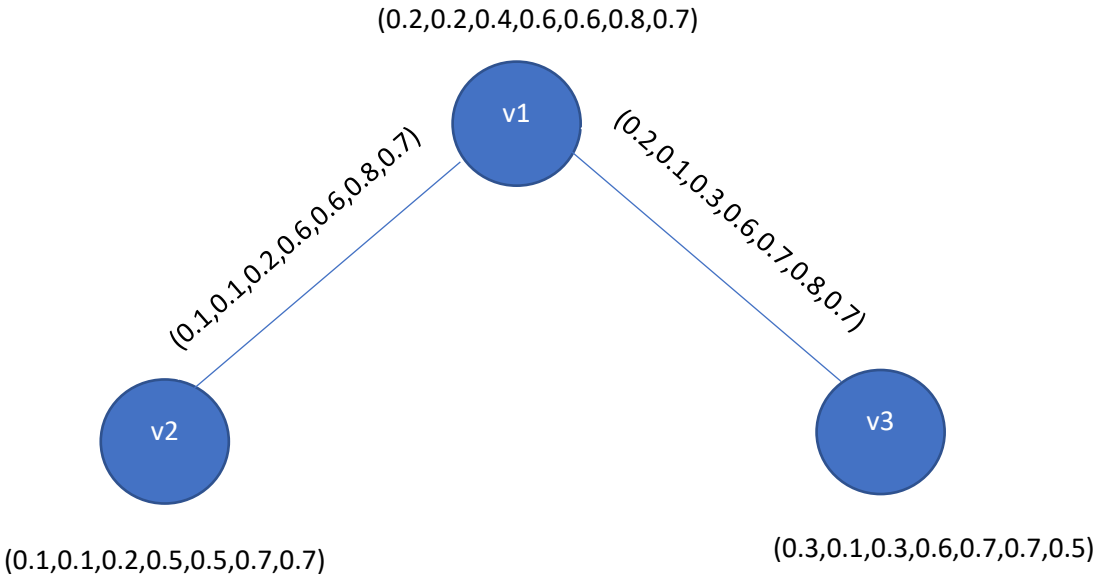
The above table is shown in the following diagram.



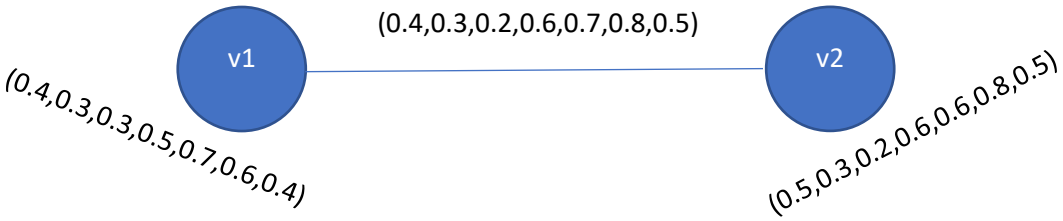
Example 3.3. $\mathcal{G}_1$ is a SVHN-graph. ${H_1}^*$ is a partial SVHN-subgraph and ${H_2}^*$ is a SVHN-subgraph of $\mathcal{G}_1$.



$\mathcal{G}_1$: SVHN-graph



H_1^* : partial SVHN-subgraph



H_2^* : SVHN-subgraph of $\mathcal{G}_1$

Definition 3.4. Let $\mathcal{G} = (H_1, H_2)$ be an SVHN-graph of $\tilde{\mathcal{G}} = (\mathfrak{V}, \mathcal{E})$. Afterward, the complement of $\mathcal{G} = (H_1, H_2)$ is an SVHN-graph $\bar{\mathcal{G}}$ of $\tilde{\mathcal{G}} = (\mathfrak{V}, \mathcal{E})$ where

(i) $\bar{T}_{H_1}(v_i) = T_{H_1}(v_i), \bar{M}_{H_1}(v_i) = M_{H_1}(v_i), \bar{C}_{H_1}(v_i) = C_{H_1}(v_i), \bar{I}_{H_1}(v_i) = I_{H_1}(v_i), \bar{U}_{H_1}(v_i) = U_{H_1}(v_i), \bar{F}_{H_1}(v_i) = F_{H_1}(v_i), \bar{K}_{H_1}(v_i) = K_{H_1}(v_i)$;

(ii) $\bar{T}_{H_2}(v_i, v_j) = \min\{T_{H_1}(v_i), T_{H_1}(v_j)\} - T_{H_2}(v_i, v_j), \bar{M}_{H_2}(v_i, v_j) = \min\{M_{H_1}(v_i), M_{H_1}(v_j)\} - M_{H_2}(v_i, v_j), \bar{C}_{H_2}(v_i, v_j) = \min\{C_{H_1}(v_i), C_{H_1}(v_j)\} - C_{H_2}(v_i, v_j), \bar{I}_{H_2}(v_i, v_j) = \max\{I_{H_1}(v_i), I_{H_1}(v_j)\} - I_{H_2}(v_i, v_j), \bar{U}_{H_2}(v_i, v_j) = \max\{U_{H_1}(v_i), U_{H_1}(v_j)\} - U_{H_2}(v_i, v_j),$

$\bar{F}_{H_2}(v_i, v_j) = \max\{F_{H_1}(v_i), F_{H_1}(v_j)\} - F_{H_2}(v_i, v_j),$

$\bar{K}_{H_2}(v_i, v_j) = \max\{K_{H_1}(v_i), K_{H_1}(v_j)\} - K_{H_2}(v_i, v_j)$, for every $(v_i, v_j) \in \mathcal{E}$.

Definition 3.5. Let $\mathcal{G} = (H_1, H_2)$ be an SVHN-graph. Afterwards, the vertices v_i and v_j are called adjacent in $\mathcal{G} = (H_1, H_2)$ if and only if $T_{H_2}(v_i, v_j) = \min\{T_{H_1}(v_i), T_{H_1}(v_j)\}, M_{H_2}(v_i, v_j) = \min\{M_{H_1}(v_i), M_{H_1}(v_j)\}, C_{H_2}(v_i, v_j) = \min\{C_{H_1}(v_i), C_{H_1}(v_j)\}, I_{H_2}(v_i, v_j) = \max\{I_{H_1}(v_i), I_{H_1}(v_j)\}, U_{H_2}(v_i, v_j) = \max\{U_{H_1}(v_i), U_{H_1}(v_j)\}, F_{H_2}(v_i, v_j) = \max\{F_{H_1}(v_i), F_{H_1}(v_j)\}, K_{H_2}(v_i, v_j) = \max\{K_{H_1}(v_i), K_{H_1}(v_j)\}$.

Example 3.4. Let $\mathcal{G} = (H_1, H_2)$ be an SVHN graph, which is specified in the following table values.

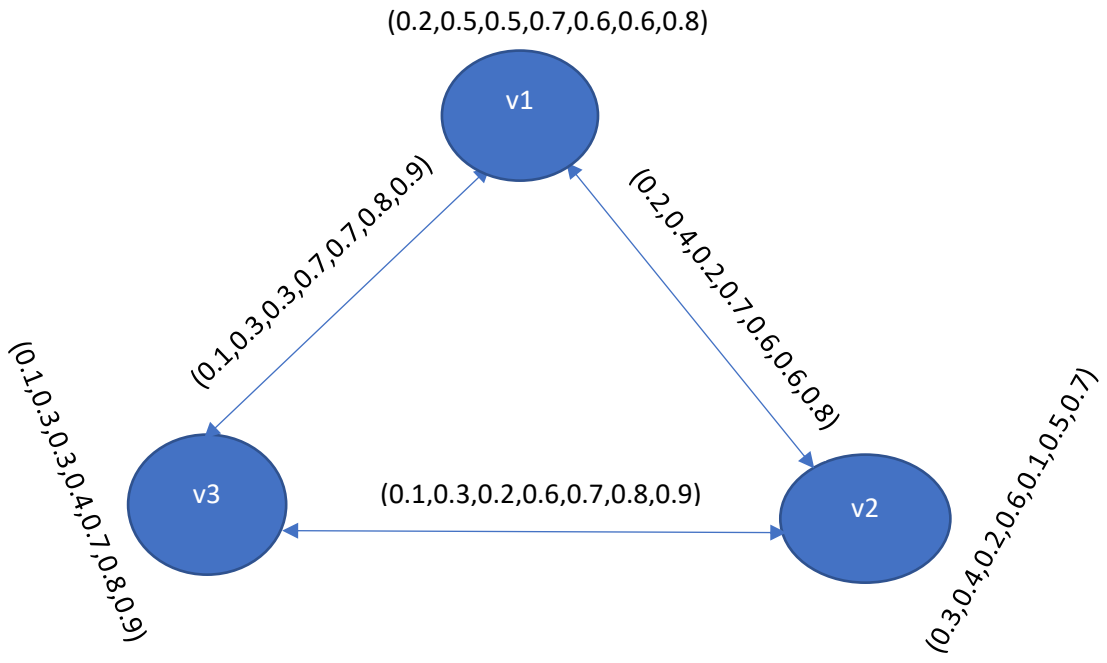
Table 5

	v_1	v_2	v_3
T_{H_1}	0.2	0.3	0.1
M_{H_1}	0.5	0.4	0.3
C_{H_1}	0.5	0.2	0.3
I_{H_1}	0.7	0.6	0.4
U_{H_1}	0.6	0.1	0.7
F_{H_1}	0.6	0.5	0.8
K_{H_1}	0.8	0.7	0.9

Table 6

	(v_1, v_2)	(v_2, v_3)	(v_3, v_1)
T_{H_2}	0.2	0.1	0.1
M_{H_2}	0.4	0.3	0.3
C_{H_2}	0.2	0.2	0.3
I_{H_2}	0.7	0.6	0.7
U_{H_2}	0.6	0.7	0.7
F_{H_2}	0.6	0.8	0.8
K_{H_2}	0.8	0.9	0.9

The above table is shown in the following diagram.



Definition 3.6. In an SVHN-graph, $\mathcal{G} = (H_1, H_2)$, a vertex $\mathcal{v}_j \in \mathfrak{V}$ is called an isolated vertex if there exists no edge incident at $\mathcal{v}_j$.

Example 3.5. Let $\mathcal{G} = (H_1, H_2)$ be an SVHN graph, which is specified by the following table values.

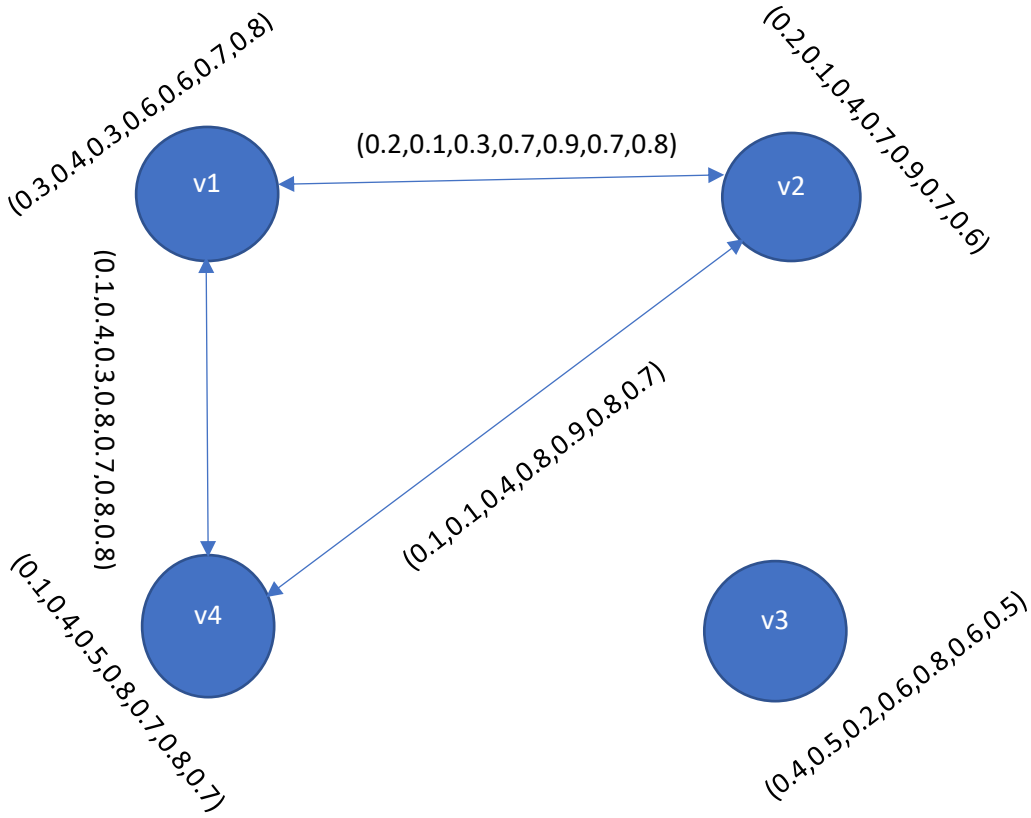
Table 7

	$\mathcal{v}_1$	$\mathcal{v}_2$	$\mathcal{v}_3$	$\mathcal{v}_4$
T_{H_1}	0.3	0.2	0.4	0.1
M_{H_1}	0.4	0.1	0.5	0.4
C_{H_1}	0.3	0.4	0.2	0.5
I_{H_1}	0.6	0.7	0.6	0.8
U_{H_1}	0.6	0.9	0.8	0.7
F_{H_1}	0.7	0.7	0.6	0.8
K_{H_1}	0.8	0.6	0.5	0.7

Table 8

	$(\mathcal{v}_1, \mathcal{v}_2)$	$(\mathcal{v}_2, \mathcal{v}_4)$	$(\mathcal{v}_4, \mathcal{v}_1)$
T_{H_2}	0.2	0.1	0.1
M_{H_2}	0.1	0.1	0.4
C_{H_2}	0.3	0.4	0.3
I_{H_2}	0.7	0.8	0.8
U_{H_2}	0.9	0.9	0.7
F_{H_2}	0.7	0.8	0.8
K_{H_2}	0.8	0.7	0.8

The above table is shown in the following diagram.



Definition 3.7. Let $\mathcal{G} = (H_1, H_2)$ is an SVHN graph. Here $\mathcal{v}_0$ and $\mathcal{v}_n$ be two vertices in $\mathcal{G} = (H_1, H_2)$. Then an SVHN path $H_1(\mathcal{v}_0, \mathcal{v}_n)$ in an SVHN-graph $\mathcal{G} = (H_1, H_2)$ is a sequence of different vertices $k_0, k_1 \ldots \ldots k_n$ such that $T_{H_2}(\mathcal{v}_{i-1}, \mathcal{v}_i) > 0, M_{H_2}(\mathcal{v}_{i-1}, \mathcal{v}_i) > 0, C_{H_2}(\mathcal{v}_{i-1}, \mathcal{v}_i) > 0, I_{H_2}(\mathcal{v}_{i-1}, \mathcal{v}_i) > 0, U_{H_2}(\mathcal{v}_{i-1}, \mathcal{v}_i) > 0, F_{H_2}(\mathcal{v}_{i-1}, \mathcal{v}_i) > 0, K_{H_2}(\mathcal{v}_{i-1}, \mathcal{v}_i) > 0$, where $0 \leq i \leq n$. Here $n(\geq 1)$ is called the length of the path $H_1(\mathcal{v}_0, \mathcal{v}_n)$. The consecutive pairs $(\mathcal{v}_{i-1}, \mathcal{v}_i)$ $(0 \leq i \leq n)$ are called the edges of the path $H_1(\mathcal{v}_0, \mathcal{v}_n)$. The path $H_1(\mathcal{v}_0, \mathcal{v}_n)$ is called a cycle if $\mathcal{v}_0 = \mathcal{v}_n$, where $n \geq 3$.

Definition 3.8. Let $\mathcal{G} = (H_1, H_2)$ be an SVHN-graph. Afterwards, $\mathcal{G} = (H_1, H_2)$ is said to be an SVHN-connected graph if there exists at least one SVHN path between two vertices.

Remark 3.2. If an SVHN-graph $\mathcal{G} = (H_1, H_2)$ is not an SVHN-C-graph, then it is called an SVHN Dis-connected graph.

Definition 3.9. A pendant vertex is a vertex with exactly one edge incident on it in a single-valued neutrosophic graph $\mathcal{G} = (H_1, H_2)$. It is referred to as a non-pendent vertex otherwise.

A pendant edge is an edge with a pendent vertex in a single-valued neutrosophic graph. $\mathcal{G} = (H_1, H_2)$ incident. It is referred to as a non-pendent edge otherwise.

A support of the pendent edge is a vertex that is next to the pendent vertex in a single-valued neutrosophic graph.

Example 3.6. Let $\mathcal{G} = (H_1, H_2)$ be an SVHN-graph, which is specified by the following table values.

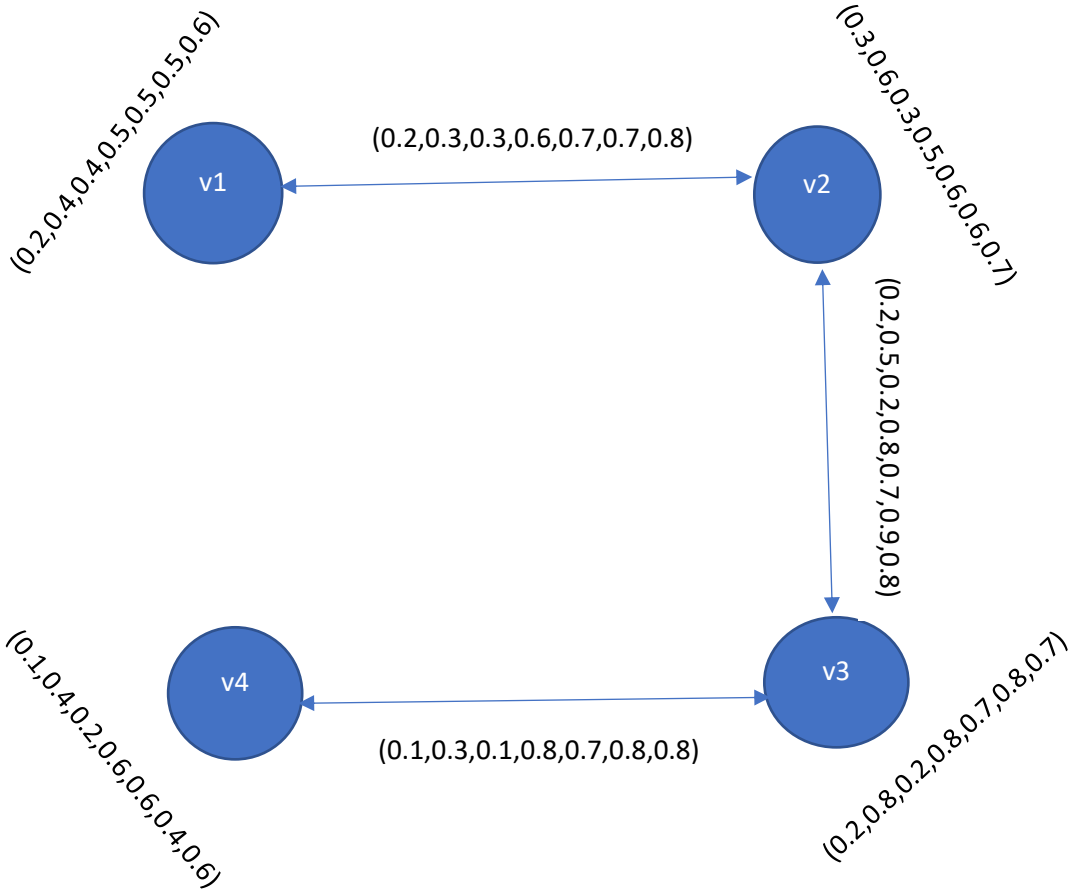
Table 9

	v_1	v_2	v_3	v_4
T_{H_1}	0.2	0.3	0.2	0.1
M_{H_1}	0.4	0.6	0.8	0.4
C_{H_1}	0.4	0.3	0.2	0.2
I_{H_1}	0.5	0.5	0.8	0.6
U_{H_1}	0.5	0.6	0.7	0.6
F_{H_1}	0.5	0.6	0.8	0.4
K_{H_1}	0.6	0.7	0.7	0.6

Table 10

	(v_1, v_2)	(v_2, v_3)	(v_3, v_4)
T_{H_2}	0.2	0.2	0.1
M_{H_2}	0.3	0.5	0.3
C_{H_2}	0.3	0.2	0.1
I_{H_2}	0.6	0.8	0.8
U_{H_2}	0.7	0.7	0.7
F_{H_2}	0.7	0.9	0.8
K_{H_2}	0.8	0.8	0.8

The above table is shown in the following diagram.



Definition 3.10. A SVHN-graph $\mathcal{G} = (H_1, H_2)$ of $\tilde{\mathcal{G}} = (\mathfrak{V}, \mathcal{E})$ is said to be a complete SVHN graph if $T_{H_2}(v_i, v_j) = \min\{T_{H_1}(v_i), T_{H_1}(v_j)\}, M_{H_2}(v_i, v_j) = \min\{M_{H_1}(v_i), M_{H_1}(v_j)\}, C_{H_2}(v_i, v_j) = \min\{C_{H_1}(v_i), C_{H_1}(v_j)\}, I_{H_2}(v_i, v_j) = \max\{I_{H_1}(v_i), I_{H_1}(v_j)\}, U_{H_2}(v_i, v_j) = \max\{U_{H_1}(v_i), U_{H_1}(v_j)\}, F_{H_2}(v_i, v_j) = \max\{F_{H_1}(v_i), F_{H_1}(v_j)\}, K_{H_2}(v_i, v_j) = \max\{K_{H_1}(v_i), K_{H_1}(v_j)\}$ for every $v_i, v_j \in \mathfrak{V}$.

Example 3.7. Let $\tilde{\mathcal{G}} = (\mathfrak{V}, \mathcal{E})$ is a graph, where $\mathfrak{V} = \{v_1, v_2, v_3\}$ and $\mathcal{E} = \{(v_1, v_2), (v_2, v_3), (v_3, v_1)\}$ be specified by the following table values.

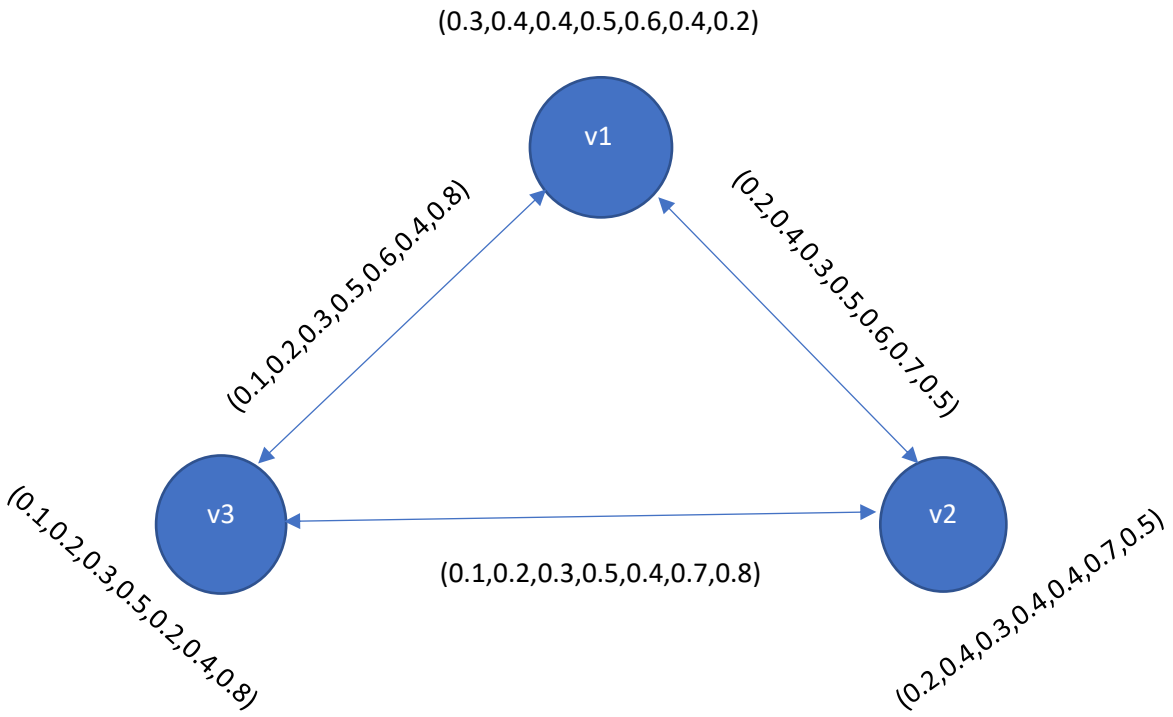
Table 11

	$\mathcal{v}_1$	$\mathcal{v}_2$	$\mathcal{v}_3$
T_{H_1}	0.3	0.2	0.1
M_{H_1}	0.4	0.4	0.2
C_{H_1}	0.4	0.3	0.3
I_{H_1}	0.5	0.4	0.5
U_{H_1}	0.6	0.4	0.2
F_{H_1}	0.4	0.7	0.4
K_{H_1}	0.2	0.5	0.8

Table 12

	$(\mathcal{v}_1,\mathcal{v}_2)$	$(\mathcal{v}_2,\mathcal{v}_3)$	$(\mathcal{v}_3,\mathcal{v}_1)$
T_{H_2}	0.2	0.1	0.1
M_{H_2}	0.4	0.2	0.2
C_{H_2}	0.3	0.3	0.3
I_{H_2}	0.5	0.5	0.5
U_{H_2}	0.6	0.4	0.6
F_{H_2}	0.7	0.7	0.4
K_{H_2}	0.5	0.8	0.8

The above table is shown in the following diagram.



Definition 3.11. An SVHN-graph $\mathcal{G} = (H_1, H_2)$ of $\tilde{\mathcal{G}} = (\mathfrak{V}, \mathcal{E})$ is called bipartite SVHN-graph if the graph $\tilde{\mathcal{G}} = (\mathfrak{V}, \mathcal{E})$ is a bipartite graph.

Example 3.8. Let $\tilde{\mathcal{G}} = (\mathfrak{V}, \mathcal{E})$ be a graph, where $\mathfrak{V} = \{\mathcal{v}_1, \mathcal{v}_2, \mathcal{v}_3, \mathcal{v}_4, \mathcal{v}_5\}$ and $\mathcal{E} = \{(\mathcal{v}_1,\mathcal{v}_2), (\mathcal{v}_1,\mathcal{v}_3), (\mathcal{v}_1,\mathcal{v}_4), (\mathcal{v}_1,\mathcal{v}_5), (\mathcal{v}_2,\mathcal{v}_3), (\mathcal{v}_2,\mathcal{v}_4), (\mathcal{v}_2,\mathcal{v}_5), (\mathcal{v}_3,\mathcal{v}_5)\}$ and $\mathcal{G} = (H_1, H_2)$ be an SVHN graph specified by the following table values.

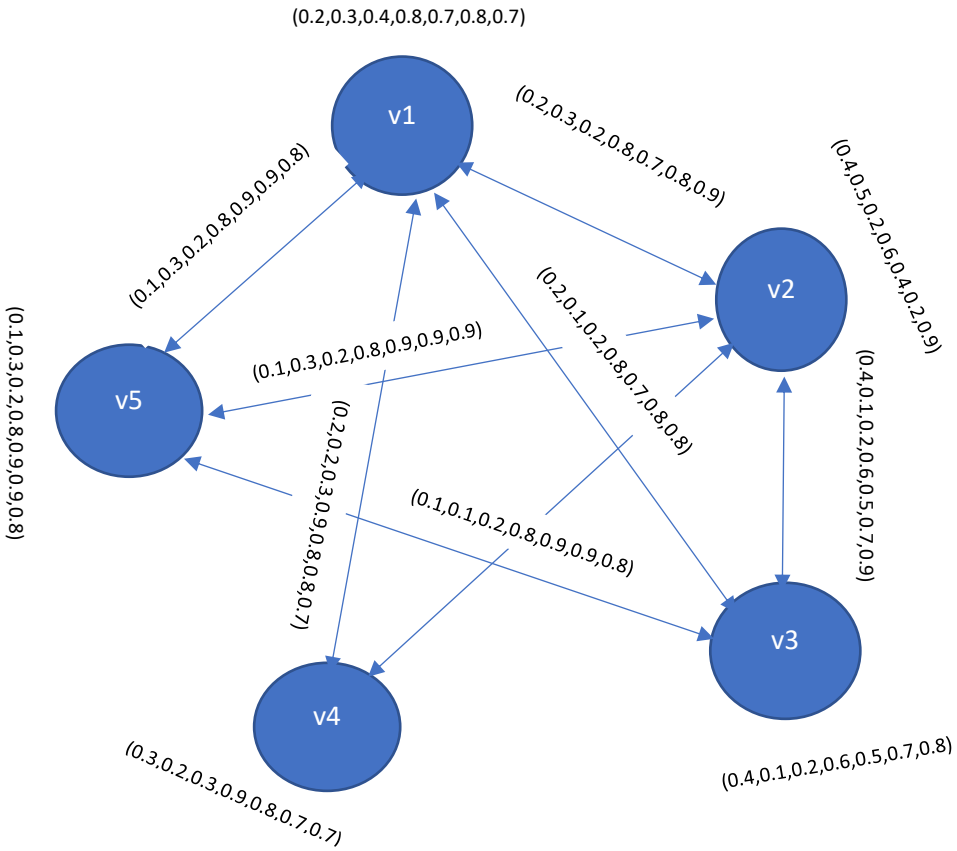
Table 13

	v_1	v_2	v_3	v_4	v_5
T_{H_1}	0.2	0.4	0.4	0.3	0.1
M_{H_1}	0.3	0.5	0.1	0.2	0.3
C_{H_1}	0.4	0.2	0.2	0.3	0.2
I_{H_1}	0.8	0.6	0.6	0.9	0.8
U_{H_1}	0.7	0.4	0.5	0.8	0.9
F_{H_1}	0.8	0.2	0.7	0.7	0.9
K_{H_1}	0.7	0.9	0.8	0.7	0.8

Table 14

	(v_1,v_2)	(v_1,v_3)	(v_1,v_4)	(v_1,v_5)	(v_2,v_3)	(v_2,v_4)	(v_2,v_5)	(v_3,v_5)
T_{H_2}	0.2	0.2	0.2	0.1	0.4	0.3	0.1	0.1
M_{H_2}	0.3	0.1	0.2	0.3	0.1	0.2	0.3	0.1
C_{H_2}	0.2	0.2	0.3	0.2	0.2	0.2	0.2	0.2
I_{H_2}	0.8	0.8	0.9	0.8	0.6	0.9	0.8	0.8
U_{H_2}	0.7	0.7	0.8	0.9	0.5	0.8	0.9	0.9
F_{H_2}	0.8	0.8	0.8	0.9	0.7	0.7	0.9	0.9
K_{H_2}	0.9	0.8	0.7	0.8	0.9	0.9	0.9	0.8

The above table is shown in the following diagram.



Definition 3.12. Let $\mathcal{G} = (H_1, H_2)$ be an SVHN-graph. Afterwards, the degree of the vertex $\mathcal{v}$ is specified by.

$$d(\mathcal{v}) = \left(d_T(\mathcal{v}), d_M(\mathcal{v}), d_C(\mathcal{v}), d_I(\mathcal{v}), d_U(\mathcal{v}), d_F(\mathcal{v}), d_K(\mathcal{v})\right)$$

where $d_T(\mathcal{v})$ = degree of membership in the absolute truth membership vertex $\mathcal{v}$

= total of all edges' absolute truth memberships that are incident on it

= $\sum_{\mathcal{u}\neq\mathcal{v}} T_{H_2}(\mathcal{u}, \mathcal{v})$

$d_M(\mathcal{v})$ = degree of membership in the relative truth vertex $\mathcal{v}$

= total of all edges' relative truth memberships that are incident on it

= $\sum_{\mathcal{u}\neq\mathcal{v}} M_{H_2}(\mathcal{u}, \mathcal{v})$

$d_C(\mathcal{v})$ = degree of membership in the contradiction vertex $\mathcal{v}$

= total of all edges' contradiction memberships that are incident on it

= $\sum_{\mathcal{u}\neq\mathcal{v}} C_{H_2}(\mathcal{u}, \mathcal{v})$

$d_I(\mathcal{v})$ = degree of membership in the ignorance vertex $\mathcal{v}$

= total of all edges' ignorance memberships that are incident on it

= $\sum_{\mathcal{u}\neq\mathcal{v}} I_{H_2}(\mathcal{u}, \mathcal{v})$

$d_U(\mathcal{v})$ = degree of membership in the unknown vertex $\mathcal{v}$

= total of all edges' unknown memberships that are incident on it

= $\sum_{\mathcal{u}\neq\mathcal{v}} U_{H_2}(\mathcal{u}, \mathcal{v})$

$d_F(\mathcal{v})$ = degree of membership in the absolute falsity vertex $\mathcal{v}$

= total of all edges' absolute falsity memberships that are incident on it

= $\sum_{\mathcal{u}\neq\mathcal{v}} F_{H_2}(\mathcal{u}, \mathcal{v})$

$d_K(\mathcal{v})$ = degree of membership in the relative falsity vertex $\mathcal{v}$

= total of all edges' relative falsity memberships that are incident on it

= $\sum_{\mathcal{u}\neq\mathcal{v}} K_{H_2}(\mathcal{u}, \mathcal{v})$

Example 3.9. Let $\mathcal{G} = (H_1, H_2)$ Be an SVHN-graph of $\tilde{\mathcal{G}} = (\mathfrak{V}, \mathcal{E})$ Be specified by the following table values.

Table 15

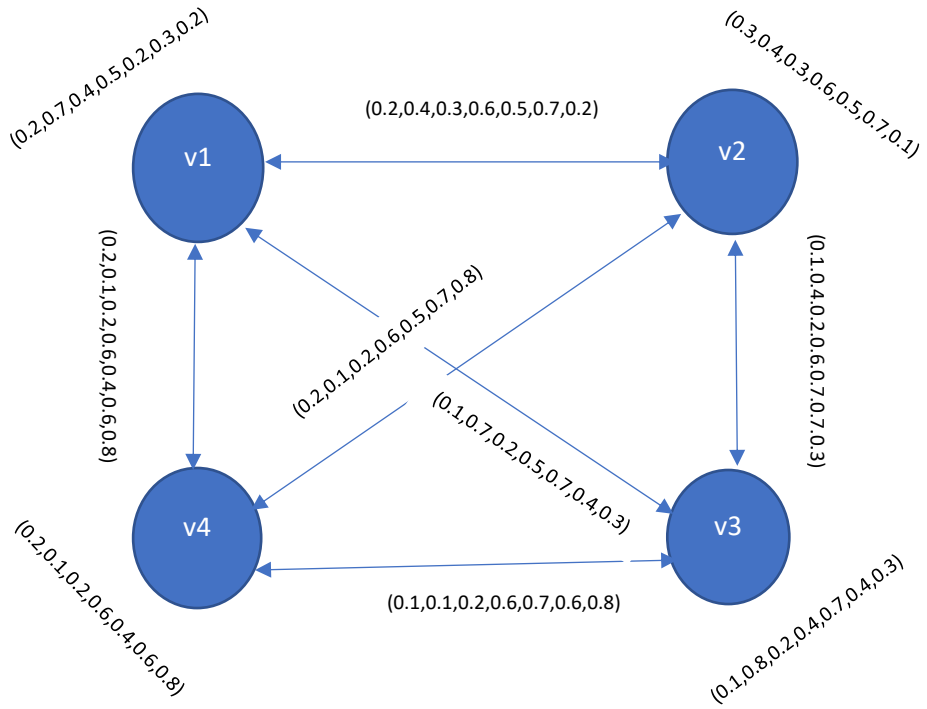
	$\mathcal{v}_1$	$\mathcal{v}_2$	$\mathcal{v}_3$	$\mathcal{v}_4$
T_{H_1}	0.2	0.3	0.1	0.2
M_{H_1}	0.7	0.4	0.8	0.1

C_{H_1}	0.4	0.3	0.2	0.2
I_{H_1}	0.5	0.6	0.4	0.6
U_{H_1}	0.2	0.5	0.7	0.4
F_{H_1}	0.3	0.7	0.4	0.6
K_{H_1}	0.2	0.1	0.3	0.8

Table 16

	(v_1,v_2)	(v_1,v_3)	(v_1,v_4)	(v_2,v_3)	(v_2,v_4)	(v_3,v_4)
T_{H_2}	0.2	0.1	0.2	0.1	0.2	0.1
M_{H_2}	0.4	0.7	0.1	0.4	0.1	0.1
C_{H_2}	0.3	0.2	0.2	0.2	0.2	0.2
I_{H_2}	0.6	0.5	0.6	0.6	0.6	0.6
U_{H_2}	0.5	0.7	0.4	0.7	0.5	0.7
F_{H_2}	0.7	0.4	0.6	0.7	0.7	0.6
K_{H_2}	0.2	0.3	0.8	0.3	0.8	0.8

The above table is shown in the following diagram.



Definition 3.13. Let $\mathcal{G} = (H_1, H_2)$ Be an SVHN-graph of $\tilde{\mathcal{G}} = (\mathfrak{V}, \mathcal{E})$. Afterwards, $\mathcal{G} = (H_1, H_2)$ Is called a constant SVHN graph if the degree of each vertex is the same. i.e.,$d(v) = (p_1, p_2, p_3, p_4, p_5, p_6, p_7)$, for every $v \in \mathfrak{V}$.

Example 3.10. Let $\mathcal{G} = (H_1, H_2)$ Be an SVHN-graph of $\tilde{\mathcal{G}} = (\mathfrak{V}, \mathcal{E})$ Be specified by the following table values.

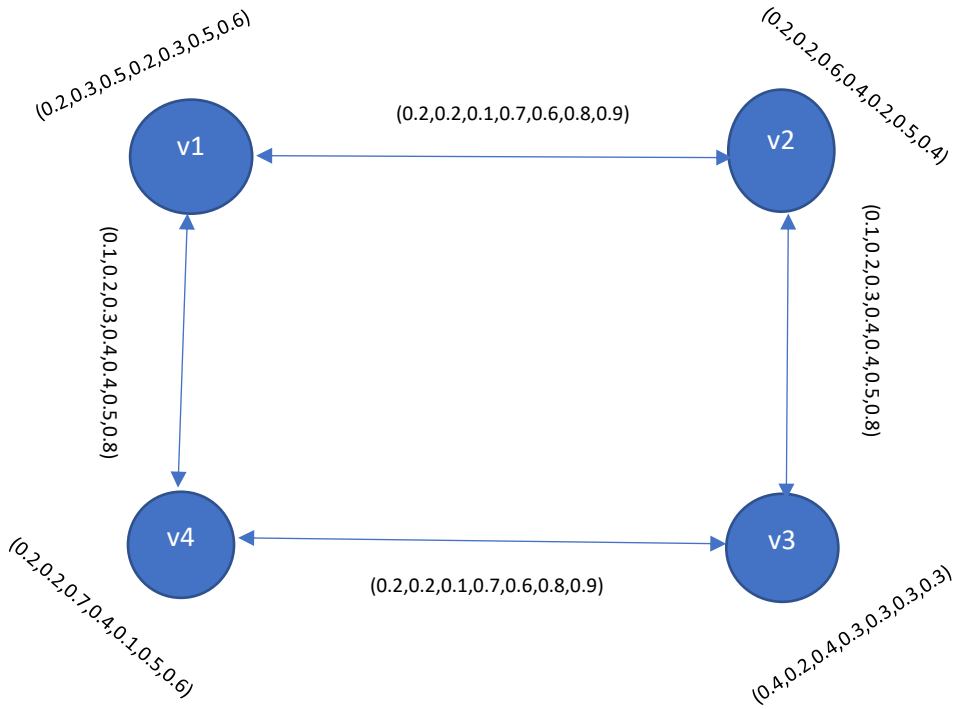
Table 17

	$\mathcal{v}_1$	$\mathcal{v}_2$	$\mathcal{v}_3$	$\mathcal{v}_4$
T_{H_1}	0.2	0.2	0.4	0.2
M_{H_1}	0.3	0.2	0.2	0.2
C_{H_1}	0.5	0.6	0.4	0.7
I_{H_1}	0.2	0.4	0.3	0.4
U_{H_1}	0.3	0.2	0.3	0.1
F_{H_1}	0.5	0.5	0.3	0.5
K_{H_1}	0.6	0.4	0.3	0.6

Table 18

	$(\mathcal{v}_1,\mathcal{v}_2)$	$(\mathcal{v}_2,\mathcal{v}_3)$	$(\mathcal{v}_3,\mathcal{v}_4)$	$(\mathcal{v}_4,\mathcal{v}_1)$
T_{H_2}	0.2	0.1	0.2	0.1
M_{H_2}	0.2	0.2	0.2	0.2
C_{H_2}	0.1	0.3	0.1	0.3
I_{H_2}	0.7	0.4	0.7	0.4
U_{H_2}	0.6	0.4	0.6	0.4
F_{H_2}	0.8	0.5	0.8	0.5
K_{H_2}	0.9	0.8	0.9	0.8

The above table is shown in the following diagram.



Definition 3.14. Let $\mathcal{G} = (H_1, H_2)$ Be an SVHN-graph. Afterward, the order of $\mathcal{G} = (H_1, H_2)$, described by $O(\mathcal{G})$ is specified by $O(\mathcal{G}) = \big(O_T(\mathcal{G}), O_M(\mathcal{G}), O_C(\mathcal{G}), O_I(\mathcal{G}), O_U(\mathcal{G}), O_F(\mathcal{G}), O_K(\mathcal{G})\big)$,where

$O_T(\mathcal{G}) = \sum_{\mathcal{v}\in\mathfrak{V}} T_{H_1}$ represents the T-order of $\mathcal{G} = (H_1, H_2)$

$O_M(\mathcal{G}) = \sum_{v \in \mathfrak{V}} M_{H_1}$ represents the M-order of $\mathcal{G} = (H_1, H_2)$

$O_C(\mathcal{G}) = \sum_{v \in \mathfrak{V}} C_{H_1}$ represents the C-order of $\mathcal{G} = (H_1, H_2)$

$O_I(\mathcal{G}) = \sum_{v \in \mathfrak{V}} I_{H_1}$ represents the I-order of $\mathcal{G} = (H_1, H_2)$

$O_U(\mathcal{G}) = \sum_{v \in \mathfrak{V}} U_{H_1}$ represents the U-order of $\mathcal{G} = (H_1, H_2)$

$O_F(\mathcal{G}) = \sum_{v \in \mathfrak{V}} F_{H_1}$ represents the F-order of $\mathcal{G} = (H_1, H_2)$

$O_K(\mathcal{G}) = \sum_{v \in \mathfrak{V}} K_{H_1}$ represents the K-order of $\mathcal{G} = (H_1, H_2)$

Example 3.11. Let $\mathcal{G} = (H_1, H_2)$ Be an SVHN-graph of $\tilde{\mathcal{G}} = (\mathfrak{V}, \mathcal{E})$ As shown in Example 3.7. Here, the order of SVHN-graph $\mathcal{G} = (H_1, H_2)$ is $O(\mathcal{G}) = (0.6,1.0,1.0,1.4,1.2,1.5,1.5)$.

Definition 3.15. Let $\mathcal{G} = (H_1, H_2)$ Be an SVHN-graph. Here, the size of $\mathcal{G} = (H_1, H_2)$,

described by $S(\mathcal{G})$ is specified by $S(\mathcal{G}) = \big(S_T(\mathcal{G}), S_M(\mathcal{G}), S_C(\mathcal{G}), S_I(\mathcal{G}), S_U(\mathcal{G}), S_F(\mathcal{G}), S_K(\mathcal{G})\big)$, where

$S_T(\mathcal{G}) = \sum_{u \neq v} T_{H_2}(u, v)$ represents the T-size of $\mathcal{G} = (H_1, H_2)$

$S_M(\mathcal{G}) = \sum_{u \neq v} M_{H_2}(u, v)$ represents the M-order of $\mathcal{G} = (H_1, H_2)$

$S_C(\mathcal{G}) = \sum_{u \neq v} C_{H_2}(u, v)$ represents the C-order of $\mathcal{G} = (H_1, H_2)$

$S_I(\mathcal{G}) = \sum_{u \neq v} I_{H_2}(u, v)$ represents the I-order of $\mathcal{G} = (H_1, H_2)$

$S_U(\mathcal{G}) = \sum_{u \neq v} U_{H_2}(u, v)$ represents the U-order of $\mathcal{G} = (H_1, H_2)$

$S_F(\mathcal{G}) = \sum_{u \neq v} F_{H_2}(u, v)$ represents the F-order of $\mathcal{G} = (H_1, H_2)$

$S_K(\mathcal{G}) = \sum_{u \neq v} K_{H_2}(u, v)$ represents the K-order of $\mathcal{G} = (H_1, H_2)$

Example 3.12. Let $\mathcal{G} = (H_1, H_2)$ Be an SVHN-graph of $\tilde{\mathcal{G}} = (\mathfrak{V}, \mathcal{E})$ As shown in Example 3.7. Then, the order of the SVHN-graph $\mathcal{G} = (H_1, H_2)$ is $S(\mathcal{G}) = (0.4,0.8,0.9,1.5,1.6,1.8,2.1)$.

Definition 3.16. A single valued heptapartitioned neutrosophic graph $\mathcal{G} = (H_1, H_2)$ of $\tilde{\mathcal{G}} = (\mathfrak{V}, \mathcal{E})$ is called strong single valued heptapartitioned neutrosophic graph if

$T_{H_2}(v_i, v_j) = \min\{T_{H_1}(v_i), T_{H_1}(v_j)\}, M_{H_2}(v_i, v_j) = \min\{M_{H_1}(v_i), M_{H_1}(v_j)\}, C_{H_2}(v_i, v_j) = \min\{C_{H_1}(v_i), C_{H_1}(v_j)\}, I_{H_2}(v_i, v_j) = \max\{I_{H_1}(v_i), I_{H_1}(v_j)\}, U_{H_2}(v_i, v_j) = \max\{U_{H_1}(v_i), U_{H_1}(v_j)\}, F_{H_2}(v_i, v_j) = \max\{F_{H_1}(v_i), F_{H_1}(v_j)\}, K_{H_2}(v_i, v_j) = \max\{K_{H_1}(v_i), K_{H_1}(v_j)\}$ for every $v_i, v_j \in \mathfrak{V}$.

Example 3.13. Let $\tilde{\mathcal{G}} = (\mathfrak{V}, \mathcal{E})$ Be a graph, where $\mathfrak{V} = \{v_1, v_2, v_3, v_4\}$ and $\mathcal{E} = \{(v_1, v_2), (v_2, v_3), (v_3, v_4), (v_4, v_1)\}$. Let H_1 be a SVHN subset of $\mathfrak{V}$ and let H_2 be a single valued neutrosophic subset of $\mathcal{E}$ denoted by the following table.

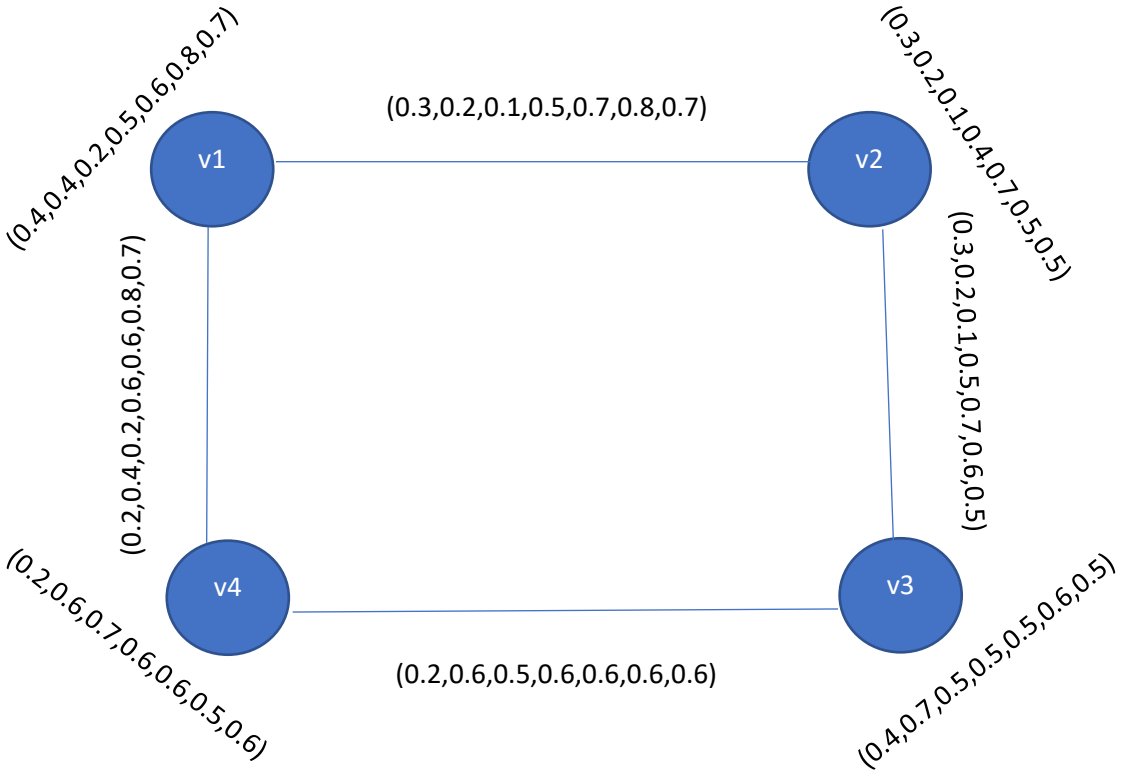
Table 19

	v_1	v_2	v_3	v_4
T_{H_1}	0.4	0.3	0.4	0.2
M_{H_1}	0.4	0.2	0.7	0.6
C_{H_1}	0.2	0.1	0.5	0.7
I_{H_1}	0.5	0.4	0.5	0.6

U_{H_1}	0.6	0.7	0.5	0.6
F_{H_1}	0.8	0.5	0.6	0.5
K_{H_1}	0.7	0.5	0.5	0.6

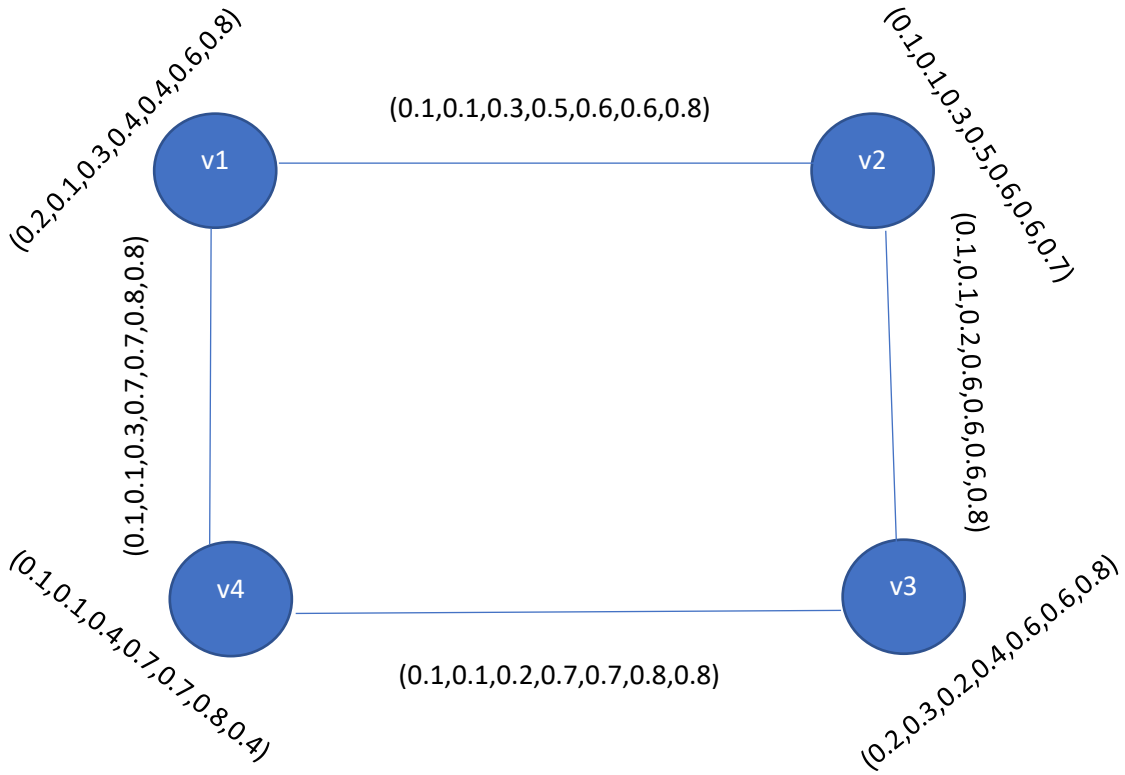
Table 20

	(v_1,v_2)	(v_2,v_3)	(v_3,v_4)	(v_4,v_1)
T_{H_2}	0.3	0.3	0.2	0.2
M_{H_2}	0.2	0.2	0.6	0.4
C_{H_2}	0.1	0.1	0.5	0.2
I_{H_2}	0.5	0.5	0.6	0.6
U_{H_2}	0.7	0.7	0.6	0.6
F_{H_2}	0.8	0.6	0.6	0.8
K_{H_2}	0.7	0.5	0.6	0.7

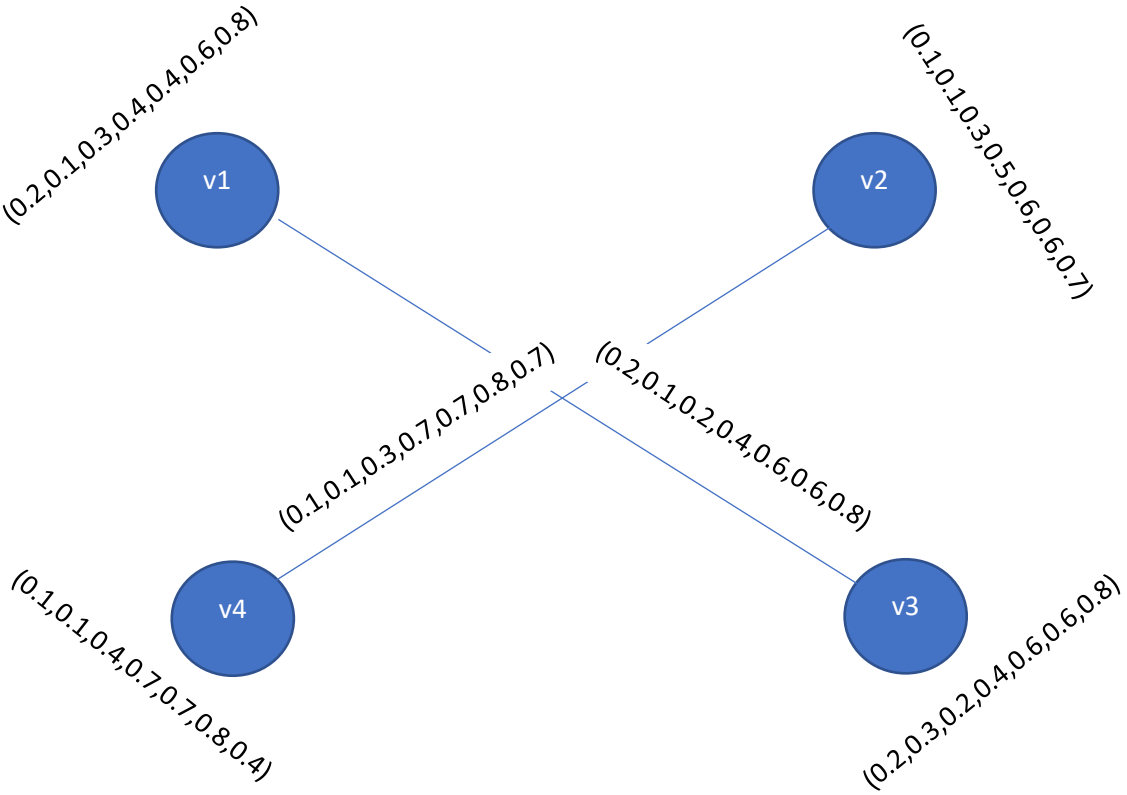


It is easy to see that $\mathcal{G}$ is a strong single valued heptapartitioned neutrosophic graph of $\tilde{\mathcal{G}}$.

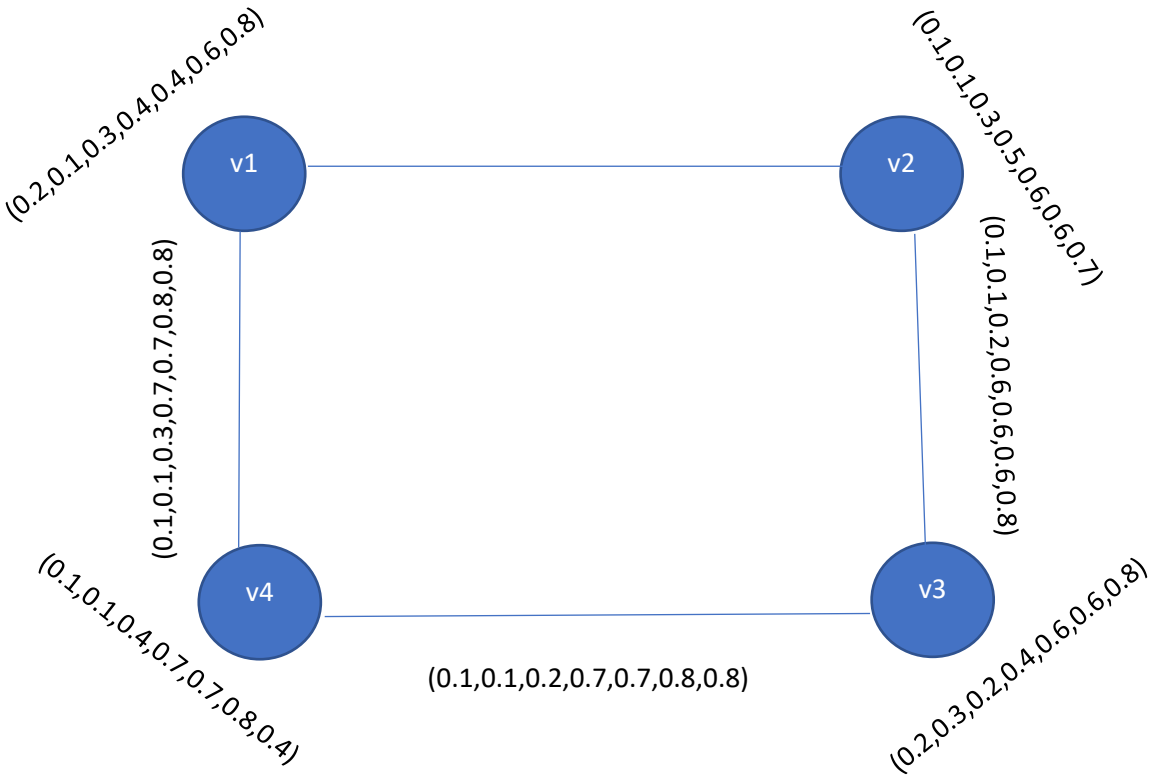
Example 3.14. When $\mathcal{G} = \bar{\bar{\tilde{\mathcal{G}}}}$ if and only if $\mathcal{G}$ is a strong single valued heptapartitioned neutrosophic graph.



$\mathcal{G}$: strong single valued heptapartitioned neutrosophic graph



$\overline{\mathcal{G}}$: strong single valued heptapartitioned neutrosophic graph



$\mathcal{G} = \overline{\overline{\mathcal{G}}}$: strong single valued heptapartitioned neutrosophic graph

Conclusion: The Single-Valued Heptapartitioned Neutrosophic (SVHN) graph is introduced in this study, along with definitions for its degree, order, and size, and an examination of its characteristics. These definitions serve as the foundation for several conclusions that are backed up by instances that confirm the ideas and conclusions put forth. Within the neutrosophic framework, we hope that the method outlined here will stimulate more investigation into SVHN graphs and their applications to practical issues.

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Mapping Barriers to Sustainable Fashion Consumption: Insights from Neutrosophic-Z Number and Delphi-DEMATEL Integration

Phi-Hung Nguyen[1,*], Lan-Anh Thi Nguyen[1], The-Vu Pham[1], Tra-Giang Vu[1], Duc-Minh Vu[1], Thu-Hoai Thi Nguyen[1] , Hong-Quan Le[2], Huong-Giang Hoang Le[1]

[1] FPT University, Vietnam, (Lan-Anh Thi Nguyen, Email: anhntl84@fpt.edu.vn; The-Vu Pham, Email: phamthevu20052002@gmail.com; Tra-Giang Vu, Email: giangvths176156@fpt.edu.vn; Duc-Minh Vu, Email: minhvdhs170908@fpt.edu.vn; Thu-Hoai Thi Nguyen, Email: hoaintths160910@fpt.edu.vn; : Huong-Giang Hoang Le, Email: hoanglehuonggiang2005@gmail.com)
[2] National Cheng Kung University, Tainan, 701, Taiwan, Email: w16137027@gs.ncku.edu.tw
***** Correspondence: hungnp30@fe.edu.vn; https://orcid.org/0000-0003-3372-8287

Abstract: This study investigates the barriers to sustainable consumption within the Vietnamese fashion industry, a critical issue given the sector's significant environmental impact and growing consumer awareness. The research employs an innovative combination of Neutrosophic Z-number sets (NZN), Delphi, and Decision-Making Trial and Evaluation Laboratory (DEMATEL) methodologies to systematically identify and analyze 47 barriers across five key dimensions: Social and Cultural Factors, Pricing and Affordability, Policy and Education, Perception and Image, and Supply Chain and Industry Practices. Through expert evaluations and causal analysis, the study reveals that Policy and Education are the most influential factors, highlighting the need for robust policy frameworks and educational initiatives to drive sustainable consumption. The NZN addresses the limitations of traditional decision-making models by capturing the uncertainty and complexity of consumer behaviors. The findings provide valuable insights for both academics and practitioners, offering theoretical contributions to sustainable consumption research and practical recommendations for businesses and policymakers. These include enhancing product variety and affordability, promoting sustainable practices through education, and improving supply chain transparency. Future research should explore longitudinal studies and comparative analyses across different markets to validate these findings and further refine approaches to fostering sustainability within the fashion industry.

Keywords: Sustainable Consumption; Fashion; Delphi; DEMATEL; Neutrosophic; Z-number; NZN

1. Introduction

1.1 Global fashion industry context

The fashion and textile sector, a colossal entity valued at $2.4 trillion, holds a vital position in the global economy, employing nearly 300 million individuals across the globe (*UN Alliance for Sustainable Fashion*, 2018). However, this economic significance is overshadowed by its profound environmental footprint, positioning it as the world's second-largest polluter (Bailey et al., 2022). The industry's carbon emissions are staggering, representing 8-10% of global totals—surpassing the emissions from both maritime shipping and international aviation (*UN Alliance for Sustainable Fashion*, 2018). Water consumption is another critical issue, with the industry's annual water consumption of around 93 billion cubic meters, a volume that could otherwise fulfill the needs of 5 million people (Ellen MacArthur Foundation, 2017). If current trends persist, greenhouse gas emissions are anticipated to rise by 50% by 2030, signaling a troubling environmental trajectory (Bailey et al., 2022). Additionally, the sector is known as a significant contributor to ocean pollution, generating about 35% of the microplastics that contaminate marine environments and threaten aquatic life (Boucher & Friot, 2017). Beyond ecological damage, the industry faces immense material inefficiencies, resulting in annual losses estimated at $100 billion due to poor recycling and resource underuse (Ellen MacArthur Foudation, 2017).

The rise of fast fashion has exacerbated these issues, encouraging the production of low-cost, short-lived garments (Bick et al., 2018). Over the last 15 years, global clothing consumption has surged by 60%, while the lifespan of individual items has halved, leading to 92 million tons of textile waste annually (Amed et at, 2019). This unsustainable model relies heavily on non-renewable resources, intensive energy consumption, and exploitative labor practices, with workers in key garment-producing countries earning less than half the calculated living wage (H&M Group, 2021). To address these challenges, the industry must adopt sustainable practices such as using eco-friendly materials, circular economy models, and ethical labor standards, alongside reducing water and energy consumption (Bailey et al., 2022; Brydges, 2021). Companies like H&M and Patagonia have pioneered initiatives such as garment recycling programs and repair-resale systems, while countries like Sweden have introduced clothing swaps and second-hand libraries to encourage sustainable consumption (Bailey et al., 2022).

1.2 Vietnam's sustainability challenges

Vietnam, a leading fashion manufacturing hub in Southeast Asia, has made strides toward sustainability through policies like Decision 687 and updates to the Law on Environmental Protection (LEP) (Thu Nguyen & Cece Nguyen, 2022). However, the nation faces significant challenges balancing rapid industrial growth with global sustainability standards (Nayak, Panwar, et al., 2019). These issues are mirrored in other developing economies such as Spain, Chile, Bangladesh, and India, which grapple with barriers like economic constraints, low consumer awareness, and limited availability of sustainable products (Adamkiewicz et al., 2022). A critical gap exists between consumers' concerns about unethical practices and purchasing behaviors. Factors such as price sensitivity, skepticism toward sustainability claims, and perceptions of unfashionability exacerbate this disparity (Bianchi & Gonzalez, 2021; Tahniyath Fatima & Saïd Elbanna, 2023). While sustainable fashion research has gained momentum, most studies focus on developed markets such as Europe, Australia, and Sweden (Brydges, 2021; Riemens et al., 2021; Sandvik & Stubbs, 2019). In Vietnam, research often emphasizes supply chain initiatives (Nayak et al., 2022) or luxury brands' perspectives on sustainability (K. T. Nguyen et al., 2023), neglecting the behaviors and perceptions of average consumers. Methodologically, many studies rely on small sample sizes or qualitative methods, limiting their scope and generalizability. This research aims to bridge these gaps by exploring the barriers to sustainable fashion consumption in Vietnam, examining consumer attitudes and behaviors, and proposing tailored strategies that consider local economic and cultural dynamics (Brydges, 2021; Riemens et al., 2021; Sandvik & Stubbs, 2019).

Recognizing the unique challenges and opportunities in Vietnam, where traditional models often fall short in capturing the complexities of innovation, this study adapts and enhances these models to suit the local context. To address the multifaceted barriers to sustainable fashion consumption in Vietnam, the Multi-Criteria Decision-Making (MCDM) approach is applied, integrating Neutrosophic Sets (NS) with Z-Numbers (NZN). NS marks a significant improvement over traditional Fuzzy Sets (FS) by enhancing the ability to model uncertainty. Unlike FS, which uses a single membership grade to represent uncertainty, NS employs three distinct components—truth, indeterminacy, and falsity—offering a more robust and flexible approach (P. H. Nguyen et al., 2024). This structure provides a richer and more nuanced framework to address hesitation, ambiguity, and contradiction inherent in the perception of sustainability among consumers. Compared to fuzzy set extensions such as Triangular Fuzzy Numbers (TFNs) (Zadeh, 1965), Intuitionistic Fuzzy Sets (IFS) (Atanassov, 1986), Pythagorean Fuzzy Sets (PFS) (Yager, 2013), Picture Fuzzy Sets (Bui, 2015), and Spherical Fuzzy Sets (Gündoğdu, 2020), NS offers unparalleled flexibility. TFNs rely on rigid triangular distributions (α with $\alpha \in [0,1]$) that fail to capture non-membership or hesitation, while IFS adds non-membership degrees but does not account for indeterminacy (P. H. Nguyen et al., 2023). In addressing this, Type-2 fuzzy sets (Sepúlveda et al., 2007) tackle uncertainty, and they fail to capture indeterminacy, which is crucial in modeling consumer behaviors influenced by diverse social and environmental factors. PFS and Spherical Fuzzy Sets incorporate degrees of hesitation but impose summation or squared constraints that limit adaptability in real-world scenarios (P. H. Nguyen, 2024). In contrast, NS treats truth, indeterminacy, and falsity as independent variables ranging freely between 0 and 1, allowing for more accurate modeling of consumer behaviors where ambiguity, skepticism, and contradictory perceptions coexist.

Previous NS and MCDM methods applications have primarily focused on domains such as supply chain management, renewable energy, risk assessment, and urban planning. Hosny et al. (2021) employed a Neutrosophic Analytical Hierarchy Process (AHP) with Multiplicative Multi-Objective Optimization based on Ratio Analysis (MULTIMOORA) to optimize sustainable supply chains, while Kalantari et al. (2022) applied a neutrosophical model for closed-loop supply chain networks, addressing inflation and carbon emissions. Yel et al. (2023) utilized Interval-Valued NZN to assess critical success factors in software projects, demonstrating the framework's ability to combine subjective judgments with quantitative evaluations. Additionally, Borah et al. (2023) introduced Quadripartitioned Single-Valued Neutrosophic Z-Numbers (QSVNZN) to model diverse

COVID-19 scenarios, further expanding NZN's applicability in risk management. While these studies demonstrate the versatility of NZN and NS in addressing sustainability challenges, their application in sustainable fashion remains limited. Notably, existing research often neglects the critical role of reliability in modeling consumer perceptions and behaviors, particularly in markets like Vietnam, where cultural, economic, and social factors uniquely influence attitudes toward sustainable fashion. Furthermore, the gap between consumer intent and action, compounded by skepticism toward brand sustainability claims and the high perceived costs of eco-friendly products, highlights the need for a framework that addresses both uncertainty and data reliability.

1.3 Research questions, objectives, and contributions

To achieve this goal, the primary objective is to explore and provide answers to these central research questions:

(i) What factors hinder the successful adoption of sustainable consumption practices in the Vietnamese fashion industry?
(ii) How are these barriers connected within the context of implementing sustainable consumption initiatives?
(iii) How significant are these barriers in affecting the adoption of sustainable consumption practices within the Vietnamese industry?

As a consequence of the previously outlined research questions in the Vietnamese context, this study will strive to achieve the following objectives:

(i) Identify the major barriers hindering sustainable consumption.
(ii) Examine the interconnections among these significant barriers and their cause-effect relationship.
(iii) To provide recommendations for overcoming major barriers and promoting sustainability in the Vietnamese Fashion Industry.

This research offers significant contributions to both theoretical understanding and practical application in sustainable consumption, specifically emphasizing the Vietnamese fashion industry. Theoretically, it introduces a novel approach by integrating NZN with Delphi and DEMATEL methods. This combination offers a robust framework for addressing the complexity and uncertainty of consumer behavior, which traditional models often overlook. Methodologically, the NZN application represents an important advancement, improving the precision of decision-making models by accommodating the inherent uncertainty and vagueness associated with consumer perceptions. Furthermore, integrating expert-driven methods (Delphi) with causal analysis (DEMATEL) provides a systematic approach for prioritizing barriers, offering a replicable framework for future research in other industries or regions. Practically, the findings highlight the critical role of Policy and Education as the most influential drivers of sustainable consumption in the fashion industry. These insights are particularly relevant for designing targeted interventions and developing comprehensive policies that support sustainability. Finally, this study contributes context-specific insights by focusing on the Vietnamese fashion industry, an emerging market with unique sustainability challenges.

The remainder of this study is organized as follows: Section 2 provides an overview of relevant literature on sustainable fashion, highlighting key developments and challenges in the fashion industry. Section 3 introduces the foundational concepts of NZN and their leading operators in decision-making frameworks. Section 4 analyzes the data collected, contextualized within Vietnam's fashion industry, and discusses the key findings. Finally, Section 5 outlines the study's implications for theory and practice, addresses its limitations, and proposes potential directions for future research.

2. Literature Review

2.1. Literature Review on Neutrosophic Set, Z-Numbers and NZN applications.

MCDM methods have been widely utilized to analyze the complex interdependencies among factors in uncertain systems. Vishwakarma et al. (2022) emphasize that DEMATEL is particularly effective for distinguishing cause-and-effect relationships, as demonstrated in their identification of critical barriers in sustainable supply chain management. Similarly, Chen et al. (2021) applied DEMATEL to assess key obstacles to implementing the circular economy, showcasing its ability to map relational structures. Riemens et al. (2021) extended these approaches by employing the Delphi-Régnier method to investigate challenges in textile recycling, such as insufficient eco-design practices and policy gaps, thereby offering insights into enhancing circular practices. When combined with expert-driven techniques such as the Delphi method, as shown by Ruano et al. (2023) in their study on sustainability indicators in Belize's ecotourism sector, MCDM methods enhance decision-making frameworks by incorporating expert consensus and systematic prioritization of factors

(Kilic & Yalcin, 2021). Additionally, these methods have been employed to evaluate sustainable supply chains. Ruano et al. (2023) applied Fuzzy DEMATEL to analyze sustainability indicators, emphasizing their adaptability in varied contexts.

Table 1. *Related works*

Authors	**Methods**	**Domain**
Chen et al. (2024)	NZN, Logarithmic Similarity Measure	Evaluates teaching quality.
Diznarda Álvarez Gómez et al. (2022)	Neutrosophic AHP, DEMATEL	Sustainable tourism development.
Hosny et al. (2021)	Neutrosophic AHP, MULTIMOORA	Sustainable suppliers.
Kilic & Yalcin (2021)	Neutrosophic DEMATEL, TOPSIS	Municipalities' environmental sustainability performance
Veeramani et al. (2022)	Neutrosophic DEMATEL	Financial ratios for portfolio optimization in the NASDAQ exchange.
Nguyen et al.(2024)	NZN, DEMATEL, COCOOSO	Cybersecurity risks.
Meza Taipe et al. (2024)	Fuzzy Set Theory, Neutrosophic Sets	Public service quality for native communities in Ucayali.
Ruano et al. (2023)	Fuzzy Delphi, Fuzzy DEMATEL	Sustainable ecotourism.
Vishwakarma et al. (2022)	ISM, Fuzzy MICMAC, DEMATEL	Critical barriers impacting SSCM in the apparel sector.
Riemens et al. (2021)	Delphi-Régnier Method	Challenges in textile recycling to enhance circular practices.
Chen et al. (2021)	DEMATEL, Fuzzy Delphi	Prioritizes barriers to implementing circular economy in the textile industry.

As shown in **Table 1**, the evolution of uncertainty modeling has seen significant advancements with the introduction of NS and Z-numbers; both provide sophisticated frameworks for addressing indeterminacy and imprecision in complex decision-making scenarios. NS, introduced by Smarandache, generalizes classical fuzzy set theory by incorporating three independent membership functions: truth (T), indeterminacy (I), and falsity (F). Unlike traditional fuzzy sets, NS allows these membership functions to operate independently within the interval $]^{-}0, 1^{+}[$, providing a comprehensive approach to managing vagueness, inconsistency, and contradiction in information. To enhance its practicality, Wang et al. (2010) developed Single-Valued Neutrosophic Sets (SVNS), which constrain the membership values to the interval [0,1], making NST more suitable for real-world applications. NST has been successfully applied across diverse fields, including supply chain management, human resources, medical diagnosis, and project management. For instance, Şahin & Yigider (2014) applied NS to optimize sustainable supply chains, while Comas Rodríguez et al. (2022) utilized it to address uncertainties in project parameters like costs and resource availability, demonstrating the adaptability of NS to intricate decision-making challenges.

Z-numbers, proposed by Zadeh (2011a), offer another significant advancement by explicitly incorporating the reliability of information. A Z-number is an ordered pair ($Z = (A, B)$), where (A) represents a fuzzy restriction on a variable and (B) denotes the reliability of (A). This dual-layer framework provides a more nuanced representation of uncertainty, addressing both the degree of imprecision and the trustworthiness of the data. Zhang et al. (2023) demonstrated Z-numbers' utility in system reliability assessments, showing their capability to integrate probabilistic and possibilistic uncertainties for more realistic reliability estimates.

The integration of NS and Z-numbers has led to the development of NZN, which combines the multi-dimensional uncertainty modeling of NS with the reliability measures of Z-numbers. NZN frameworks have effectively addressed complex decision-making problems by integrating degrees of truth, falsity, and indeterminacy with reliability scores (Borah & Dutta, 2023). Applications of NZN include cybersecurity risk evaluation (P. H. Nguyen et al., 2024), where a Delphi-DEMATEL-COCOSO approach combined with NZN effectively prioritized strategies in Vietnam's finance sector. Du et al.(2021) introduced NZN-weighted aggregation operators to enhance decision-making precision, while Ye et al.(2021) expressed the inconsistent, incomplete, and indeterminate knowledge and judgments of human cognitions in the real world to refine multi-criteria decision-making methods.

Despite these advancements, the application of NZN in sustainable fashion remains limited. The industry faces unique challenges, including consumer skepticism, greenwashing, and cost barriers, which require nuanced approaches to uncertainty modeling. NZN offers significant advantages in this context by capturing the complexities of consumer attitudes—such as acceptance (truth), doubts about brand authenticity (falsity), and hesitation due to cost or availability (indeterminacy)—and integrating reliability measures to address issues like the credibility of sustainability certifications and brand claims. This dual capability makes NZN particularly suited for analyzing and overcoming barriers to sustainable fashion consumption. By leveraging NZN's robust uncertainty modeling features, this study provides a targeted framework for addressing barriers to sustainable fashion in Vietnam, setting a precedent for advanced decision-making in sustainability efforts across various domains.

The integration of the Delphi (Diznarda Álvarez Gómez et al., 2022), DEMATEL (Torres & Vargas, 2014), and NZN (Du et al., 2021) represents a sophisticated and synergistic approach to addressing complex, uncertain, and interdependent challenges sustainably. The Delphi method, developed in the 1950s, facilitates consensus-building among experts through iterative rounds of questionnaires. While effective for prioritizing issues and identifying critical factors, traditional Delphi methods face limitations in addressing uncertainty and variability in expert judgments. The NZN-enhanced Delphi method overcomes these shortcomings by incorporating the multi-dimensional uncertainty modeling capabilities of NZNs. By integrating truth, indeterminacy, falsity, and reliability measures, NZN Delphi provides a nuanced and robust framework for analyzing ambiguous expert opinions, ensuring a focused and precise identification of the most critical barriers to sustainable fashion adoption. Complementing Delphi, the DEMATEL method maps and analyzes interdependencies among factors within complex systems. By constructing cause-and-effect frameworks, DEMATEL reveals root causes and cascading effects, enabling decision-makers to identify primary drivers and prioritize interventions. Unlike other MCDM methods, such as AHP, which focuses on short-term prioritization, DEMATEL provides long-term insights by clarifying causal relationships among factors. The DEMATEL process—from constructing direct-relation matrices to visualizing results in cause-and-effect diagrams—offers actionable insights highlighting critical elements and root causes. Its application in sustainability-focused research has proven invaluable, particularly in identifying systemic barriers and designing strategies with cascading benefits.

Integrating NZN, Delphi, and DEMATEL creates a comprehensive decision-making framework. NZNs enhance the process by addressing the uncertainty and reliability often inherent in data and expert judgments, ensuring a more accurate representation of barriers and interdependencies. The combined approach is particularly suitable for the sustainable fashion sector, where challenges such as consumer skepticism, greenwashing, and cost barriers require nuanced analysis. For instance, NZN captures consumer skepticism (falsity), hesitation due to availability or cost (indeterminacy), and trust in certifications (reliability), enabling a detailed examination of how these factors interact within the broader system.

This integrated methodology provides several advantages. First, it enhances uncertainty modeling through NZN, enabling a deeper understanding of multi-dimensional challenges. Second, it leverages Delphi's structured consensus-building and DEMATEL's causal analysis to identify root causes and prioritize impactful interventions. Third, it offers actionable insights tailored to Vietnam's sustainable fashion sector's unique cultural, economic, and social context. By combining expert-driven consensus, systemic analysis, and robust uncertainty modeling, this framework bridges gaps in existing research, setting a new standard for addressing sustainability challenges in dynamic environments. Ultimately, this approach facilitates targeted, data-driven strategies to overcome barriers, drive consumer engagement, and promote systemic change in sustainable fashion practices.

2.2 Sustainability in the Fashion Industry

The fashion industry, valued at $2.4 trillion, is recognized as one of the most environmentally unsustainable sectors due to the prevalence of "fast fashion," drastically altering consumer buying and disposal behaviors. Fast fashion prioritizes low-cost, trendy garments, resulting in mass production and consumption that significantly contributes to global textile waste (Vladimirova et al., 2022). This unsustainable business model has led to a surge in material throughput, with brands producing nearly double the clothing output compared to pre-2000 levels. The environmental consequences are extensive, spanning all production phases and including issues such as excessive water use, toxic chemicals, and significant energy consumption (Niinimäki et al., 2020). These practices harm ecosystems and pose health risks to factory workers and consumers.

As sustainability awareness grows, the fashion industry gradually adopts practices emphasizing ethical and eco-friendly standards, reflecting a shift toward responsible consumerism (Amed et at, 2019; Khandual & Pradhan, 2019). This includes a holistic approach to sustainable fashion, focusing on eco-friendly materials,

reduced carbon footprints, water conservation, fair labor practices, and extended garment lifecycles (Henninger et al., 2016). Sustainable fashion encompasses diverse approaches, including using organic fabrics, promoting recycling, encouraging clothing swaps, and adopting circular economy principles [38]. For instance, brands like H&M and Patagonia have pioneered initiatives such as garment recycling and resale programs to reduce waste and promote sustainability (Patagonia, 2022).

In Vietnam, companies like An Phuoc are implementing eco-friendly collections using organic and recycled materials, while denim brands are researching water-saving technologies and reducing harmful chemical use in collaboration with the Ministry of Science and Technology (*Sustainable Denim Production Using Lasers and Ozone*, 2024). Furthermore, this study employs the Theory of Planned Behavior (TPB) and the Black Box Model of Consumer Behavior to explore barriers to sustainable fashion consumption in Vietnam. TPB posits that behavior is influenced by attitudes, subjective norms, and perceived behavioral control, providing a robust framework for analyzing consumer intentions (Ajzen, 1991). Saricam et al. (Saricam & Okur, 2019) emphasize the adaptability of TPB across scenarios, including sustainable consumption, by incorporating additional factors like self-identity, ethical values, and past experiences. By integrating TPB with the Black Box Model of Consumer Behavior, this study broadens the analytical lens to include both external and internal factors influencing consumer decision-making. The Black Box Model focuses on the internal cognitive processes that drive consumer behavior, such as perception, motivation, and decision-making, while accounting for external stimuli like marketing messages, product availability, and pricing (Kotler & Keller, 2016). This combined approach allows a nuanced understanding of how external environmental cues interact with internal psychological factors to shape consumer attitudes and behaviors toward sustainable fashion. In the context of Vietnam, where cultural, economic, and social factors uniquely influence consumer decisions, the integration of these models provides a comprehensive framework for identifying and addressing the key barriers to sustainable fashion adoption.

Despite these advancements, significant barriers remain to sustainable fashion adoption. Practical challenges, such as high costs, limited product variety, and the dominance of fast fashion trends, hinder consumers' ability to make sustainable choices (Brandão & Costa, 2021). Although many consumers are willing to purchase from sustainable brands, actual adoption rates remain low due to cost, accessibility, and style preferences (Lambert, 2019). These challenges are particularly evident in Vietnam, where younger consumers show greater interest in sustainability while older generations prioritize cost and convenience (Lambert, 2019). Younger consumers are more receptive to these initiatives, while older generations are less engaged. The effectiveness of these campaigns is influenced by factors such as awareness, accessibility, and budget-friendliness of sustainable clothing options. Local initiatives, such as Green Yarn, are actively fostering sustainable fashion through educational programs via CSR initiatives and workshops such as Fashion Rob My Future (*Fashion Rob My Future – Sustainable Fashion Talkshow*, 2020). These efforts are designed to educate consumers on the environmental implications and offer practical advice for sustainable living, such as using natural materials like Pineapple Leaf Fiber (*Exploring the Applications of Pineapple Yarn in Everyday Life*, 2024).

2.3 Literature Review on Barriers to Sustainable Fashion

The adoption of sustainable practices in fashion remains a significant global challenge, with a pronounced gap between consumer intentions and actual behaviors. Understanding these barriers—categorized into internal and external factors—is essential for promoting sustainable practices, particularly in Vietnam's fashion sector, where cultural, economic, and infrastructural influences shape unique consumption patterns.

2.3.1 Internal Barriers: Social, Cultural, and Perceptual Challenges

Internal barriers to sustainable fashion adoption encompass social and cultural influences, cost concerns, and consumer perceptions of affordability and image. Social and cultural dynamics significantly shape consumer attitudes. While high pricing is often perceived as a primary obstacle, research by Ronda (2024) highlights that consumers are willing to spend more on sustainable products but struggle to find options that align with their tastes and styles. This suggests that availability and design are critical barriers rather than cost alone. Furthermore, a lack of awareness about the environmental harm caused by fast fashion limits consumer action (Becker-Leifhold & Iran, 2018; Lorena Ronda, 2024). Despite growing awareness of sustainability, many consumers remain unaware of the ecological damage their fashion choices cause, further hindering their shift toward sustainable practices.

Perception issues also play a crucial role. Many consumers consider sustainable fashion less stylish or appealing, associating it with subpar design or limited variety (Moon et al., 2014; Sehnem et al., 2024; Vishwakarma et al., 2022). This misconception, compounded by inadequate training in sustainable practices among designers, results in a scarcity of attractive eco-friendly options (Harris et al., 2016; Moon et al., 2014).

Addressing these barriers requires educating consumers and empowering designers to create innovative, stylish, sustainable collections.

2.3.2 External Barriers: Policy, Education, and Industry Practices

External barriers include policy limitations, gaps in education, and unsustainable supply chain practices. The lack of sustainability-focused education in fashion design courses limits professionals' ability to advocate for and implement sustainable practices effectively. For instance, curricula often omitted crucial concepts like upcycling and eco-friendly fashion(Mizrachi & Alon, 2022). Similarly, government policies, such as VAT structures that discourage second-hand clothing purchases or rentals, fail to incentivize circular economy practices.

Additionally, the absence of standardized, transparent certification programs makes it difficult for consumers to identify sustainable products, perpetuating greenwashing (Adamkiewicz et al., 2022; Lorena Ronda, 2024). Studies in markets like the UK have shown that many sustainability claims fail to meet established guidelines, leading to consumer distrust. Establishing clear certification criteria, as proposed by Mizrachi et al. (2022), could address this issue and promote informed consumer choices.

Industry practices, particularly in supply chains, exacerbate barriers to sustainability. Outsourcing production to regions with cheaper labor and lax environmental regulations drives fast fashion's growth, often at significant environmental and social costs (Bick et al., 2018; Nayak, Panwar, et al., 2019). Greenwashing further complicates the landscape, as companies make misleading claims about the sustainability of their products, undermining trust in ethical brands and hindering their success.

2.2.3 A Multidimensional Framework for Understanding Barriers

Effectively addressing barriers to sustainable fashion consumption requires a comprehensive and nuanced approach. This study adopts a multidimensional framework, categorizing these barriers into five key dimensions: Social and Cultural Factors, Pricing and Affordability, Policy and Education, Perception and Image, and Supply Chain and Industry Practices. This structure offers an in-depth analysis of the challenges that hinder the widespread adoption of sustainable fashion practices.

(i) **Social and Cultural Factors:** The pervasive influence of fast fashion culture and societal norms that encourage frequent wardrobe updates and disposable consumption presents a significant barrier to sustainable fashion. Misconceptions that sustainable clothing lacks style or prestige further deter adoption (Bianchi & Gonzalez, 2021). Media portrayals and social influences often reinforce these attitudes, making realigning sustainability with cultural preferences and values imperative.

(ii) **Pricing and Affordability**: Economic barriers, particularly the higher costs of sustainable fashion compared to fast fashion, limit accessibility for many consumers. The fast fashion industry's reliance on low-cost, rapid production exacerbates these affordability challenges (Brandão & Costa, 2021). To address this issue, supply chains must be optimized to reduce costs, and the long-term value of sustainable fashion—such as durability and environmental benefits—must be effectively communicated (Harris et al., 2016).

(iii) **Policy and Education**: Comprehensive policy frameworks and educational initiatives are critical in fostering sustainable practices. Governments can drive change by implementing subsidies, developing transparent certification standards, and enforcing stricter regulations (Mizrachi & Alon, 2022). At the same time, integrating sustainability concepts into educational programs is vital for equipping future professionals and consumers with the knowledge and skills needed to adopt eco-friendly practices (Martín-Sánchez et al., 2022).

(iv) **Perception and Image**: Consumer perceptions significantly influence the adoption of sustainable fashion. Misconceptions about the quality, style, and practicality of sustainable clothing remain widespread, creating resistance to change. Overcoming these barriers requires reshaping public perceptions through positive branding, enhancing product design, and communicating sustainable fashion's environmental and societal benefits(Munir, 2020).

(v) **Supply Chain and Industry Practices**: Systemic issues within supply chains, such as greenwashing, lack of transparency, and high production costs, are substantial barriers to sustainability. Unsustainable practices, often driven by outsourcing to regions with lower labor costs and lax environmental regulations, exacerbate these challenges. Collaboration among stakeholders—including brands, manufacturers, and suppliers—is essential to improve sustainability, enforce accountability, and address financial and technological constraints (W. K. Chen et al., 2021).

2.3.4 Research Contribution and Context in Vietnam

This research identifies 47 key barriers to sustainable fashion consumption through an extensive literature survey validated by expert feedback (as outlined in **Table 2**). These barriers are classified into the five dimensions discussed above, enabling a deeper understanding of their interplay and cumulative impact.

Table 2. Barriers to Sustainable Consumption in the Fashion Industry.

No	Dimensions	Code	Barrier	Definition	References
1	Social and Cultural Factors	SC1	Emergence of Affordable Fashion	Affordable fashion makes it harder for consumers to choose sustainable options.	(Riemens et al., 2021)
2		SC2	Sustainable Fashion Stereotypes	Preconceived notions that sustainable fashion could be less stylish, expensive, or impractical.	(Bianchi & Gonzalez, 2021)
3		SC3	Fashion as a Status Symbol	Consumers prioritize fashion as a status symbol over sustainability.	(Vishwakarma et al., 2022)
4		SC4	Social and Media-Induced Wardrobe Updates	Social media pressure leads consumers to buy more often, reducing sustainable choices.	(Harris et al., 2016)
5	Pricing and Affordability	E01	Economic Growth Impact on Consumption	Increased income drives more spending on fast fashion, which is not a sustainable option.	(Ozdamar Ertekin et al., 2020)
6		E02	High Prices of Sustainable Fashion	Higher costs deter consumers from purchasing sustainable fashion.	(Brandão & Costa, 2021)
7		E03	Consumer Budget Constraints	Limited budgets lead consumers to choose cheaper, less sustainable fashion.	(Harris et al., 2016)
8		E04	Balancing Sustainability and Profitability	Businesses struggle to offer sustainable products that appeal to cost-conscious consumers.	(Monyaki & Cilliers, 2023)
9		E05	Revenue Decline from Long-Lasting Products	Consumers buy less frequently when products last longer, reducing business revenue.	(Harris et al., 2016)
10	Policy and Education	PE1	Social and Cultural Norms Against Sustainability	Cultural norms discourage sustainable consumption habits among consumers.	(Harris et al., 2016) ; (Mizrachi & Alon, 2022)

11		PE2	Changing Definitions of Waste Laws in Fashion	Evolving laws on waste affect consumer access to sustainable fashion.	(Mizrachi & Alon, 2022)
12		PE3	Legal Accountability for Corporations	Corporate responsibility impacts consumer trust sustainably.	(Mizrachi & Alon, 2022)
13		PE4	Advertising Regulations in Fashion	Regulations on advertising affect consumer awareness of sustainable options.	(Mizrachi & Alon, 2022)
14		PE5	Educational Campaigns on Sustainable Consumption	Lack of effective education reduces consumer motivation to choose sustainable fashion.	(Mizrachi & Alon, 2022)
15		PE6	Low Consumer Awareness and Education About Green Consumption	Consumers are less likely to buy sustainably due to a lack of knowledge.	(Harris et al., 2016)
16		PE7	Carbon Taxation on Apparel	Carbon taxes could increase prices, making sustainable fashion less affordable for consumers.	(Mizrachi & Alon, 2022)
17		PE8	Subsidies for Unsustainable Products	Subsidies lower the cost of unsustainable fashion, discouraging sustainable choices.	(Lueg et al., 2015)
18		PE9	Lack of Standards for Sustainable Fashion	Inconsistent standards confuse consumers and hinder sustainable purchasing decisions.	(Bly et al., 2015)
19		PE10	Limited Education Level of a Consumer Segment	Low education levels prevent some consumers from understanding the benefits of sustainable fashion.	(Monyaki & Cilliers, 2023)
20	Perception and Image	PI1	Price Sensitivity Among Consumers	Price-sensitive consumers often choose cheaper, less sustainable options.	(Harris et al., 2016)
21		PI2	Brand-Conscious Consumer Behavior	Consumers prefer well-known brands, even if they are not sustainable.	(Harris et al., 2016)

22	PI3	Focus on Bargain Shopping	The focus on finding bargains leads consumers away from sustainable options.	(Bly et al., 2015); (Medalla et al., 2021)
23	PI4	Shopping as a Leisure Activity	Viewing shopping as entertainment increases the consumption of unsustainable fashion.	(Bly et al., 2015); (Medalla et al., 2021)
24	PI5	Influence of Fashion Trends	Fast-changing trends push consumers to buy frequently, making sustainable choices less appealing.	(Legere & Kang, 2020)
25	PI6	Degree of Consumer Involvement in Fashion	Consumers highly involved in fashion may prioritize trends over sustainability.	(Bly et al., 2015)
26	PI7	Perceived Unattractiveness and Quality of Sustainable Fashion	Negative perceptions about style and quality reduce consumer interest in sustainability.	(Harris et al., 2016)
27	PI8	Misunderstanding of Sustainable Fashion	Misconceptions about sustainability discourage consumers from buying sustainable fashion.	(Harris et al., 2016)
28	PI9	Lack of Focus on Durability	Consumers often overlook durability, preferring cheaper, short-lived fashion.	(Bly et al., 2015)
29	PI10	Aspirational Luxury Fashion	A desire for luxury brands drives consumers towards non-sustainable options.	(Bly et al., 2015)
30	PI11	Impact of Income Levels on Fashion Choices	Lower income limits consumer access to sustainable fashion.	(Harris et al., 2016)
31	PI12	Perception of Sustainable Fashion as Inconvenient	The perceived inconvenience of sustainable fashion reduces consumer interest.	(Harris et al., 2016)
32	PI13	Perceived Lack of Variety and Styles in Sustainable Fashion	Limited style options in sustainable fashion deter consumers.	(Harris et al., 2016)

33		PI14	Complexity and Confusion Around Sustainability	Confusion about sustainability concepts makes consumers hesitant to choose sustainable fashion.	(Bly et al., 2015)
34		PI15	Materialistic Consumer Behavior	Materialism drives consumers to buy more, often choosing unsustainable options.	(Medalla et al., 2021)
35		PI16	Consumer Overconsumption	Excessive buying habits prevent consumers from embracing sustainability.	(Harris et al., 2016)
36		PI17	Disposable Attitude Toward Clothing	Viewing clothes as disposable leads to frequent purchases of unsustainable fashion.	(Harris et al., 2016)
37		PI18	Skepticism Towards the Authenticity of Producers' Sustainability Claims	Doubts about the authenticity of sustainability claims reduce consumer trust.	(Adamkiewicz et al., 2022)
38	Supply Chain and Industry Practices	SI1	Limited Product Variety	The lack of options makes it difficult for consumers to find sustainable fashion that suits their needs.	(Monyaki & Cilliers, 2023)
39		SI2	Supply Chain Transparency Issues	Lack of transparency makes it hard for consumers to verify sustainability claims.	(Bick et al., 2018)
40		SI3	Fair Wages and Working Conditions Challenges	Ethical concerns about production conditions affect consumer choices.	(Nayak et al., 2022)
41		SI4	Greenwashing Practices in the Industry	Misleading claims about sustainability confuse consumers and erode trust.	(Adamkiewicz et al., 2022)
42		SI5	Limited Promotion of Sustainable Brands	Insufficient marketing makes consumers less aware of sustainable options.	(Monyaki & Cilliers, 2023)
43		SI6	Financial Constraints and Scalability Concerns Due to Insufficient Investment in Sustainable Infrastructure	Lack of investment hinders the availability and accessibility of sustainable fashion.	(Monyaki & Cilliers, 2023)
44		SI7	Technological Challenges in Production	Technology limitations restrict the production and variety of sustainable fashion.	(W. K. Chen et al., 2021)

45	SI8	Difficulty in Scaling Local Production	Challenges in expanding sustainable fashion from local to larger markets affect availability.	(Sandvik & Stubbs, 2019);
46	SI9	High Production Costs	High costs of sustainable production limit consumer options and affordability.	(W. K. Chen et al., 2021)
47	SI10	Shift in the Fashion Industry Toward the Circular Economy Trend	The transition to circular economy practices is slow, limiting sustainable fashion choices for consumers.	(W. K. Chen et al., 2021); (Sehnem et al., 2024)

Vietnam's unique cultural and economic context underscores the importance of targeted solutions. For instance, younger Vietnamese consumers exhibit greater openness to sustainable practices, motivated by heightened awareness of environmental issues and a desire for value-driven purchases. In contrast, older generations often prioritize affordability and convenience, presenting a distinct set of challenges for sustainable fashion adoption. Moving beyond a simplistic internal-external categorization, this study's multidimensional framework captures the complexity of these barriers and their interconnections. This approach enhances the ability to design and implement targeted interventions that address the root causes of resistance to sustainable practices. It provides actionable insights for businesses, policymakers, and educators to align their strategies with consumer needs and industry demands. Ultimately, this study contributes to developing a more sustainable fashion industry in Vietnam by bridging the gap between consumer intentions and actual practices. By addressing barriers across social, economic, educational, perceptual, and systemic dimensions, this framework equips stakeholders with the tools to promote sustainable consumption effectively and foster long-term change.

3. Proposed Methods

3.1 Research process

Building on current research in sustainable consumption, this study presents an all-encompassing model to investigate the barriers to sustainable fashion within the Vietnamese context.
It employs an MCDM approach, integrating the DELPHI and DEMATEL methods along with NZN, as shown in **Figure 1**.

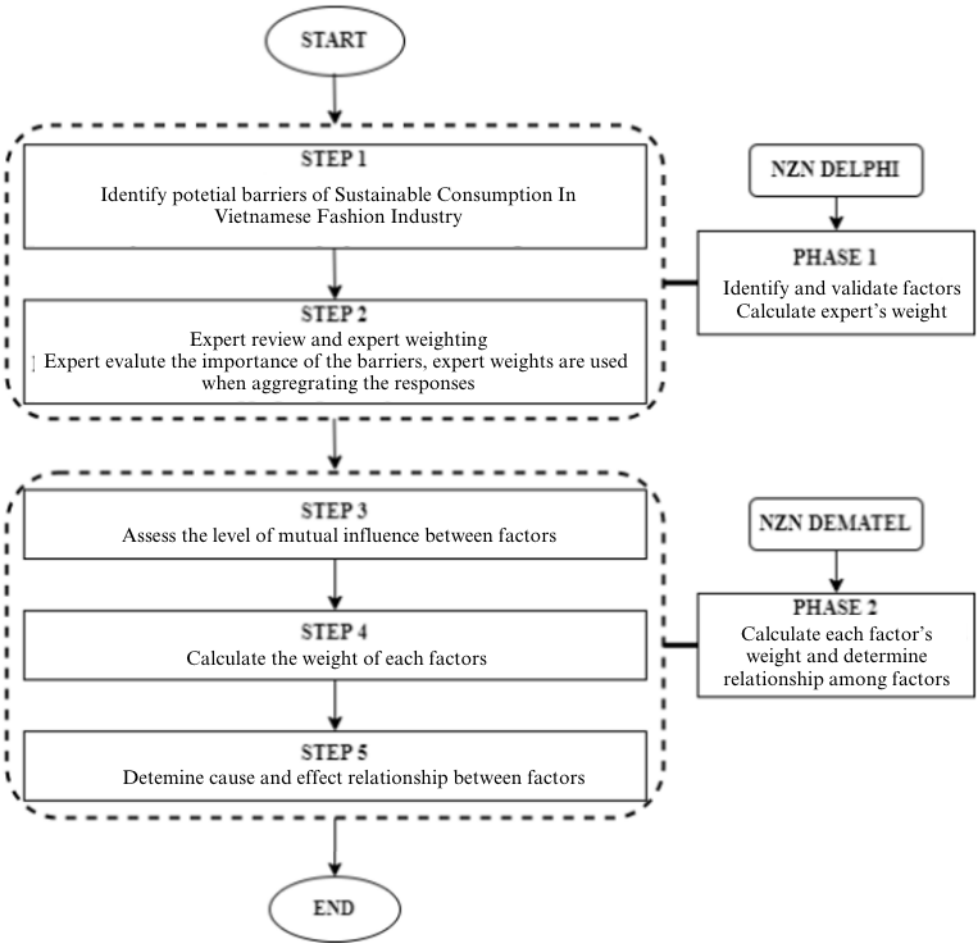
START
STEP 1
Identify potetial barriers of Sustainable Consumption In Vietnamese Fashion Industry
STEP 2
Expert review and expert weighting
Expert evalute the importance of the barriers, expert weights are used when aggregrating the responses
NZN DELPHI
PHASE 1
Identify and validate factors
Calculate expert's weight
STEP 3
Assess the level of mutual influence between factors
STEP 4
Calculate the weight of each factors
STEP 5
Detemine cause and effect relationship between factors
NZN DEMATEL
PHASE 2
Calculate each factor's weight and determine relationship among factors
END

Figure 1. Research Framework

The first phase focuses on identifying key barriers to sustainable practices within the fashion industry in Vietnam. This is achieved through a comprehensive literature review, categorizing the barriers into five main groups: Social and Cultural Factors, Affordability and Pricing, Policy and Education, Image and Perception, and Supply Chain and Industry Practices. The relevance and appropriateness of these barriers are assessed using the NZN DELPHI method. In this phase, structured questionnaires are distributed to experts who evaluate the significance and impact of each barrier. Expert evaluations are weighted based on their expertise and credentials, with these weights converted into NZN values. The data is then aggregated and defuzzified to yield precise and consistent results, ensuring reliable expert input throughout the research. Barriers that do not meet the established criteria are excluded from further analysis.

The second phase investigates the relationships among the validated barriers using the DEMATEL method in combination with the NZN framework. This phase addresses the complex interdependencies between factors, mapping out cause-and-effect relationships. The DEMATEL analysis creates an influence matrix that quantifies the impact of each barrier. The influence matrix undergoes defuzzification to maintain accuracy, ensuring all calculations align with the NZN format.

Data analysis was carried out using Microsoft Excel 2016 and SPSS version 26. Each evaluation, which involved processing input from 30 experts, examining 47 barriers, and ranking various strategies, required approximately 180 minutes. The findings from this study provide valuable guidance for researchers and industry professionals alike, delivering a systematic, evidence-based approach to overcoming obstacles to sustainable fashion consumption in Vietnam.

3.2 Preliminaries

Zadeh (2011b) is defined as a structured pair of fuzzy numbers (A, C), where A conveys the fuzzy value linked to an uncertain variable X, and C reflects the reliability of that value. However, this model does not capture the inherent aspects of indeterminacy and falsity in Z-numbers. To overcome this shortcoming and to provide a more comprehensive framework that includes elements of truth, indeterminacy, and falsity, the NZN has been proposed as an advanced extension of the original framework (Du et al., 2021b).

Definition 1 (Du et al., 2021) : Consider X represents a universe set. An NZN set within X is described by Equation (1):

$$N_z = \{[x, \alpha(A,C)(x), \beta(A,C)(x), \gamma(A,C)(x) | x \epsilon X\} \quad (1)$$

In this context, we define $\alpha(A,C)(x) = \big(\alpha_A(x), \alpha_C(x)\big)$; $\beta(A,C)(x) = \big(\beta_A(x), \beta_C(x)\big)$; $\gamma(A,C)(x) = \big(\gamma_A(x), \gamma_C(x)\big)$ where each component maps X to the interval $X \to [0,1]^2$. These components represent fuzzy values corresponding to truth, indeterminacy, and falsity. Here, A stands for the neutrosophic values associated with elements of the set X, and C is the measure of reliability for those values within the neutrosophic. Each component satisfies a set of predefined conditions, as specified below: $0 \le \alpha_A(x) + \beta_A(x) + \gamma_A(x) \le 3$ and $0 \le \alpha_C(x) + \beta_C(x) + \gamma_C(x) \le 3$

For simplicity and ease of understanding, the element $[x, \alpha(A,C)(x), \beta(A,C)(x), \gamma(A,C)(x)$ in N_Z can be rewritten in a more compact notation as $N_Z = [\alpha(A,C), \beta(A,C), \gamma(A,C)] = [(\alpha_A, \alpha_C), (\beta_A, \beta_C), (\gamma_A, \gamma_C)$, and is referred to as NZN.

Definition 2 (Du et al., 2021; Yazdani et al., 2021): Let $N_{Z1} = [\alpha_1(A,C), \beta_1(A,C), \gamma_1(A,C)] = [(\alpha_{A1}, \alpha_{C1}), (\beta_{A1}, \beta_{C1}), (\gamma_{A1}, \gamma_{C1})$ and $N_{Z2} = [\alpha_2(A,C), \beta_2(A,C), \gamma_2(A,C)] = [(\alpha_{A2}, \alpha_{C2}), (\beta_{A2}, \beta_{C2}), (\gamma_{A2}, \gamma_{C2})$ considers two NZNs, with ϵ>0. The following correspondences are established through Equations (2)-(10):

1. $N_{Z1} \supseteq N_{Z2} \iff \alpha_{A1} \ge \alpha_{A2}$, $\alpha_{C1} \ge \alpha_{C2}$, $\beta_{A1} \le \beta_{A2}$, $\beta_{C1} \le \beta_{C2}$, $\gamma_{A1} \le \gamma_{A2}$ and $\gamma_{C1} \le \gamma_{C2}$ (2)
2. $N_{Z1} = N_{Z2} \iff N_{Z1} \supseteq N_{Z2}$ and $N_{Z2} \supseteq N_{Z1}$ (3)
3. $N_{Z1} \cup N_{Z2} \iff [(\alpha_{A1} \vee \alpha_{A2}, \alpha_{C1} \vee \alpha_{C2}), (\beta_{A1} \wedge \beta_{A2}, \beta_{C1} \wedge \beta_{C2}), (\gamma_{A1} \wedge \gamma_{A2}, \gamma_{C1} \wedge \gamma_{C2})$ (4)
4. $N_{Z1} \cap N_{Z2} \iff [(\alpha_{A1} \wedge \alpha_{A2}, \alpha_{C1} \wedge \alpha_{C2}), (\beta_{A1} \vee \beta_{A2}, \beta_{C1} \vee \beta_{C2}), (\gamma_{A1} \vee \gamma_{A2}, \gamma_{C1} \vee \gamma_{C2})$ (5)
5. $(N_{Z1})^C = [(\gamma_{A1}, \gamma_{C1}), (1-\beta_{A1}, 1-\beta_{C1}), (\alpha_{A1}, \alpha_{C1})$ (Complement of N_{Z1}) (6)
6. $N_{Z1} \oplus N_{Z2} = [(\alpha_{A1} + \alpha_{A2} - \alpha_{A1}\alpha_{A2}, \alpha_{C1} + \alpha_{C2} - \alpha_{C1}\alpha_{C2}), (\beta_{A1}\beta_{A2}, \beta_{C1}\beta_{C2}), (\gamma_{A1}\gamma_{A2}, \gamma_{C1}\gamma_{C2})$ (7)
7. $N_{Z1} \otimes N_{Z2} = [(\alpha_{A1}\alpha_{A2}, \alpha_{C1}\alpha_{C2}), (\beta_{A1} + \beta_{A2} - \beta_{A1}\beta_{A2}, \beta_{C1} + \beta_{C2} - \beta_{C1}\beta_{C2}), (\gamma_{A1} + \gamma_{A2} - \gamma_{A1}\gamma_{C2}, \gamma_{C1} + \gamma_{C2} - \gamma_{C1}\gamma_{C2})$ (8)
8. $\varepsilon N_{Z1} = [(1-(1-\alpha_{A1})^{\varepsilon}, 1-(1-\alpha_{A1})^{\varepsilon}), (\beta_{A1}^{\varepsilon}, \beta_{C1}^{\varepsilon}), (\gamma_{A1}^{\varepsilon}, \gamma_{C1}^{\varepsilon})$ (9)
9. $(N_{Z1})^{\varepsilon} = [\,(\alpha_{A1}^{\varepsilon}, \alpha_{C1}^{\varepsilon}), (1-(1-\beta_{A1})^{\varepsilon}, 1-(1-\beta_{C1})^{\varepsilon}), (1-(1-\gamma_{A1})^{\varepsilon}, 1-(1-\gamma_{C1})^{\varepsilon})$ (10)

The defuzzification process aims to convert these fuzzy values into a single crisp value for clearer interpretation and decision-making; the defuzzification process of

$N_{Z1} = [\alpha_1(A,C), \beta_1(A,C), \gamma_1(A,C)] = [(\alpha_{A1}, \alpha_{C1}), (\beta_{A1}, \beta_{C1}), (\gamma_{A1}, \gamma_{C1})$, using Equation (11):

$$DEF(N_{Z1}) = \frac{2 + \alpha_{A1}\,\alpha_{C1} - \beta_{A1}\,\beta_{C1} - \gamma_{A1}\,\gamma_{C1}}{3} \; for\; DEF(N_{Z1}) \epsilon [0,1 \quad (11)$$

Example illustration1: We have two numbers NZN be expressed as: N_{Z1} = [0.7,0.9),(0.2,0.1),(0.3,0.2) and N_{Z2} = [(0.5,0.7),(0.4,0.3),(0.2,0.1), ε = 0.6, an example of Equations (7) - (11) are calculated as follows:

$N_{Z1} \oplus N_{Z2}$ = 9(0.7,0.9),(0.2,0.1),(0.3,0.2) ⊕ [(0.5,0.7),(0.4,0.3),(0.2,0.1)

= [(0.85,0.97),(0.12,0.07),(0.06,0.02)

$N_{Z1} \otimes N_{Z2}$ =

[(0.7,0.9),(0.2,0.1),(0.3,0.2)⊗[(0.5,0.7),(0.4,0.3),(0.2,0.1)=[(0.35,0.63),(0.56,0.37),(0.46,0.28)

εN_{Z1} =0.6·[(0.7,0.9),(0.2,0.1),(0.3,0.2)=[(0.42,0.54),(0.12,0.06),(0.18,0.12

$(N_{Z1})^{\varepsilon}$ =([(0.7,0.9),(0.2,0.1),(0.3,0.2))0.6=[(0.7487,0.8748),(0.1429,0.0859),(0.2004,0.1348)

$$DEF(N_{Z1}) = \frac{2 + 0.7 \cdot 0.9 - 0.2 \cdot 0.1 - 0.3 \cdot 0.2}{3} = 0.85$$

Definition 3 (Du et al., 2021): **Weighted Aggregation Arithmetic Mean (NZNWAAM) and Weighted Aggregation Geometric Mean (NZNWAGM)**

Consider $N_{Zi} = [\alpha_i(A,C), \beta_i(A,C), \gamma_i(A,C)] = [(\alpha_{Ai}, \alpha_{Ci}), (\beta_{Ai}, \beta_{Ci}), (\gamma_{Ai}, \gamma_{Ci})]$, (i = 1, 2, 3, ... n) as a set of NZN. Let NZNWAA: $\Omega^n \longrightarrow \Omega$. This formulation facilitates the combination of multiple NZNs, offering a thorough depiction of consolidated information. It proves especially valuable in decision-making scenarios requiring the merging of uncertain and inconsistent data. The formal definition of the NZNWAA equation is presented in Equation (12), which captures the collective behavior of the given set of neutrosophic Z-numbers

through weighted averaging. This method ensures that each component of the NZN is considered proportionally according to its significance in the aggregation process (Du et al., 2021b), as presented in Equation (12):

$$\text{NZNWAA}(N_{Z1}, N_{Z2}, \dots, N_{Zn}) = \sum_{i=1}^{n} \varepsilon_i N_{Zi} \quad (12)$$

$$= [(1 - \prod_{i=1}^{n}(1 - \alpha_{Ai})^{\varepsilon_i}, 1 - \prod_{i=1}^{n}(1 - \alpha_{Ci})^{\varepsilon_i}), (\prod_{i=1}^{n} \beta_{Ai}{}^{\varepsilon_i}, \prod_{i=1}^{n} \beta_{Ci}{}^{\varepsilon_i}), (\prod_{i=1}^{n} \gamma_{Ai}{}^{\varepsilon_i}, \prod_{i=1}^{n} \gamma_{Ci}{}^{\varepsilon_i})$$

In which $\varepsilon_i (i = 1, 2, 3, \dots n)$ was defined as a weight of N_{Zi} under the condition $0 \leq \varepsilon_i \leq 1$ and $\sum_{i=1}^{n} \varepsilon_i = 1$

Similarly, by employing Equations (8) and (10) from Definition 2, the Weighted Aggregation Geometric Mean (NZNWAGM) equation for NZNs can be established.

Allow $N_{Zi} = [\alpha_i(A, C), \beta_i(A, C), \gamma_i(A, C)] = [(\alpha_{Ai}, \alpha_{Ci}), (\beta_{Ai}, \beta_{Ci}), (\gamma_{Ai}, \gamma_{Ci})], (\text{i} = 1, 2, \dots \text{n})$. This defines the capability to aggregate a group of NZNs, represented as NZNWGA: $\Omega^n \longrightarrow \Omega$. Representation of NZNWGA, as presented in Equation (13):

$$\text{NZNWGA}(N_{Z1}, N_{Z2}, \dots, N_{Zn}) = \prod_{i=1}^{n} (N_{Zi})^{\varepsilon_i} \quad (13)$$

$$= [(\prod_{i=1}^{n} (\alpha_{Ai})^{\varepsilon_i}, \prod_{i=1}^{n} (\alpha_{Ci})^{\varepsilon_i}), (1 - \prod_{i=1}^{n}(1 - \beta_{Ai})^{\varepsilon_i}, 1 - \prod_{i=1}^{n}(1 - \beta_{Ci})^{\varepsilon_i}), 1 - \prod_{i=1}^{n}(1 - \gamma_{Ai})^{\varepsilon_i}, 1 - \prod_{i=1}^{n}(1 - \gamma_{Ci})^{\varepsilon_i})$$

In which $\varepsilon_i (i = 1, 2..n)$ was defined as the weight of N_{Zi} when $0 \leq \varepsilon_i \leq 1$ and $\sum_{i=1}^{n} \varepsilon_i = 1$

Example Illustration 2:

NZ1 = [(0.7,0.6),(0.3,0.2),(0.4,0.3)

NZ2 = [(0.5,0.7),(0.4,0.3),(0.5,0.4)

NZ3 = [(0.8,0.9),(0.2,0.1),(0.3,0.2)

Given corresponding weights ε_i={0.4,0.3,0.3}

The weighted aggregation arithmetic mean of these NZNs calculated by NZNWAA (12) and NZNWGA (13) equations are determined as follows:

$\text{NZNWAA}(N_{Z1}, N_{Z2}, N_{Z3})$ = [(0.7444,0.8331),(0.2797,0.2105),(0.3985,0.3012)

$\text{NZNWGA}(N_{Z1}, N_{Z2}, N_{Z3})$ = [(0.6476,0.7169),(0.4114,0.2992),(0.5129,0.3924)

Definition 4 (Ye, 2021)**: Measures of distance and similarity for NZN sets**

Researchers have increasingly emphasized the similarity and distance between sets, as they are valuable instruments for facilitating decision-making (Zavadskas et al., 2017).

Allow $N_{Z1} = \{N_{Z11}, N_{Z12} \dots, N_{Z1n}\}$ and $N_{Z2} = \{N_{Z21}, N_{Z22} \dots, N_{Z2n}\}$, where $N_{Z1k} = [\alpha_{1k}(A, C), \beta_{1k}(A, C), \gamma_{1k}(A, C)] = [(\alpha_{A1k}, \alpha_{C1k}), (\beta_{A1k}, \beta_{C1k}), (\gamma_{A1k}, \gamma_{C1k})]$, and $N_{Z2k} = [\alpha_{2k}(A, C), \beta_{2k}(A, C), \gamma_{2k}(A, C)] = [(\alpha_{A2k}, \alpha_{C2k}), (\beta_{A2k}, \beta_{C2k}), (\gamma_{A2k}, \gamma_{C2k})]$ to be two NZNs, under the condition of giving any integer as $\vartheta \geq 1$. The corresponding weights assigned to the n pairs of NZN $w_k = (w_1, w_2, \dots w_n), \sum_{k=1}^{n} w_k = 1$. Following this, Equation (14) is used to calculate the generalized distance between N_{Z1} and N_{Z2}:

$$D_{w\vartheta}(N_{Z1}, N_{Z2}) = \frac{1}{2}\left\{\sqrt[\vartheta]{\frac{1}{3}\sum_{k=1}^{n} w_k(|\alpha_{A1k} - \alpha_{A2k}|^{\vartheta} + |\beta_{A1k} - \beta_{A2k}|^{\vartheta} + |\gamma_{A1k} - \gamma_{A2k}|^{\vartheta})} + \sqrt[\vartheta]{\frac{1}{3}\sum_{k=1}^{n} w_{k(|\alpha_{C1k} - \alpha_{C2k}|^{\vartheta} + |\beta_{C1k} - \beta_{C2k}|^{\vartheta} + |\gamma_{C1k} - \gamma_{C2k}|^{\vartheta})}}\right\} \quad (14)$$

When ϑ = 1, the generalized distance formula is simplified to the Hamming distance formula, which was D_{w1} as shown in Equation (15):

$$D_{w1}(N_{Z1}, N_{Z2}) = \frac{1}{6}\left\{\sum_{k=1}^{n} w_k(|\alpha_{A1k} - \alpha_{A2k}| + |\beta_{A1k} - \beta_{A2k}| + |\gamma_{A1k} - \gamma_{A2k}|) + \sum_{k=1}^{n} w_k(|\alpha_{C1k} - \alpha_{C2k}| + |\beta_{C1k} - \beta_{C2k}| + |\gamma_{C1k} - \gamma_{C2k}|)\right\} \quad (15)$$

When ϑ = 2, the generalized distance formula is D_{w2}, as shown in Equation (16):

$$D_{w2}(N_{Z1}, N_{Z2}) = \frac{1}{2}\left\{\sqrt{\frac{1}{3}\sum_{k=1}^{n} w_k(|\alpha_{A1k} - \alpha_{A2k}|^2 + |\beta_{A1k} - \beta_{A2k}|^2 + |\gamma_{A1k} - \gamma_{A2k}|^2)} + \sqrt{\frac{1}{3}\sum_{k=1}^{n} w_k(|\alpha_{C1k} - \alpha_{C2k}|^2 + |\beta_{C1k} - \beta_{C2k}|^2 + |\gamma_{C1k} - \gamma_{C2k}|^2)}\right\} \quad (16)$$

Example Illustration 3: Consider a pair of two NZN numbers:

$N_{Z1} = \{[(0.8,0.8),(0.15,0.15),(0.2,0.2), [(0.6,0.8),(0.35,0.15),(0.4,0.2)\}$

$N_{Z2} = \{[(0.6,0.4),(0.35,0.65),(0.4,0.6)], [(0.2,0.2),(0.85,0.85),(0.8,0.8)\}$

Under the condition of corresponding weights $\varepsilon_i = (0.65, 0.35)$, Hamming and Euclidean distance was calculated by applying Equations (15)-(16) :

Hamming Distance:

$D_{w1}(N_{Z1}, N_{Z2}) = \frac{1}{6}\{[0.65 \cdot (|0.8 - 0.6| + |0.15 - 0.35| + |0.2 - 0.4|) + 0.35 \cdot (|0.6 - 0.2| + |0.35 - 0.85| + |0.4 - 0.8|)] + [0.65 \cdot (|0.8 - 0.4| + |0.15 - 0.65| + |0.2 - 0.6|) + 0.35 \cdot (|0.8 - 0.2| + |0.15 - 0.85| + |0.2 - 0.8|)]\} =$ **0.3925**

Euclidean Distance:

$$D_{w2}(N_{Z1}, N_{Z2}) = \frac{1}{2}\left\{\sqrt{\frac{1}{3}\left[0.65 \cdot \left(|0.8 - 0.6|^2 + |0.15 - 0.35|^2 + |0.2 - 0.4|^2\right) + 0.35 \cdot \left(|0.6 - 0.2|^2 + |0.35 - 0.8\right.\right.\right.} + \sqrt{\frac{1}{3}\left[0.65 \cdot \left(|0.8 - 0.4|^2 + |0.15 - 0.65|^2 + |0.2 - 0.6|^2\right) + 0.35 \cdot \left(|0.8 - 0.2|^2 + |0.15 - 0.85|^2\right.\right.\right.}$$

$=$ **0,4093**

3.3 NZN DELPHI Method

In a scenario with k experts evaluating n factors, each rates the significance using a linguistic scale converted into NZN numbers via NZN sets. Expert weights, based on education and experience, enhance analysis precision. The calculation is shown below:

Step 1: The expert's weights.

Expert weights will be determined using NZNs, consisting of two key elements: A, representing the expert's qualifications based on education and experience, and C, indicating the certainty level based on the research team's familiarity with the expert. These NZN values, reflecting evaluations of experience and credentials, are combined into an exact score using Equation (11). **Table 3** presents the assessments given by the experts alongside the corresponding linguistic scales (P. H. Nguyen et al., 2024).

Table 3. Experts evaluation scale

Education (A)	Experience (A)	Linguistic scale (C)	NZN	Code
Doctor	Over 20 years	Very high	(0.8,0.15,0,2)	VH
Master	10 – 20 years	High	(0.6,0.35,0.4)	H
Bachelor	5 – 10 years	Medium	(0.4,0.65,0.6)	M
Under Bachelor	Under five years	Low	(0.2,0.85,0.8)	L
		Very low	(0,1,1)	VL

As an illustration, Expert 3 holds a bachelor's degree and over 20 years of relevant professional experience. This expert is well-acquainted with the authors, which implies high confidence in their evaluation (VH). As a result, the weighted score for this expert based on their experience will also be (VH; VH), while the evaluation corresponding to their qualifications will be (H; VH). Their evaluations will be defined as NZNs: [(0.4,0.8),(0.65,0.15),(0.6,0.2)] for experience and [(0.8,0.8),(0.15,0.15),(0.2,0.2)] for qualifications.

The two assessments presented as NZN numbers will be aggregated using Equation (7). The resulting value will then be transformed into a crisp score through Equation (11) to derive the evaluation value. In the aforementioned illustration, the calculation is as follows:

[(0.7, 0.9) , (0.4 , 0.2) , (0.5 , 0.3)] ⊕ [(0.6 , 0.7) , (0.3 , 0.2) , (0.4 , 0.1)] = [(0.88 , 0.97) , (0.12 , 0.04) , (0.2 , 0.03)]

Equation (11) was used to convert NZN [(0.88,0.97),(0.12,0.04),(0.2,0.03)] to a crisp score, and the result obtained is 0.9476.

Determine the assessment scores for k experts, generating k distinct result EK: $ek_j = \{ek_1, ek_2, \dots ek_k\}$. The weight of expert EW: $ew_j = \{ew_1, ew_2, \dots ew_k\}$ is calculated as Equation (17) below:

$$ew_j = \frac{ek_j}{\sum_{j=1}^{k} ek_j} \tag{17}$$

Step 2: Construction of expert weight evaluation matrix

To determine the relevance of n factors, experts provide initial evaluations using linguistic descriptors. These assessments are then converted into NZN numbers, generating a matrix $\otimes EM = \left[em_{ij}\right]_{nxk}$, where n refers to the number of factors involved and k represents the experts' numbers. Linguistic scales and their corresponding NZN representations are detailed in Table 4 (Abdullah et al., 2021; Kutlu Gündoğdu & Kahraman, 2019) .

Table 4. NZN Delphi linguistic significant scale

Important level	Code	Membership		
		α	**β**	**γ**
Very low	VL	0	1	1
Low	L	0.2	0.85	0.8
Medium	M	0.4	0.65	0.6
High	H	0.6	0.35	0.4
Very high	VH	0.8	0.15	0.2

The expert's weighted evaluation matrix $\otimes EMW = \left[emw_{ij}\right]_{nxk}$ is created using Equation (18) as below.

$$emw_{ij} = em_{ij} \otimes ew_j \tag{18}$$

In condition of $i = 1, 2, 3, \dots n$ and $j = 1, 2, 3, \dots k$; $ew_j = \{ew_1, ew_2, \dots ew_k\}$ is the j of weight

Step 3: Factors validation and Threshold calculations

A panel of k experts evaluates each factor. By implementing Equation (12), their assessments are merged to form comprehensive NZN evaluations for m factors. These aggregated evaluations are then converted into crisp values using Equation (11), producing n EV score: $ev_i = \{ev_1, ev_2, \dots ev_n\}$. To determine the threshold, Equation (19) is applied as:

$$\delta = \frac{\sum_{i=1}^{n} ev_i}{n} \tag{19}$$

Once the threshold is calculated, a factor i is accepted if it meets the criteria $ev_i \geq \delta$. In contrast, if $ev_i < \delta$, the factor is rejected.

3.4 NZN DEMATEL Method

Consider k experts, each assigned weights ew, tasked with evaluating the interrelationships among n factors. The initial assessments, provided in linguistic terms, are then transformed into NZN values. **Table 5** presents the rating scale alongside the corresponding NZN conversions (Abdullah et al., 2021).

Table 5. NZN DEMATEL linguistic influence scale

Influence level	Code	Membership		
		a	**b**	**c**
Equal influence	EI	0	1	1
Week influence	WI	0.2	0.85	0.8
Fair influence	FI	0.4	0.65	0.6
Very influence	VI	0.6	0.35	0.4
Absolute influence	AI	0.8	0.15	0.2

The evaluations were converted into NZN numbers and analyzed using the DEMATEL method. The computation steps are detailed in the following section.

Step 1: Forming the Direct Relationship Matrix $\otimes$ D

The mutual influence evaluations among the n factors (where factor i impacts factor j), provided by k experts and denoted as d_{ij}^k, are converted into NZN values using the expert weights, ew_t. These evaluations are then aggregated using Equation (12) to generate the direct influence matrix $\otimes D = \left[\otimes d_{ij}\right]_{nxn}$, where:

$$d_{ij} = \text{NZNWAA}\left(d_{ij}^1, d_{ij}^2, \dots, d_{ij}^k\right) = \sum_{t=1}^{k} ew_t d_{ij}^k \quad (20)$$

In which $i = 1, 2, \dots n,\ \ j = 1, 2, \dots n$, t = 1, 2, …k; $\otimes d_{ij} = \left[\left(d_{ij}^{\alpha_A}, d_{ij}^{\alpha_C}\right), \left(d_{ij}^{\beta_A}, d_{ij}^{\beta_C}\right), \left(d_{ij}^{\gamma_A}, d_{ij}^{\gamma_C}\right)\right]$. Note that the diagonal elements are 0, i.e., $\otimes d_{ij} = 0$ (with i = j).

Step 2: Normalizing the Direct Relationship Matrix $\otimes$ D*

The matrix $\otimes D = \left[\otimes d_{ij}\right]_{nxn}$ is normalized into $\otimes D^* = \left[\otimes d_{ij}^*\right]_{nxn}$ according Equation (21):

$$d_{ij}^* = (\theta_A, \theta_C).d_{ij} = \left[\left(\theta_A d_{ij}^{\alpha_A}, \theta_C d_{ij}^{\alpha_C}\right), \left(\theta_A d_{ij}^{\beta_A}, \theta_C d_{ij}^{\beta_C}\right), \left(\theta_A d_{ij}^{\gamma_A}, \theta_C d_{ij}^{\gamma_C}\right)\right] \quad (21)$$

In which $\otimes d_{ij}^* = \left[\left(d_{ij}^{*\alpha_A}, d_{ij}^{*\alpha_C}\right), \left(d_{ij}^{*\beta_A}, d_{ij}^{*\beta_C}\right), \left(d_{ij}^{*\gamma_A}, d_{ij}^{*\gamma_C}\right)\right]$

$$\theta_A = Min\left\{\frac{1}{\sum_{j=1}^{n} d_{ij}^{\alpha_A}}; \frac{1}{\sum_{j=1}^{n} d_{ij}^{\beta_A}}; \frac{1}{\sum_{j=1}^{n} d_{ij}^{\gamma_A}}\right\} \quad (22)$$

$$\theta_C = Min\left\{\frac{1}{\sum_{j=1}^{n} d_{ij}^{\alpha_C}}; \frac{1}{\sum_{j=1}^{n} d_{ij}^{\beta_C}}; \frac{1}{\sum_{j=1}^{n} d_{ij}^{\gamma_C}}\right\}$$

Step 3: Constructing the total influence matrix $\otimes$T

The normalized direct relationship matrix $\otimes$T derives the total influence matrix. This matrix accounts for both direct and indirect influence paths by aggregating impacts over an infinite range, spanning from minimal to maximal levels of influence. The detailed computation process is described below:

$$\otimes \text{T} = \left[\otimes t_{ij}\right]_{nxn}\ ,\ i = j = 1, 2, \dots n \quad (23)$$

$$\otimes t_{ij} = \left[\left(t_{ij}^{\alpha_A}, t_{ij}^{\alpha_C}\right), \left(t_{ij}^{\beta_A}, t_{ij}^{\beta_C}\right), \left(t_{ij}^{\gamma_A}, t_{ij}^{\gamma_C}\right)\right]$$

$$\otimes T = \otimes N^* + \otimes N^{*2} + \cdots + \otimes N^{*\infty} \quad (24)$$
$$= \otimes T(I + \otimes N^* + \otimes N^{*2} + \cdots + \otimes N^{*\infty - 1})$$
$$= \otimes D^*(I - \otimes N^{*\infty})(I - \otimes N^*)^{-1} = \otimes N^*(I - \otimes N^*)^{-1}$$

In which $\otimes T^{\infty} = [0]_{nxn}$ with I as the identification matrix

Matrix $\otimes T$' elements as NZNs are converted to crisp NZNs using Equation (11), creating a matrix $\otimes T^* = \left[\otimes t_{ij}^*\right]_{nxn}$

Step 4: Construct an Influence Network Relation Map (INRM) to assess the interdependencies among development projects.

$\otimes D$ is obtained by summing the values in each column of the total influence matrix $\otimes T^*$, while $\otimes R$ is calculated by summing the values in each row of $\otimes T^*$.

$$\otimes D = [\otimes D_i]_{nx1} = (\otimes D_1, \otimes D_2, \dots, \otimes D_i, \dots, \otimes D_n) \quad (25)$$

$$[\otimes D_i]_{nx1} = \left[\sum_{j=1}^{n} \otimes t_{ij}^*\right]_{nx1} \quad (26)$$

$$\otimes R = [\otimes R_i]_{1xn} = (\otimes R_1, \otimes R_2, \dots, \otimes R_j, \dots, \otimes R_n)^T \quad (27)$$

$$\left[\otimes R_j\right]_{1xn} = \left[\sum_{i=1}^{n} \otimes t_{ij}^*\right]_{1xn} = [\otimes R_i]_{nx1}^T \quad (28)$$

Note: "The superscript T" denotes the transpose of the matrix.

The measure of influence strength imparted and received is symbolized by $\otimes D_i + \otimes R_i$. The net influence is calculated as $\otimes D_i - \otimes R_i$. A higher $\otimes D_i + \otimes R_i$ value suggests that criterion i exerts a substantial influence on the evaluation system. If $\otimes D_i - \otimes R_i > 0$ (positive), criterion i significantly affects others. Conversely, if $\otimes D_i - \otimes R_i < 0$ (negative), it indicates that criterion i is predominantly influenced by other factors.

The total effect of each indicator on the evaluation system is given by $\otimes D_i + R$ and the weight of the indicator's impact is calculated using Equation (29):

$$\phi_i = \frac{(Di + R_i)}{\sum_{i=1}^{n}(D_i + R_i)} \quad (29)$$

4. Results and Discussions

4.1. Results of NZN Delphi

A set of 47 factors was compiled through an extensive literature review and expert consultations. The survey was structured into two parts and designed to be completed within 30 minutes. The first section focused on gathering demographic details such as industry affiliation, job title, education level, and years of professional experience. In the subsequent section, participants were asked to evaluate their level of agreement with various criteria and express the certainty of their responses based on their expertise and confidence. Invitations were emailed, and the questionnaire was accessible only after consent. Conducted in English via Jotform, the data collection took place over three months, from May to August 2024. Sixteen experts representing both industry and academia participated in this study.

Following the guidelines of Veugelers et al. (2020), a panel of 10 to 18 experts is recommended to achieve a consensus among participants. Table 2 presents the detailed profiles of the participating experts, while Table 4 illustrates how each expert assessed the significance of the critical criteria using NZN Delphi linguistic scales outlined in **Table 6**.

Table 6. NZN Delphi results

	NZN Weight	Crisp Weight	Validate
SC1	(0.6342;0.6149);(0.326;0.3391);(0.3658;0.3851)	0.7129	Accept
SC2	(0.5459;0.6781);(0.4149;0.2684);(0.4541;0.3219)	0.7042	Accept
SC3	(0.4986;0.6651);(0.4744;0.2817);(0.5014;0.3349)	0.6767	Accept
SC4	(0.5801;0.7304);(0.3938;0.216);(0.4199;0.2696)	0.7418	Accept
E01	(0.5092;0.7044);(0.4809;0.2418);(0.4908;0.2956)	0.6991	Accept
E02	(0.5612;0.7232);(0.4158;0.2247);(0.4388;0.2768)	0.7303	Accept
E03	(0.6673;0.7052);(0.2822;0.2444);(0.3327;0.2948)	0.7678	Accept
E04	(0.5826;0.6502);(0.3853;0.2969);(0.4174;0.3498)	0.7061	Accept
E05	(0.3356;0.6749);(0.689;0.2759);(0.6644;0.3251)	0.6068	Reject
PE1	(0.4793;0.7407);(0.4872;0.206);(0.5207;0.2593)	0.7065	Accept
PE2	(0.2872;0.5123);(0.7341;0.4683);(0.7128;0.4877)	0.4852	Reject
PE3	(0.379;0.7019);(0.618;0.2441);(0.621;0.2981)	0.6433	Reject
PE4	(0.5391;0.4545);(0.4247;0.5507);(0.4609;0.5455)	0.5866	Reject
PE5	(0.615;0.6783);(0.3475;0.2681);(0.385;0.3217)	0.7334	Accept
PE6	(0.4052;0.6639);(0.6157;0.2831);(0.5948;0.3361)	0.6316	Reject
PE7	(0.4725;0.6555);(0.4773;0.2937);(0.5275;0.3445)	0.6626	Reject
PE8	(0.5654;0.7061);(0.3936;0.2401);(0.4346;0.2939)	0.7257	Accept
PE9	(0.5065;0.7046);(0.4613;0.2415);(0.4935;0.2954)	0.6999	Accept
PE10	(0.4479;0.7099);(0.5043;0.2382);(0.5521;0.2901)	0.6792	Accept
PI1	(0.4838;0.4329);(0.4829;0.5713);(0.5162;0.5671)	0.5469	Reject
PI2	(0.5282;0.4329);(0.4406;0.5713);(0.4718;0.5671)	0.5698	Reject
PI3	(0.4106;0.7019);(0.572;0.2441);(0.5894;0.2981)	0.6576	Reject
PI4	(0.4231;0.7091);(0.5512;0.239);(0.5769;0.2909)	0.6668	Reject
PI5	(0.5578;0.6035);(0.4121;0.3555);(0.4422;0.3965)	0.6716	Accept
PI6	(0.5786;0.6648);(0.3882;0.282);(0.4214;0.3352)	0.7113	Accept
PI7	(0.597;0.6111);(0.3617;0.3496);(0.403;0.3889)	0.6939	Accept
PI8	(0.3977;0.5163);(0.6138;0.4665);(0.6023;0.4837)	0.5426	Reject
PI9	(0.6317;0.5777);(0.323;0.3836);(0.3683;0.4223)	0.6952	Accept
PI10	(0.3977;0.5163);(0.6138;0.4665);(0.6023;0.4837)	0.5426	Reject

PI11	(0.5684;0.6661);(0.3974;0.2872);(0.4316;0.3339)	0.7068	Accept
PI12	(0.5431;0.6031);(0.4325;0.3571);(0.4569;0.3969)	0.6639	Reject
PI13	(0.517;0.6358);(0.4596;0.3193);(0.483;0.3642)	0.6687	Accept
PI14	(0.2305;0.5914);(0.7983;0.3772);(0.7695;0.4086)	0.5069	Reject
PI15	(0.6636;0.6165);(0.2879;0.3467);(0.3364;0.3835)	0.7268	Accept
PI16	(0.5403;0.6051);(0.4187;0.3545);(0.4597;0.3949)	0.6657	Reject
PI17	(0.5121;0.6535);(0.4478;0.301);(0.4879;0.3465)	0.6769	Accept
PI18	(0.5209;0.7049);(0.4486;0.2412);(0.4791;0.2951)	0.7059	Accept
SI1	(0.3505;0.6673);(0.6663;0.2858);(0.6495;0.3327)	0.6091	Reject
SI2	(0.4406;0.661);(0.5551;0.291);(0.5594;0.339)	0.6467	Reject
SI3	(0.5737;0.7092);(0.3937;0.2388);(0.4263;0.2908)	0.7296	Accept
SI4	(0.617;0.6512);(0.3391;0.2998);(0.383;0.3488)	0.7222	Accept
SI5	(0.5636;0.6548);(0.4055;0.2946);(0.4364;0.3452)	0.6996	Accept
SI6	(0.5214;0.6747);(0.4536;0.2801);(0.4786;0.3253)	0.6897	Accept
SI7	(0.606;0.6263);(0.3545;0.3319);(0.394;0.3737)	0.7049	Accept
SI8	(0.5083;0.6218);(0.4678;0.3352);(0.4917;0.3782)	0.6578	Reject
SI9	(0.5492;0.6215);(0.4158;0.3347);(0.4508;0.3785)	0.6772	Accept
SI10	(0.6943;0.6176);(0.2576;0.3443);(0.3057;0.3824)	0.7411	Accept

The comprehensive results of the NZN Delphi technique are displayed in Table 5. Factors scoring higher than the threshold (0.668) were accepted, reflecting strong expert agreement on their significance in influencing consumer behavior sustainably. Those factors below the threshold were deemed less impactful and rejected for further consideration. All of the social and cultural factors were accepted. SC4 and SC1 received the highest validation scores, with 0.7418 and 0.7129, respectively, because they reflect current consumer trends where affordability and social media pressures drive purchasing decisions. SC2 gets a score immediately, followed by SC1 (0.7042). SC3 scored 0.6767, barely passing the threshold, suggesting its moderate influence in the context of sustainable purchasing.

For the Pricing and Affordability dimension, all factors except E05 were accepted, with E03 receiving the highest score of 0.7678, underscoring that price sensitivity remains a dominant barrier to sustainable fashion adoption. Next are factors E02 (0.7303), E04 (0.7061), E01 (0.6991), E05 (0.6068). The rejection of E05 indicates that experts saw this as more of a business concern than a direct factor influencing sustainable purchasing decisions.

The Policy and Education dimension showed mixed results. The Policy and Education dimension showed mixed results. While factors like PE5 and PE8 were strongly validated, with scores of 0.7334 and 0.7257, followed by PE1 (0.7065), PE9 (0.6999), PE10 (0.6792), factors such as PE2 (0.4852) and PE3 (0.6433), PE4 (0.5866), PE6 (0.6316), PE7 (0.6626) were rejected. PE2 and PE3 were rejected, indicating that while the lack of standards and subsidies for unsustainable products are important, they do not have an immediate or direct influence on consumer behavior compared to cultural and educational factors. PE4, PE6, and PE7 were rejected because these factors are more relevant to regulatory and corporate practices than individual consumer choices.

In the Perception and Image dimension, factors PI1 (0.5469), PI2 (0.5698), PI3 (0.6576), PI8 (0.5426), PI10 (0.5426), PI12 (0.6639), PI14 (0.5069), and PI16 (0.6657) were all rejected with scores below the threshold of 0.668. These factors were likely deemed less critical or redundant than more substantial barriers, such as economic constraints and trend-following behavior, which overlap with accepted factors like E02 and E03. In contrast, factors such as PI5 (0.6716), PI6 (0.7113), PI7 (0.6939), PI9 (0.6952), PI11 (0.7068), PI13 (0.6687), PI15 (0.7268), PI17 (0.6769), and PI18 (0.7059) were accepted, as they scored above the threshold. These factors highlight the importance of fashion trends, personal involvement, perceptions of quality, and durability in shaping consumer behavior, as well as the impact of income levels and skepticism regarding sustainability claims. These scores point to deep-rooted social and psychological influences on fashion consumption, making these factors critical obstacles to embracing sustainable fashion.

Finally, the Supply Chain and Industry Practices dimension strongly validated factors like SI3 and SI10, which scored 0.7296 and 0.7411, respectively. SI4 (0.7222), SI5 (0.6996), SI6 (0.6897), SI7 (0.7049), and SI9

(0.6772) were accepted because they reflect critical issues directly impacting the affordability, availability, and perception of sustainable fashion. These elements underscore the ethical and operational challenges preventing consumers from embracing sustainable fashion more widely. Conversely, SI1 (0.6091), SI2 (0.6467), and SI8 (0.6578) were rejected, as experts likely saw them as secondary issues with less direct impact on consumer behavior compared to more pressing barriers like cost, promotion, and greenwashing concerns. These rejected factors, while relevant, were not viewed as immediate deterrents to sustainable purchasing decisions.

4.2. Results of NZN DEMATEL Technique

4.2.2. NZN DEMATEL Results of Main Dimensions

Table 7. The NZN DEMATEL results of the main dimensions

	Crisp Di+Ri	Crisp Di-Ri	Weight	Rank	Classification
SC	35.5217	8.1087	0.185214	4	Cause
E0	35.1841	-7.7555	0.183453	5	Effect
PE	42.6309	13.3693	0.222282	1	Cause
PI	42.2226	-13.8778	0.220153	2	Effect
SI	36.2285	0.1553	0.188899	3	Cause

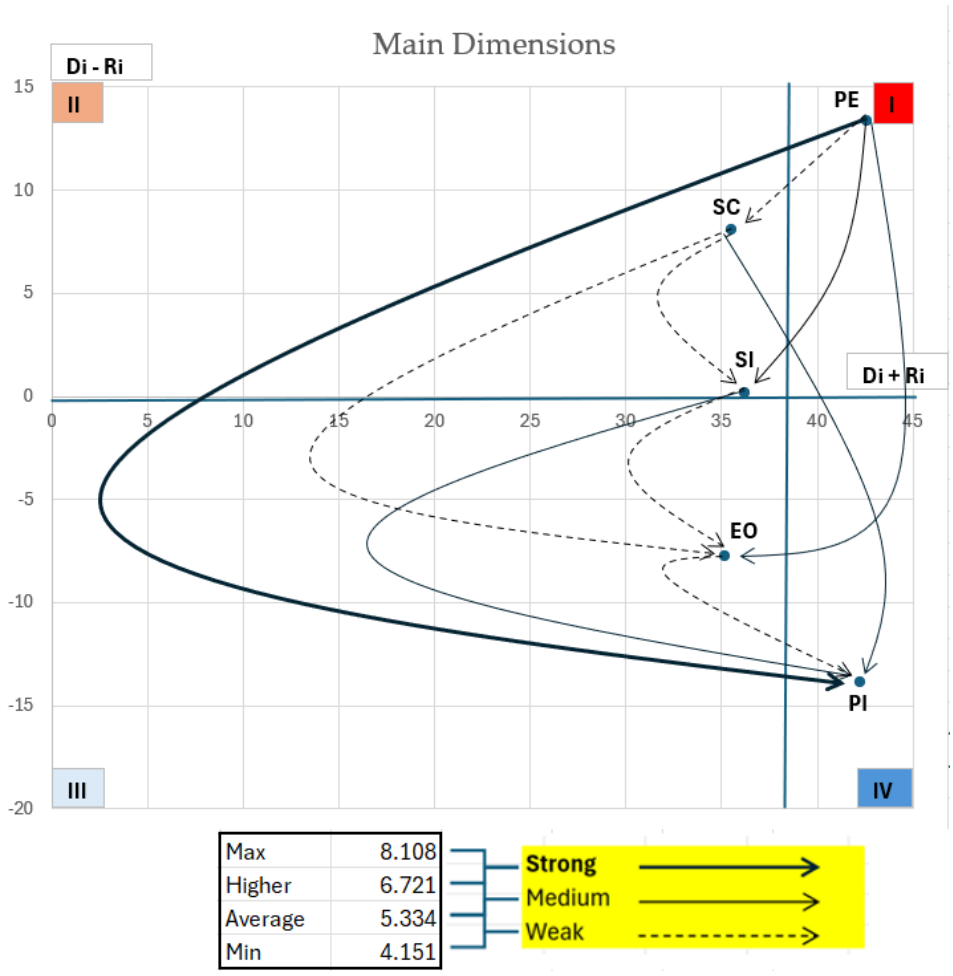

Figure 2. Impact-Relation Map Main Dimension.

The DEMATEL analysis findings, as presented in Table 7 and Figure 2, provide a comprehensive depiction of the interactions among the principal dimensions influencing sustainable consumption within the fashion industry. Notably, Policy and Education (PE) emerges as the most dominant causal factor, exhibiting

the highest combined (Di+Ri) score and a positive (Di–Ri) score, situating it within quadrant I. This underscores PE's pivotal role in promoting sustainable practices. Additionally, Social and Cultural (SC) factors, illustrated in Figure 2, hold the position as the second most influential causal component, impacting all dimensions except PE. Supply Chain and Industry Practices (SI) are the third most significant factor, showcasing substantial causal influence.

On the other hand, perception and image (PI) and pricing and affordability (E0) are identified as effect factors, with low (Di−Ri) values, indicating that the causal variables largely shape them. PI is positioned in quadrant IV, meaning multiple factors impact it but do not directly drive change. E0 is located in quadrant III, reflecting its independent nature and lower prominence in the system. PE, SC, and SI primarily influence both PI and E0. Enhancing PE and SC will lead to broader improvements in sustainable fashion consumption, as they serve as the foundational drivers of change across the system.

4.2.2. NZN DEMATEL Results of Sub Dimensions

4.2.2.1 NZN DEMATEL Results of Social and Culture

Table 8. The NZN DEMATEL results of social and cultural factors

	Crisp Di+Ri	Crisp Di-Ri	Weight	Rank	Classification
SC	35.5217	8.1087	0.185214	4	Cause
E0	35.1841	-7.7555	0.183453	5	Effect
PE	42.6309	13.3693	0.222282	1	Cause
PI	42.2226	-13.8778	0.220153	2	Effect
SI	36.2285	0.1553	0.188899	3	Cause

The NZN DEMATEL analysis highlights the relationships among the factors in the Social and Cultural Factors dimension, as shown in **Table 8**. SC1 (Emergence of Affordable Fashion) and SC3 (Fashion as a Status Symbol) are noted as the primary causal factors, as shown by the positive (Di−Ri) values and placement in the first quadrant of the analysis. This quadrant signifies that these factors influence other factors and are highly important within the system, reaffirming their significance in driving sustainable purchasing behavior. SC1, with the highest (Di+Ri) value of 0.2644, emerges as the most influential factor, emphasizing its role in shaping consumer choices through affordability. SC3 similarly plays a key role, with a weight of 0.2548, reinforcing the cultural value of fashion as a status symbol.

Further analysis reveals that SC3 influences SC1, SC2 (Sustainable Fashion Stereotypes), and SC4 (Social and Media-Induced Wardrobe Updates), though varying degrees. SC3 has the most decisive impact on SC4, indicating a significant role in driving frequent wardrobe updates through social media pressures. Its influence on SC2 is moderate, shaping perceptions around sustainable fashion but to a lesser extent. The effect on SC1 is weaker, suggesting that while status-driven consumption is linked to affordability, the two factors operate somewhat independently.

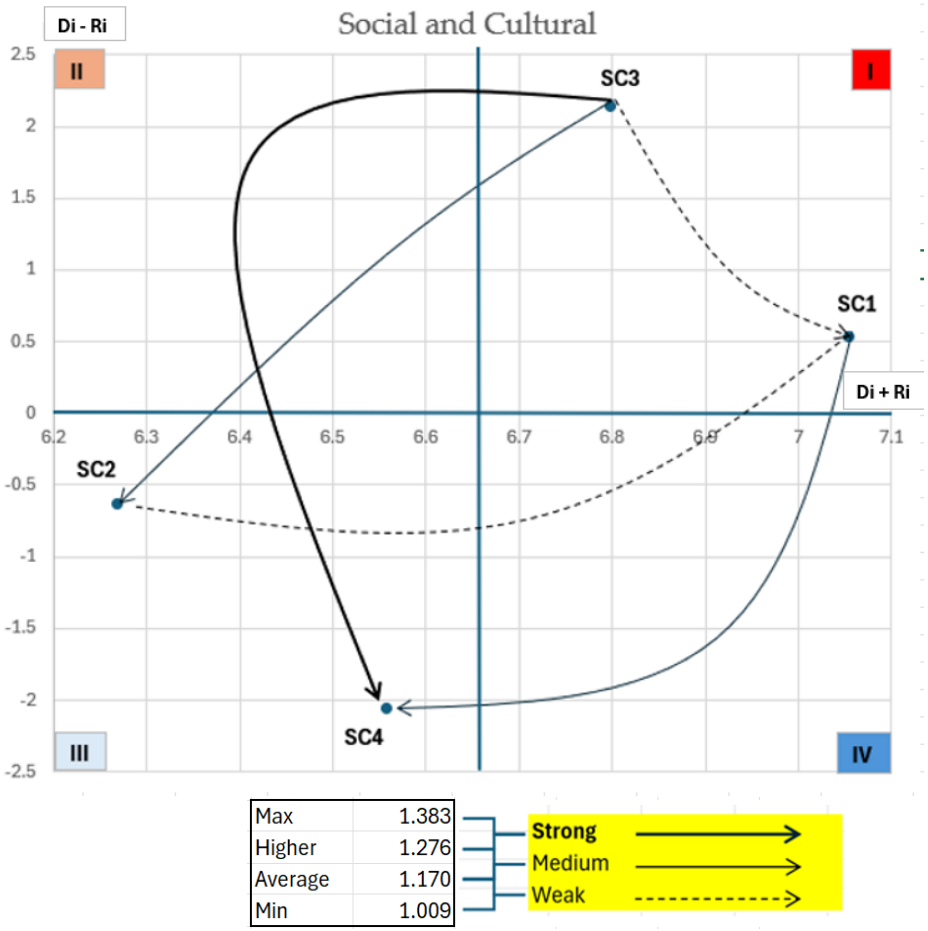

Figure 3. Impact-Relation Map Social and Culture Dimension.

Conversely, SC2 and SC4 are classified as effect factors, with negative (Di−Ri) values and weights of 0.2350 and 0.2458, respectively. These factors are influenced by SC1 and SC3, reflecting the broader social and cultural dynamics. SC2 is shaped by affordability and status, while social media-driven consumption patterns heavily influence SC4.

4.2.2.2 NZN DEMATEL Results of Pricing and Affordability

Table 9. The NZN DEMATEL results of pricing and affordability factors

Factor	Crisp Di+Ri	Crisp Di-Ri	Weight	Rank	Classification
E01	4.9669	1.3677	0.268952	1	Cause
E02	4.6107	-1.2253	0.249664	3	Effect
E03	4.0575	-1.3523	0.219709	4	Effect
E04	4.8325	1.2099	0.261675	2	Cause

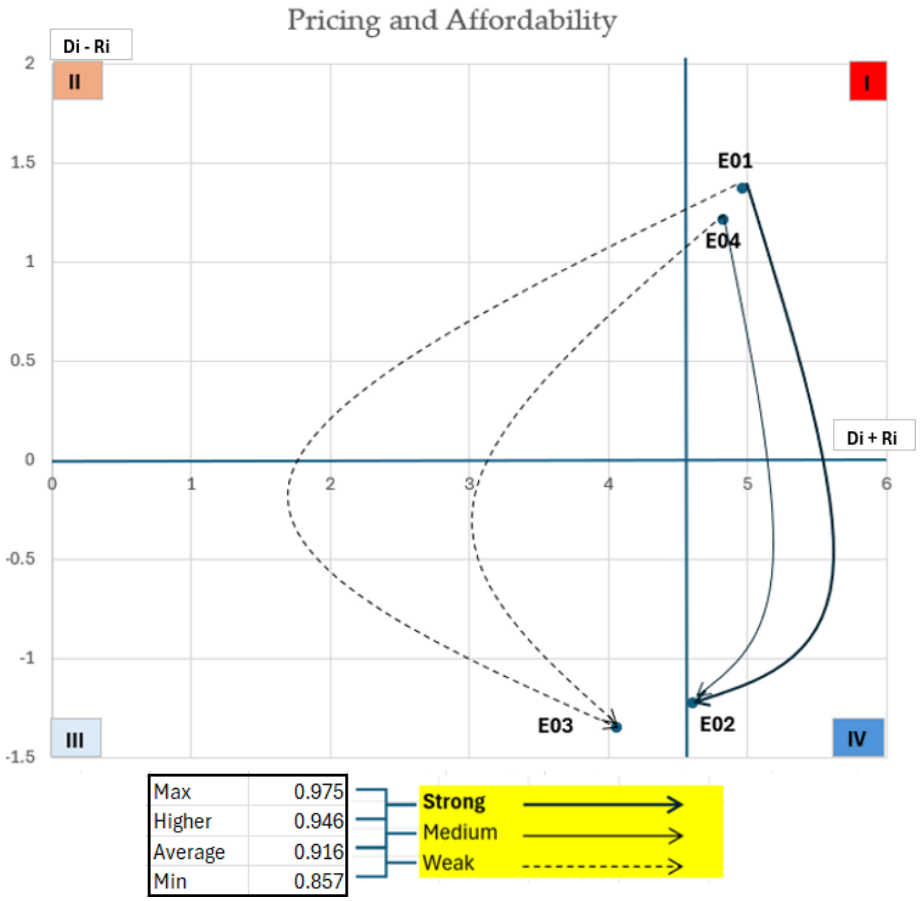

Figure 4. Impact-Relation Map Pricing and Affordability Dimensions

The Pricing and Affordability dimension, as outlined in **Table 9**, identifies key causal and effect factors shaping consumer behavior toward sustainable fashion. E01 (Economic Growth Impact on Consumption), with a weight of 0.2690, and E04 (Balancing Sustainability and Profitability), with a weight of 0.2617, are highlighted as the most influential causal factors, both positioned in quadrant I, underscoring their critical role in driving sustainable fashion through economic and profitability mechanisms. The results show that E01 and E04 exhibit similar "strong influence" curves, indicating that both factors significantly influence the system. E01, with the highest (Di−Ri) value, plays a pivotal role in shaping how economic conditions impact consumer purchasing power and choices sustainably. E04, closely following, influences the balance between sustainable production and profitability, highlighting businesses' tension in offering sustainable options at competitive prices. Conversely, E02 (High Prices of Sustainable Fashion) with a weight of 0.2497 and E03 (Consumer Budget Constraints) with a weight of 0.2197 are identified as effect factors, with negative (Di−Ri) values.

Figure 6 visually represents these relationships, with E01 and E04 positioned as driving forces in the Pricing and Affordability dimension, exerting influence over the cost-related factors E02 and E03. The placement of E01 and E04 in Quadrant I reaffirms their role as core factors driving change, while E02 and E03 respond to these broader economic and pricing pressures.

4.2.2.3 NZN DEMATEL Results of Policy and Education

Table 10. The NZN DEMATEL results of policy and education factors

Factor	Crisp Di+Ri	Crisp Di-Ri	Weight	Rank	Classification
PE1	7.1168	1.791	0.223152	2	Cause
PE5	6.9082	1.4698	0.216611	3	Cause
PE8	7.2482	1.5136	0.227272	1	Cause

PE9	5.2804	-2.2076	0.16557	5	Effect
PE10	5.3386	-2.5668	0.167395	4	Effect

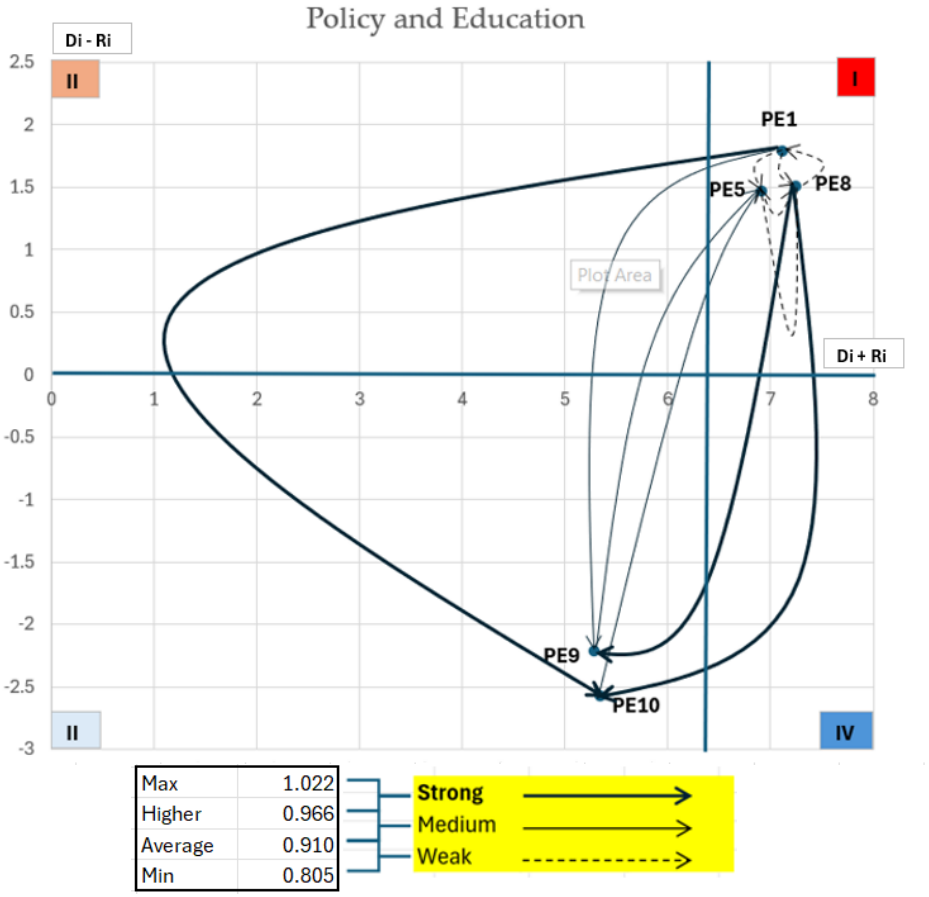

Figure 5. Impact-Relation Map Policy and Education Dimension

The Policy and Education dimension comprises five factors, as shown in **Table 10**, providing insights into the relationships among the five factors, highlighting both cause and effect factors. PE1 (Social and Cultural Norms Against Sustainability) with a weight of 0.2232, PE5 (Educational Campaigns on Sustainable Consumption) with a weight of 0.2166, and PE8 (Subsidies for Unsustainable Products) with a weight of 0.2273, are identified as the primary causal factors, positioned in quadrant I of the DEMATEL matrix. They are referred to as the "core factors" or "intertwined givers," characterized by both high prominence and strong connections to other factors, signifying that they are key drivers of the system. Among these, PE8 holds the highest influence, as indicated by its (Di+Ri) value of 0.2273, signaling its substantial role in shaping the policy and education landscape, particularly in affecting market incentives that drive unsustainable purchasing behavior.

Conversely, PE9 (Lack of Standards for Sustainable Fashion) with a weight of 0.1656 and PE10 (Limited Education Level of a Consumer Segment) with a weight of 0.1674 are positioned in quadrant IV, also known as "impact factors" or "intertwined receivers." That indicates this factor has high prominence but low relation, meaning other factors significantly influence them but are not easily improved through direct intervention. With negative (Di−Ri) values of -0.1656 for PE9 and -0.1674 for PE10, these factors are strongly impacted by the dynamics driven by the causal factors in quadrant I, particularly PE8, PE1, and PE5. These factors reflect broader systemic issues, such as unclear sustainability standards and insufficient consumer education, which respond to the influence of core policy drivers but do not independently generate change.

Figure 4 further illustrates the directional influence among these factors, showing how PE8 and PE1 considerably impact the entire system, particularly on PE9 and PE10. The prominence of PE8 as a major driver with the highest (Di−Ri) value indicates that addressing subsidies for unsustainable products would have a ripple effect throughout the system, leading to improvements in sustainability standards and consumer

education levels. Similarly, PE1 and PE5, with their positions in quadrant I, act as key influencers in shaping cultural norms and the effectiveness of educational campaigns.

4.2.2.4 NZN DEMATEL Results of Perception and Image

Table 10. The NZN DEMATEL results of perception and image factors

Factor	Crisp Di+Ri	Crisp Di-Ri	Weight	Rank	Classification
PI5	10.7206	0.8268	0.120187	1	Cause
PI6	10.6544	1.0242	0.119445	2	Cause
PI7	9.3695	-0.9781	0.10504	8	Effect
PI9	9.3876	-0.6852	0.105243	7	Effect
PI11	10.5658	0.8372	0.118451	3	Cause
PI13	9.45	-0.7094	0.105942	5	Effect
PI15	10.3525	0.9737	0.11606	4	Cause
PI17	9.2526	-0.7614	0.103729	9	Effect
PI18	9.4464	-0.5278	0.105902	6	Effect

Table 10 uncovers important cause-and-effect relationships between the factors within the Perception and Image dimension. PI5 (Influence of Fashion Trends), PI6 (Degree of Consumer Involvement in Fashion), PI11 (Impact of Income Levels on Fashion Choices), PI15 (Materialistic Consumer Behavior), and PI18 (Skepticism Towards Sustainability Claims) are recognized as causal factors based on their positive (Di−Ri) values. Among these, PI5 holds the highest (Di+Ri) value, making it the most influential factor. However, PI6 exerts a more substantial overall influence on other factors, particularly PI7, PI9, and PI13, indicating that improving PI6 would lead to significant shifts in consumer perception and behavior.

Conversely, PI7, PI9, PI13, and PI17 are categorized as effect factors with negative (Di−Ri) values, indicating that the causal variables shape them. Notably, factors such as PI5, PI11, PI6, and PI15 are positioned in the first quadrant, signifying their role as core drivers in the system, while PI18, PI13, PI9, PI17, and PI7 are located in the fourth quadrant, marking them as impact factors, heavily influenced by other variables but difficult to address directly.

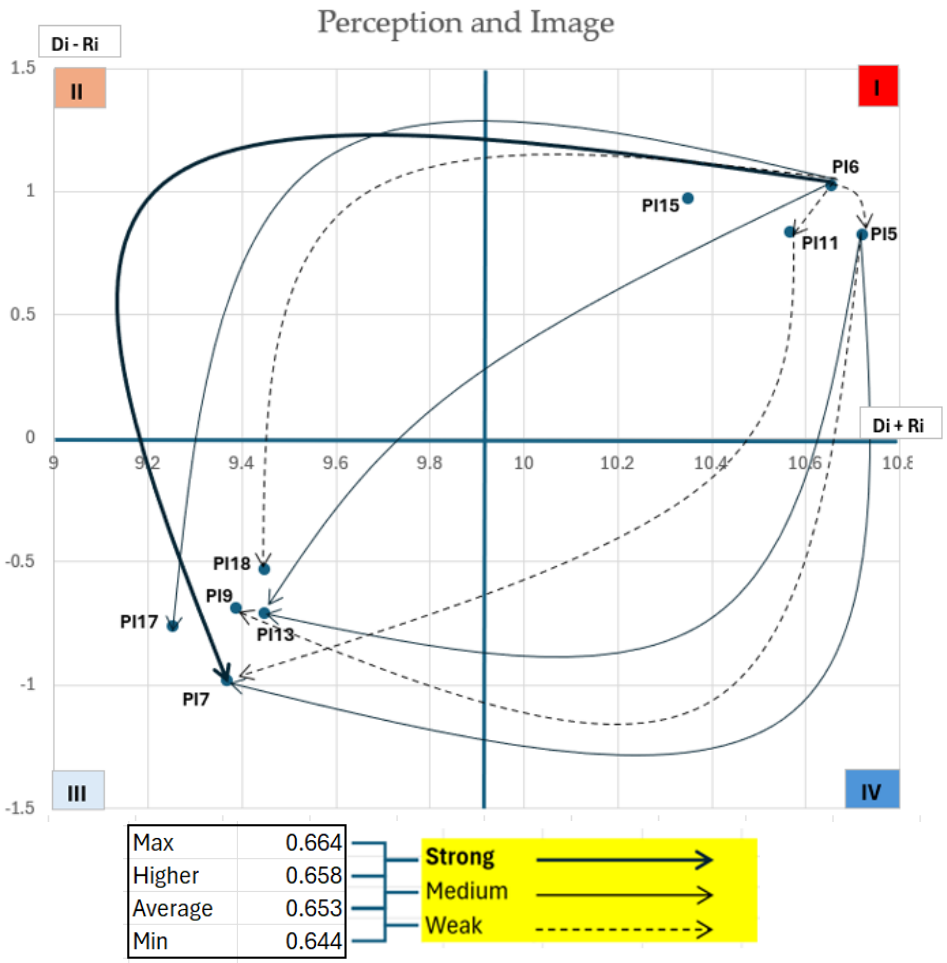

Figure 6. Impact-Relation Map Perception and Image Dimension

Additional illustrations of the correlations are shown in **Figure 6**, which demonstrates that although PI15 is more independent, PI6 and PI5 have a significant effect on the environment. In terms of fostering sustainable purchasing behavior, this underlines how important it is to concentrate on PI6 as a critical driver of change within the Perception and Image dimension inside the organization.

4.2.2.5 NZN DEMATEL Results of Supply Chain and Industry Practices

Table 11. The NZN DEMATEL results of supply chain and industry practices factors

Factor	Crisp Di+Ri	Crisp Di-Ri	Weight	Rank	Classification
SI3	9.1112	0.625	0.150426	2	Cause
SI4	8.0217	-0.6817	0.132439	6	Effect
SI5	7.9853	-1.0465	0.131838	7	Effect
SI6	9.0793	0.5953	0.1499	3	Cause
SI7	9.12	0.5484	0.150572	1	Cause
SI9	8.2573	-0.6521	0.136328	5	Effect
SI10	8.9944	0.6116	0.148498	4	Cause

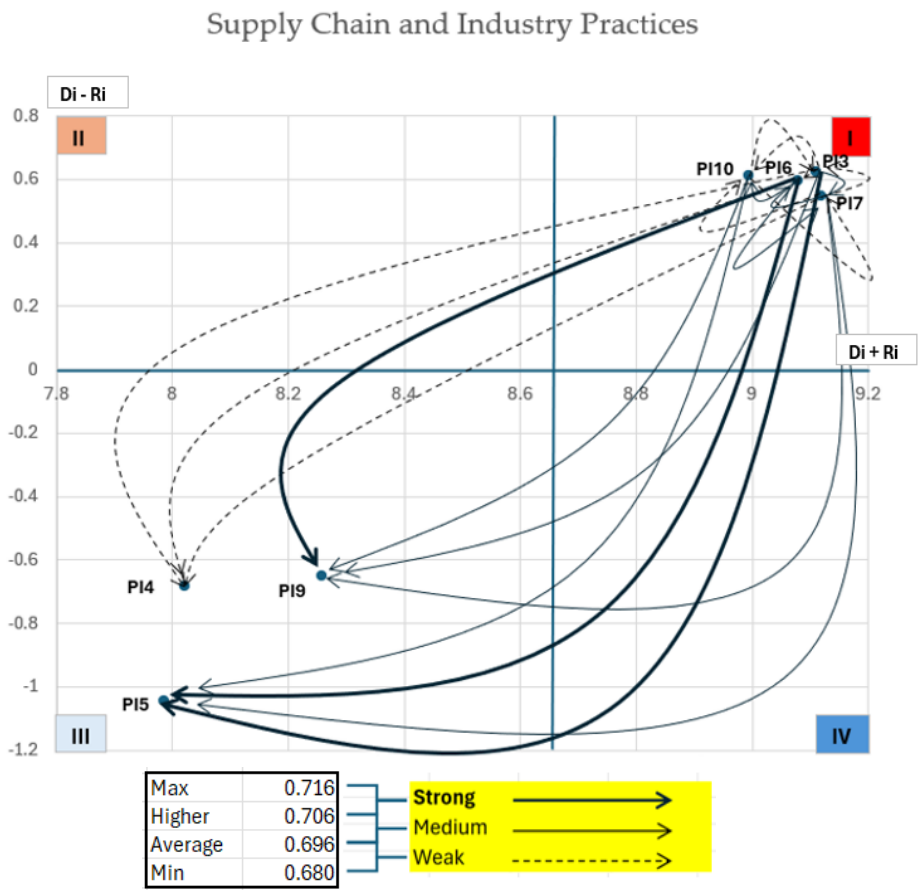

Figure 7. Impact-Relation Map Supply Chains and Industry Practices Dimension

The DEMATEL analysis for the Supply Chain and Industry Practices dimension, as outlined in **Table 11**, highlights the causal relationships among the seven factors. SI3 (Fair Wages and Working Conditions Challenges), SI6 (Financial Constraints and Scalability Concerns), SI7 (Technological Challenges in Production), and SI10 (Shift in the Fashion Industry Toward Circular Economy) are designated as causal factors based on their positive (Di−Ri) values. Among these, SI7 emerges as the most influential, with SI3 and SI6 following closely behind in impact. These factors are located in quadrant I, indicating their role as core drivers within the system, possessing high prominence and strong relations with other variables.

In contrast, SI4 (Greenwashing Practices in the Industry), SI5 (Limited Promotion of Sustainable Brands), and SI9 (High Production Costs) are categorized as effect factors, as reflected by their negative (Di−Ri) values. These factors are positioned in quadrant III, meaning they have lower prominence and relation and are influenced by the causal factors. Figure 7 demonstrates that SI4, SI5, and SI9 are significantly impacted by the core causal factors, particularly SI7 and SI3, both directly and indirectly.

The findings underscore the pivotal role of SI7, which serves as the primary catalyst in shaping supply chain practices and promoting sustainable behavior. Additionally, SI3 and SI6 significantly contribute to the impact of other factors. Prioritizing these essential elements is crucial for driving innovation and enhancing sustainability in the fashion industry.

4.3 Discussions

Employing a novel methodological framework by integrating Neutrosophic Z-number theory into the Delphi-DEMATEL approach. Unlike traditional methods, this integration allows for a more robust handling of uncertainty and reliability in expert evaluations, enabling a deeper analysis of causal relationships among barriers to sustainable consumption. The NZN Delphi study has identified key barriers and enablers for sustainable fashion adoption in Vietnam, highlighting the significant influence of social and cultural factors on consumer behavior. In Vietnam, affordability and social status are primary drivers of purchasing decisions, often taking precedence over sustainability concerns. This dynamic reflects a deeper conflict between immediate consumer priorities and long-term sustainability objectives, a pattern that is not unique to Vietnam

but resonates across various emerging markets. As highlighted by Blas Riesgo et al. (2023), economic barriers and skepticism toward green claims similarly hinder progress in Spain, suggesting a global relevance to these findings. This reflects a broader trend of conspicuous consumption driven by social norms, a phenomenon not unique to Vietnam but observed across various emerging markets (Min Kong & Ko, 2017; Singh et al., 2024; Thi et al., 2019). The influence of social media on fashion trends and consumer behavior seems stronger in Vietnam than in other Southeast Asian nations (Gazzola et al., 2020; Nayak, Akbari, et al., 2019). Social media further amplifies these influences, exerting a more pronounced impact on fashion trends and consumer behavior in Vietnam than neighboring Southeast Asian nations (Gazzola et al., 2020; Nayak, Akbari, et al., 2019). This aligns with Hong Lan & Watkins (2023), who underscore the dual role of social media in both perpetuating fast fashion and promoting sustainable alternatives, emphasizing its potential as a strategic tool for change. It suggests underscoring the necessity of leveraging digital platforms to craft narratives that align with consumer aspirations while promoting sustainability, a strategy that has proven effective in other contexts.

Although the Delphi process identified critical factors influencing sustainable fashion consumption, excluding some elements does not diminish their broader significance. Previous research predominantly focuses on barriers like high costs and technological constraints, but a holistic understanding of Vietnam requires considering sociocultural and economic factors unique to this market. For instance, consumer skepticism towards sustainability claims, compounded by the prevalence of greenwashing, is a significant barrier, as Vietnamese consumers often question the authenticity of brands' environmental claims (Adamkiewicz et al., 2022). Moreover, the prevalence of consumer skepticism, driven by widespread greenwashing, presents a significant challenge. Incorporating advanced transparency mechanisms, such as blockchain for supply chain traceability, as recommended by Mizrachi & Alon (2022), could provide a robust solution to enhance consumer confidence and accountability among industry stakeholders. This aligns with global calls for enhanced supply chain transparency as a foundation for building trust, particularly in emerging markets where regulatory oversight may be less stringent.

Applying the NZN DEMATEL method has provided nuanced insights into the complex interactions between key factors influencing sustainable fashion in Vietnam. One of the most significant insights from this study is identifying the Policy and Education dimension as the dominant causal factor. This finding underscores the transformative potential of well-designed policies and educational interventions in driving sustainable practices. Sehnem et al. (2024) advocate for embedding sustainability education within institutional frameworks, particularly in emerging economies, to cultivate a deeper societal understanding of sustainability principles. These results echo calls for targeted policy measures and community-centric educational campaigns to foster sustainable purchasing habits (Martín-Sánchez et al., 2022). In the Vietnamese context, where cultural values and economic realities significantly shape consumer behavior, integrating educational interventions that resonate with local perspectives can substantially enhance consumer awareness and involvement in sustainable fashion. This finding is particularly noteworthy as it diverges from previous studies primarily focused on economic or technological factors. For instance, Niinimäki et al. (2020) emphasized the role of technological innovation in driving sustainable fashion, while our study indicates that in the Vietnamese context, policy and education may be more powerful tools for driving change. Moreover, as Bick et al. (2018) highlighted, addressing labor and environmental injustices through policy reforms is timely and essential for advancing sustainability in the textile sector. To implement this effectively, policymakers should consider developing curricula on sustainable fashion for schools and universities, creating public awareness campaigns about the environmental impact of fast fashion, and establishing incentives for businesses adopting sustainable practices.

According to the findings of our DEMATEL investigation, the Pricing and Affordability component demonstrates that economic factors have a considerable effect on the behavior of consumers with regard to sustainable fashion. This is the second most significant discovery that was made by the research. "Economic Growth Impact on Consumption" and "Balancing Sustainability and Profitability" are two elements that play crucial roles in making sustainable fashion accessible in Vietnam, where income levels and economic stability are significant drivers of purchasing power. In this context, the phrase "economic growth impact on consumption" is particularly relevant. This conclusion is consistent with the findings of earlier research that assert that affordability and economic stability are necessary for environmentally responsible practices in emerging countries (Villa Todeschini et al., 2017). However, our research further identifies the specific relationship between economic growth and sustainable consumption in the Vietnamese context. To address this, strategies could include developing policies and business models that enhance the affordability of sustainable fashion, such as through subsidies or tax incentives, encouraging local brands to develop affordable, sustainable product lines, and promoting sharing economy models in the fashion industry, such as clothing rentals or swaps.

The classification of Social and Cultural elements as the third most significant influence underscores the substantial effect of societal norms and social media on consumer decisions. Sandvik & Stubbs (2019) emphasize the power of cultural values and media platforms in normalizing sustainable behaviors, a finding that aligns with our analysis. Forging partnerships with social media influencers to advocate for sustainable fashion in Vietnam could serve as a game-changing strategy. Compared to previous studies, such as (Min Kong & Ko, 2017) focusing on the impact of social media on fast fashion consumption in China, our research provides a unique perspective on how sociocultural factors interact with other aspects of sustainability in the Vietnamese context. The Supply Chain and Industry Practices dimension highlights systemic challenges within the industry, suggesting a need for comprehensive reforms to align industry operations with sustainability goals. There is a correlation between this finding and a study by Khurana & Ricchetti (2015), who emphasized the importance of restructuring fashion supply chains to achieve sustainability. To address these challenges, industry stakeholders could develop marketing campaigns to change perceptions of sustainable fashion, collaborate with social media influencers to promote sustainable fashion choices and invest in research and development of sustainable production technologies.

Our sub-dimension analysis reveals that factors such as "Fashion as a Status Symbol" and "Consumer Involvement in Fashion" are crucial in driving consumer engagement and shaping attitudes towards sustainable fashion. The strong influence of these factors on "Social and Media-Induced Wardrobe Updates" highlights how societal expectations and social media can drive unsustainable consumption patterns. Compared to McNeil and Moore (2015) research on the conflict between personal values and sustainable behavior in fashion, our findings emphasize the specific role of social status in the Vietnamese context. This underscores the necessity of deploying tailored advertisements that promote mindful consumption and confront the dominant culture of fast fashion. Potential intervention strategies include the creation of marketing campaigns that emphasize the unique characteristics and prestige of sustainable fashion, the persuasion of luxury brands to embrace and advocate for sustainable practices, and the instruction of consumers on the enduring value of investing in high-quality, sustainable apparel. This finding correlates with the study on eco-friendly apparel.

Examining the Perception and Image dimension highlights the significance of fashion trends and consumer engagement in influencing attitudes toward sustainable fashion. The predominance of fashion trends in Vietnam, frequently propelled by social media and peer influence, indicates that integrating sustainable fashion with current trends might substantially enhance public acceptance. Compared to Henninger's research (2016) about the role of social media in sustainable fashion consumption, our findings emphasize the importance of tailoring communication strategies to the specific cultural context of Vietnam. It is essential to implement educational initiatives that not only increase consumer engagement but also address economic inequities in order to cultivate a shift in consumer behavior that is durable. Developing interactive platforms to educate consumers about the consequences of their fashion choices, collaborating with educational institutions to incorporate sustainable fashion education into the curriculum, and creating community events that encourage garment exchanges and reuse are all potential approaches that may be taken.

Our DEMATEL analysis for the Supply Chain and Industry Practices dimension reveals significant challenges, including technological barriers, financial constraints, and ethical labor concerns. These factors directly impact the efficiency of production processes, ethical practices, and overall cost management. This echoes previous research highlighting the importance of technological adaptation and financial resources in fostering sustainable practices across industries (Karaosman et al., 2015). In comparison to Nayak & Panwar et al. (2019) on sustainable practices in Vietnamese fashion and textiles SMEs, our findings provide a more comprehensive view of how these challenges interact and influence the entire fashion ecosystem in Vietnam. Addressing these core challenges could lead to a more sustainable and transparent industry. Strategies might include facilitating technology transfer and knowledge sharing within the industry, developing financial support mechanisms for sustainable upgrades in production processes, and implementing industry-wide standards for ethical labor practices and environmental sustainability.

This study's integration of the NZN Delphi and DEMATEL techniques offers a thorough knowledge of the relationships among numerous aspects influencing sustainable fashion in Vietnam. Our results, derived from the systematic identification of key features, provide significant insights for regulators, industry leaders, and marketers to formulate more effective strategies that promote sustainable practices in consumer fashion consumption. This study not only augments existing information on sustainable fashion but also establishes a foundation for future research employing integrated decision-making procedures to tackle intricate socio-environmental issues within the fashion sector.

5. Conclusions

5.1. Conclusions

This research delivers an in-depth exploration of the challenges to sustainable consumption within the Vietnamese fashion sector, employing an integrated methodology that merges Neutrosophic Z-number sets with Delphi and DEMATEL approaches. By assessing 47 barriers across five main, the study sheds light on the complex interplay that hinders sustainable fashion adoption. The results reveal that Policy and Education (PE) emerge as the most impactful causal elements driving sustainable consumption, underlining the necessity for comprehensive policy structures and educational initiatives. Furthermore, Social and Cultural Factors (SC) and Supply Chain and Industry Practices (SI) are vital in influencing consumer behavior and attitudes, reinforcing the importance of targeted measures to advance sustainable practices.

5.2. Theoretical Implications

This research contributes significantly to the literature on sustainable consumption by integrating NZN with the Delphi and DEMATEL techniques, offering a novel framework for analyzing complex consumer behaviors and decision-making processes. Using NZN sets introduces a sophisticated method for representing uncertainty, indeterminacy, and falsity in expert judgments, effectively addressing critical limitations inherent in traditional MCDM models. By incorporating reliability alongside the three dimensions of truth, indeterminacy, and falsity, this framework provides a more nuanced and robust means of sustainably capturing the ambiguity and complexity often associated with consumer perceptions and behaviors.

A notable theoretical contribution of this research lies in its ability to map causal relationships comprehensively across multiple dimensions: Social and Cultural Factors, Pricing and Affordability, Policy and Education, Perception and Image, and Supply Chain and Industry Practices. Unlike previous studies that typically emphasize singular dimensions, this study highlights the interdependencies between these factors, offering a holistic perspective on the barriers and drivers of sustainable consumption. Identifying key causal factors, such as the transformative role of policy interventions and the persistent challenge of affordability, underscores the interconnected nature of these issues, advancing the understanding of how systemic and behavioral factors influence sustainable consumption patterns.

The integration of NZN with Delphi and DEMATEL methodologies establishes a new benchmark for robustness in causal analysis. This approach enhances the reliability and depth of insights derived from multi-dimensional studies by systematically addressing complex systems' inherent uncertainties and ambiguities. The findings validate the applicability of the NZN framework in consumer behavior research and suggest that this method could be extended to other domains where complex and interrelated factors influence decision-making.

Additionally, this research makes a significant contribution by tailoring its analysis to the unique socio-economic and cultural context of Vietnam, a setting often underrepresented in sustainability studies. By confirming and expanding upon known barriers, such as skepticism towards sustainability claims and high costs, while introducing novel insights—such as the influential role of social media and the impact of targeted policy incentives—the study exemplifies the value of a context-sensitive approach. These findings not only address gaps in the existing literature but also provide actionable insights for stakeholders seeking to advance sustainable consumption in Vietnam. Furthermore, the framework developed in this study serves as a foundation for future research, particularly in regions with similar socio-cultural dynamics, offering a replicable model for examining sustainable consumption challenges in diverse contexts.

5.3. Managerial Implications

To address the pressing challenges of sustainable consumption within the fashion industry, policymakers and industry leaders must pioneer comprehensive strategies that drive substantial changes in production processes and consumer behavior. A critical starting point lies in formulating dynamic policies that transcend mere advocacy for environmental stewardship, actively reshaping economic systems to prioritize sustainability. Incentives such as tax reductions for green enterprises and financial support for adopting eco-friendly materials are essential to lower the barriers to entry for sustainable practices. These measures not only encourage businesses to transition toward greener operations but also foster a cultural shift that integrates sustainability into mainstream economic and social frameworks.

Education is pivotal in bridging the gap between consumer awareness and actionable behavior. Comprehensive educational campaigns emphasizing the socio-environmental impacts of fashion consumption can cultivate informed and responsible consumer habits. Incorporating sustainability into school curricula early on can embed environmental consciousness as a core societal value. Public awareness initiatives should complement formal education by demystifying the detrimental effects of fast fashion and emphasizing the long-term economic and environmental benefits of sustainable consumption habits. These efforts can nurture a generation that intrinsically values sustainability, positioning it as a non-negotiable component of everyday life rather than an optional ideal.

On the supply chain front, transparency and accountability are indispensable. Implementing technologies such as blockchain can revolutionize supply chain traceability, ensuring that sustainability claims are verifiable and trustworthy. Transparent supply chains enable consumers and stakeholders to make informed decisions, fostering trust in sustainable brands. Moreover, the transition to circular economy models is imperative. Prioritizing the reuse, recycling, and upcycling of materials reduces waste and promotes a continuous resource utilization loop. The fashion industry must also collaborate actively with technologists, designers, and environmental scientists to drive innovation in sustainable materials and production techniques. Collaborative efforts can lead to groundbreaking advancements that minimize the environmental footprint of fashion production without compromising quality or style, positioning sustainable fashion as an accessible standard rather than an exclusive alternative.

Ultimately, the integration of these strategies requires a multi-faceted and coordinated approach. Legislative support, educational enhancement, technological innovation, and industry-wide collaboration must work in tandem to create a robust framework for sustainability. By addressing these dimensions holistically, the fashion industry can transition to a more sustainable future that is innovative, inclusive, and resilient. Such an approach ensures that all stakeholders—policymakers, businesses, and consumers—actively engage in the dialogue and action toward achieving a sustainable fashion ecosystem.

5.4. Limitations and Future Works

Despite its valuable insights, this study is subject to several limitations. One notable limitation is the reliance on expert opinions in the Delphi approach, which inherently involves a degree of subjectivity. Although incorporating the Neutrosophic Z-number theory mitigates this subjectivity, it may still affect the robustness and objectivity of the conclusions. Additionally, the study's focus on the Vietnamese fashion sector may restrict the generalizability of its findings to other markets with different cultural, economic, and social dynamics. The cross-sectional nature of the data provides only a snapshot of consumer behaviors and attitudes at a specific time, limiting its ability to capture temporal changes or trends in sustainable fashion consumption.

Another limitation is the lack of exploration into concrete solutions or interventions to address the identified barriers. While the study successfully highlights key obstacles to sustainable fashion, it stops short of providing actionable strategies or evaluating the effectiveness of specific policy measures or industry initiatives. Future research should address these gaps by broadening the scope and employing complementary methodologies.

Subsequent studies could expand to include additional developing markets, allowing for a comparative analysis of barriers and drivers across diverse cultural and economic contexts. Longitudinal studies would provide deeper insights into how consumer attitudes and behaviors evolve, particularly in response to targeted interventions such as public awareness campaigns, policy changes, or shifts in industry practices. Furthermore, integrating Neutrosophic Z-number theory with established behavioral models could offer a more nuanced understanding of the key psychological and contextual factors driving sustainable fashion purchases.

Future research should also explore the role of digital technology in advancing sustainability in the fashion sector. For instance, examining the impact of technologies like blockchain for supply chain transparency, artificial intelligence for personalized consumer engagement, or e-commerce platforms for promoting sustainable choices could yield practical insights. Additionally, evaluating the effectiveness of policy measures such as environmental taxes, subsidies for sustainable products, or stricter greenwashing regulations would enhance the practical relevance of future studies. By addressing these limitations and exploring these new dimensions, future research could significantly advance the understanding of how to promote sustainable practices in the fashion industry, both in Vietnam and globally.

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Trigonometric interval-valued neutrosophic set setting with reciprocal fractional function using a finite weighted method

Murugan Palanikumar[1], Nasreen kausar[2], Tonguc Cagin[3,*]

[1] Department of Mathematics, Saveetha School of Engineering, Saveetha Institute of Medical and Technical Sciences, Chennai-602105, India; palanimaths86@gmail.com

[2] Department of Mathematics, Faculty of Arts and Science, Balikesir University, 10145 Balikesir, Turkey; kausar.nasreen57@gmail.com

[3] College of Business Administration, American University of the Middle East, Kuwait; tonguc.cagin@aum.edu.kw

*Correspondence: tonguc.cagin@aum.edu.kw

Abstract. This paper presents a new method for creating a trigonometric interval-valued neutrosophic set via reciprocal fractional function. This article will cover the concepts of geometric, generalized weighted averaging, trigonometric interval-valued neutrosophic reciprocal fractional weighted averaging, and generalized weighted geometric. We used an aggregating model to get the weighted average and geometric. To further examine a number of sets with important qualities, the algebraic techniques listed below will be applied.

Keywords: weighted averaging, weighted geometric, generalized weighted averaging, generalized weighted geometric.

1. Introduction

Many theories have been proposed to explain uncertainty, such as fuzzy sets (FS) [1], which have a membership degree s (MD) between zero and one. For $\hat{b}, \vec{b} \in [0,1]$ produced an intuitionistic FS (IFS) where each element has two MDs: $0 \leq \hat{b} + \vec{b} \leq 1$ and positive $\hat{b}$ and negative $\vec{b}$ by [2] Atanassov. Yager [3] established the Pythagorean FSs (PFS) concept, which is marked by its MD and non-MD (NMD) with $\hat{b} + \vec{b} \geq 1$ to $\hat{b}^2 + \vec{b}^2 \leq 1$. Numerous studies have examined the use of IFSs and PFSs in various fields. Their information-transmission capabilities are currently restricted. Consequently, the experts were still having trouble interpreting the data in these sets and the associated data. Wang et al. investigated the concept of interval-valued IFS with DOMBI prioritized AOs and its application for trustworthy green

supplier selection [4]. Cuong et al. [5] claim that the three primary ideas of the picture FS are positive MD ($\hat{b}$), neutral MD ($\ddot{b}$), and negative MD ($\vec{b}$). In comparison to PFS and IFS, it offers additional benefits. Since $\hat{b}, \ddot{b}, and \vec{b} \in [0,1]$, it has been noted that the picture FS is an enhancement of the IFS that may handle greater inconsistency and $0 \leq \hat{b}+\ddot{b}+\vec{b} \leq 1$. Depending on the image FS description, expert opinions like "yes," "abstain," "no," and "refusal" will be supplied. The SFS for certain AOs was defined by Shahzaib et al. [6] using MADM. With SFS, $0 \leq \hat{b}^2 + \ddot{b}^2 + \vec{b}^2 \leq 1$ is required rather than $0 \leq \hat{b} + \ddot{b} + \vec{b} \leq 1$. In the beginning, Hussain et al. [7] proposed the concept of an intelligent decision support system for SFS. SFSs and their applications in DM were initially introduced by Rafiq et al. [8]. For instance, a DM problem with a property is $\hat{b}^2 + \vec{b}^2 \geq 1$.

On the condition that $0 \leq \hat{b}^3 + \vec{b}^3 \leq 1$, Fermatean FS (FFS) was first proposed by Senapati et al. [9] in 2019. Generalized orthopair FSs were initially proposed by Yager [10]. In the q-rung orthogonal pair FS (q-ROFS), both the MD and the NMD have power q; nonetheless, their total always cannot exceed one. Palanikumar et al. [11] introduced the MADM approach for Pythagorean neutrosophic normal interval-valued fuzzy AOs. Xu et al. used IFSs to produce geometric operators, such as weighted, ordered weighted, and hybrid operators [12]. Li and colleagues [13] proposed the use of generalized ordered weighted averaging operators (GOWs) in 2002. The computation of ordered weighted distances using AOs and distance measurements was described by Zeng et al. [14]. Peng et al. studied a basic PFS based on the characteristics of AOs [15]. Dombi AOs were developed by Spherical Fuzzy Ashraf and associates [16]. Additional details on SFSs and T-SFSs are given by Ullah et al. [17, 18]. The idea of interval-valued NSSs was covered by Ibraheem et al. [19] employing a variety of AOs. Type-I extension Diophantine IVNS is discussed by Al-Husban et al. [20]. Palanikumar et al. [21–23] examined a variety of algebraic structures and aggregation approaches with potential applications.

2. **AOs based on IVTNRFN**

Throughout the section $\propto = 1$. We use IVTNRFWA, IVTNRFWG, GIVTNRFWA, and GIVTNRFWG to describe the AOs. The fractional part of $\angle$, where $\angle$ is a real number, may be expressed as follows if $\mathcal{M}$ is a fractional part function: $\mathcal{M}[\angle] = \lfloor \angle \rfloor = \angle - \lfloor \angle \rfloor$. Alternatively, the difference between a real number and its largest integer value [which is determined via the greatest integer function] can be defined as a fractional part function. If $\angle$ is an integer, then the fractional component of $\angle = 0$. Suppose that $\mathcal{M}[\angle] = \frac{1}{\angle}$ is a reciprocal fractional part function. As is well known, whenever $\angle$ is an integer, its fractional part equals 0. Therefore, $\angle$ must not be an integer in order for $\mathcal{M}[\angle] = \frac{1}{\angle}$ to be defined. With the

exception of integers, all real numbers comprise the domain of $\mathcal{M}[\angle] = \frac{1}{\angle}$. We present the concept IVTNRFN and its operations were established and $\tan \pi/2 = \exists$

Definition 2.1. The IVNRFS $Ⓢ = \left\lfloor \eth, \left\langle \left\lfloor \lfloor\lfloor \exists Z^{tl} \lfloor \eth \rfloor, \exists Z^{tu} \lfloor \eth \rfloor \rfloor, \lfloor \exists Z^{il} \lfloor \eth \rfloor, \exists Z^{iu} \rfloor \lfloor \eth \rfloor, \lfloor \exists Z^{fl} \lfloor \eth \rfloor, \exists Z^{fu} \lfloor \eth \rfloor \rfloor \rfloor \right\rfloor \right\rangle \middle| \eth \in U \right\rfloor$, where $\exists Z^{tu}, \exists Z^{iu}, \exists Z^{fu} : U \to \lfloor 0, 1 \rfloor$ denote the MD, IMD and NMD of $\eth \in U$ to Ⓢ, respectively and $0 \leq \lfloor\lfloor \exists Z^{tu} \rfloor \lfloor \eth \rfloor \rfloor^{\angle -\propto} + \lfloor\lfloor \exists Z^{iu} \rfloor \lfloor \eth \rfloor \rfloor^{\angle -\propto} + \lfloor\lfloor \exists Z^{fu} \rfloor \lfloor \eth \rfloor \rfloor^{\angle -\propto} \leq 1$.
For convenience, $Ⓢ = \left\langle \lfloor \exists Z^{tl}, \exists Z^{tu} \rfloor, \lfloor \exists Z^{il}, \exists Z^{iu} \rfloor, \lfloor \exists Z^{fl}, \exists Z^{fu} \rfloor \right\rangle$ is represent a IVTNRFN.

Definition 2.2. Let $Ⓢ = \left\langle \left\lfloor \lfloor\lfloor \exists Z^{tl}, \exists Z^{tu} \rfloor, \lfloor \exists Z^{il}, \exists Z^{iu} \rfloor, \lfloor \exists Z^{fl}, \exists Z^{fu} \rfloor \rfloor \right\rfloor \right\rangle$
$Ⓢ_1 = \left\langle \left\lfloor \lfloor\lfloor \exists Z_1^{tl}, \exists Z_1^{tu} \rfloor, \lfloor \exists Z_1^{il}, \exists Z_1^{iu} \rfloor, \lfloor \exists Z_1^{fl}, \exists Z_1^{fu} \rfloor \rfloor \right\rfloor \right\rangle$
and $Ⓢ_2 = \left\langle \left\lfloor \lfloor\lfloor \exists Z_2^{tl}, \exists Z_2^{tu} \rfloor, \lfloor \exists Z_2^{il}, \exists Z_2^{iu} \rfloor, \lfloor \exists Z_2^{fl}, \exists Z_2^{fu} \rfloor \rfloor \right\rfloor \right\rangle$ be any three IVTNRFNs. Then

$$(1)\ Ⓢ_1 \vartriangle Ⓢ_2 = \begin{bmatrix} \sqrt[\angle -\propto]{\lfloor\lfloor \exists Z_1^{tl} \rfloor\rfloor^{\angle -\propto} + \lfloor\lfloor \exists Z_2^{tl} \rfloor\rfloor^{\angle -\propto}} & \sqrt[\angle -\propto]{\lfloor\lfloor \exists Z_1^{tu} \rfloor\rfloor^{\angle -\propto} + \lfloor\lfloor \exists Z_2^{tu} \rfloor\rfloor^{\angle -\propto}} \\ -\lfloor\lfloor \exists Z_1^{tl} \rfloor\rfloor^{\angle -\propto} \cdot \lfloor\lfloor \exists Z_2^{tl} \rfloor\rfloor^{\angle -\propto}, & -\lfloor\lfloor \exists Z_1^{tu} \rfloor\rfloor^{\angle -\propto} \cdot \lfloor\lfloor \exists Z_2^{tu} \rfloor\rfloor^{\angle -\propto} \\ \sqrt[\angle -\propto]{\lfloor\lfloor \exists Z_1^{il} \rfloor\rfloor^{\angle -\propto} + \lfloor\lfloor \exists Z_2^{il} \rfloor\rfloor^{\angle -\propto}} & \sqrt[\angle -\propto]{\lfloor\lfloor \exists Z_1^{iu} \rfloor\rfloor^{\angle -\propto} + \lfloor\lfloor \exists Z_2^{iu} \rfloor\rfloor^{\angle -\propto}} \\ -\lfloor\lfloor \exists Z_1^{il} \rfloor\rfloor^{\angle -\propto} \cdot \lfloor\lfloor \exists Z_2^{il} \rfloor\rfloor^{\angle -\propto}, & -\lfloor\lfloor \exists Z_1^{iu} \rfloor\rfloor^{\angle -\propto} \cdot \lfloor\lfloor \exists Z_2^{iu} \rfloor\rfloor^{\angle -\propto} \\ \lfloor\lfloor \exists Z_1^{fl} \rfloor\rfloor^{\angle -\propto} \lfloor\lfloor \exists Z_2^{fl} \rfloor\rfloor^{\angle -\propto} & \cdot \lfloor\lfloor \exists Z_1^{fu} \rfloor\rfloor^{\angle -\propto} \lfloor\lfloor \exists Z_2^{fu} \rfloor\rfloor^{\angle -\propto} \end{bmatrix},$$

$$(2)\ Ⓢ_1 \odot Ⓢ_2 = \begin{bmatrix} \lfloor\lfloor \exists Z_1^{tl} \rfloor\rfloor^{\angle -\propto} \lfloor\lfloor \exists Z_2^{tl} \rfloor\rfloor^{\angle -\propto}, & \lfloor\lfloor \exists Z_1^{tu} \rfloor\rfloor^{\angle -\propto} \lfloor\lfloor \exists Z_2^{tu} \rfloor\rfloor^{\angle -\propto}, \\ \sqrt[\angle -\propto]{\lfloor\lfloor \exists Z_1^{il} \rfloor\rfloor^{\angle -\propto} + \lfloor\lfloor \exists Z_2^{il} \rfloor\rfloor^{\angle -\propto}} & \sqrt[\angle -\propto]{\lfloor\lfloor \exists Z_1^{iu} \rfloor\rfloor^{\angle -\propto} + \lfloor\lfloor \exists Z_2^{iu} \rfloor\rfloor^{\angle -\propto}} \\ -\lfloor\lfloor \exists Z_1^{il} \rfloor\rfloor^{\angle -\propto} \cdot \lfloor\lfloor \exists Z_2^{il} \rfloor\rfloor^{\angle -\propto} & -\lfloor\lfloor \exists Z_1^{iu} \rfloor\rfloor^{\angle -\propto} \cdot \lfloor\lfloor \exists Z_2^{iu} \rfloor\rfloor^{\angle -\propto} \\ \sqrt[\angle -\propto]{\lfloor\lfloor \exists Z_1^{fl} \rfloor\rfloor^{\angle -\propto} + \lfloor\lfloor \exists Z_2^{fl} \rfloor\rfloor^{\angle -\propto}} & \sqrt[\angle -\propto]{\lfloor\lfloor \exists Z_1^{fl} \rfloor\rfloor^{\angle -\propto} + \lfloor\lfloor \exists Z_2^{fu} \rfloor\rfloor^{\angle -\propto}} \\ -\lfloor\lfloor \exists Z_1^{fl} \rfloor\rfloor^{\angle -\propto} \cdot \lfloor\lfloor \exists Z_2^{fl} \rfloor\rfloor^{\angle -\propto}, & -\lfloor\lfloor \exists Z_1^{fl} \rfloor\rfloor^{\angle -\propto} \cdot \lfloor\lfloor \exists Z_2^{fu} \rfloor\rfloor^{\angle -\propto} \end{bmatrix}$$

$$(3)\ \varrho, Ⓢ = \begin{bmatrix} \sqrt[\angle -\propto]{\propto - \lfloor \propto - \lfloor\lfloor \exists Z^{tl} \rfloor^{\angle -\propto} \rfloor^{\varrho}}, & \sqrt[\angle -\propto]{\propto - \lfloor \propto - \lfloor\lfloor \exists Z^{tu} \rfloor^{\angle -\propto} \rfloor^{\varrho}}, \\ \sqrt[\angle -\propto]{\propto - \lfloor \propto - \lfloor\lfloor \exists Z^{il} \rfloor^{\angle -\propto} \rfloor^{\varrho}}, & \sqrt[\angle -\propto]{\propto - \lfloor \propto - \lfloor\lfloor \exists Z^{iu} \rfloor^{\angle -\propto} \rfloor^{\varrho}}, \\ \lfloor\lfloor\lfloor \exists Z^{fl} \rfloor^{\angle -\propto} \rfloor^{\varrho}, & \lfloor\lfloor\lfloor \exists Z^{fu} \rfloor^{\angle -\propto} \rfloor^{\varrho} \end{bmatrix}$$

$$(4)\ Ⓢ^{\varrho} = \begin{bmatrix} \lfloor\lfloor\lfloor \exists Z^{tl} \rfloor^{\angle -\propto} \rfloor^{\varrho}, & \lfloor\lfloor\lfloor \exists Z^{tu} \rfloor^{\angle -\propto} \rfloor^{\varrho}, \\ \sqrt[\angle -\propto]{\propto - \lfloor \propto - \lfloor\lfloor \exists Z^{il} \rfloor^{\angle -\propto} \rfloor^{\varrho}}, & \sqrt[\angle -\propto]{\propto - \lfloor \propto - \lfloor\lfloor \exists Z^{iu} \rfloor^{\angle -\propto} \rfloor^{\varrho}}, \\ \sqrt[\angle -\propto]{\propto - \lfloor \propto - \lfloor\lfloor \exists Z^{fl} \rfloor^{\angle -\propto} \rfloor^{\varrho}}, & \sqrt[\angle -\propto]{\propto - \lfloor \propto - \lfloor\lfloor \exists Z^{fu} \rfloor^{\angle -\propto} \rfloor^{\varrho}} \end{bmatrix}.$$

2.1. *IVTNRFWA*

Definition 2.3. Let $Ⓢ_i = \left\langle \left\lfloor \lfloor\lfloor \exists Z_i^{tl}, \exists Z_i^{tu} \rfloor, \lfloor \exists Z_i^{il}, \exists Z_i^{iu} \rfloor, \lfloor \exists Z_i^{fl}, \exists Z_i^{fu} \rfloor \rfloor \right\rfloor \right\rangle$ be the IVTNRFNs, $W = \lfloor \sharp_1, \sharp_2, ..., \sharp_n \rfloor$ be the weight of $Ⓢ_i$, $\sharp_i \geq 0$ and $\vartriangle_{i \to 1}^{n} \sharp_i = 1$. Then IVTNRFWA $\lfloor Ⓢ_1, Ⓢ_2, ..., Ⓢ_n \rfloor = \vartriangle_{i \to 1}^{n} \sharp_i Ⓢ_i$.

Theorem 2.4. *Let* $Ⓢ_i = \left\langle \left\lfloor \lfloor\lfloor \exists Z_i^{tl}, \exists Z_i^{tu} \rfloor, \lfloor \exists Z_i^{il}, \exists Z_i^{iu} \rfloor, \lfloor \exists Z_i^{fl}, \exists Z_i^{fu} \rfloor \rfloor \right\rfloor \right\rangle$ *be the IVTNRFNs. Then IVTNRFWA* $\lfloor Ⓢ_1, Ⓢ_2, ..., Ⓢ_n \rfloor$

$$= \left[\begin{array}{c} \sqrt[\angle -\propto]{\propto -\nabla_{i\to 1}^{n}\left\lfloor \propto -\lfloor\lfloor \exists Z_i^{tl}\rfloor\rfloor^{\angle -\propto}\right\rfloor^{\sharp_i}}, \sqrt[\angle -\propto]{\propto -\nabla_{i\to 1}^{n}\left\lfloor \propto -\lfloor\lfloor \exists Z_i^{tu}\rfloor\rfloor^{\angle -\propto}\right\rfloor^{\sharp_i}}, \\ \sqrt[\angle -\propto]{\propto -\nabla_{i\to 1}^{n}\left\lfloor \propto -\lfloor\lfloor \exists Z_i^{il}\rfloor\rfloor^{\angle -\propto}\right\rfloor^{\sharp_i}}, \sqrt[\angle -\propto]{\propto -\nabla_{i\to 1}^{n}\left\lfloor \propto -\lfloor\lfloor \exists Z_i^{iu}\rfloor\rfloor^{\angle -\propto}\right\rfloor^{\sharp_i}}, \\ \nabla_{i\to 1}^{n}\lfloor\lfloor\lfloor \exists Z_i^{fl}\rfloor\rfloor^{\angle -\propto}\rfloor^{\sharp_i}, \nabla_{i\to 1}^{n}\lfloor\lfloor\lfloor \exists Z_i^{fu}\rfloor\rfloor^{\angle -\propto}\rfloor^{\sharp_i} \end{array} \right].$$

Proof If $n = 2$, then $\text{IVTNRFWA}\lfloor \text{Ⓢ}_1, \text{Ⓢ}_2\rfloor = \sharp_1\text{Ⓢ}_1 \vartriangle \sharp_2\text{Ⓢ}_2$, where

$$\sharp_1\text{Ⓢ}_1 = \left[\begin{array}{c} \sqrt[\angle -\propto]{\propto -\left\lfloor \propto -\lfloor\lfloor \exists Z_1^{tl}\rfloor\rfloor^{\angle -\propto}\right\rfloor^{\sharp_1}}, \sqrt[\angle -\propto]{\propto -\left\lfloor \propto -\lfloor\lfloor \exists Z_1^{tu}\rfloor\rfloor^{\angle -\propto}\right\rfloor^{\sharp_1}}, \\ \sqrt[\angle -\propto]{\propto -\left\lfloor \propto -\lfloor\lfloor \exists Z_1^{il}\rfloor\rfloor^{\angle -\propto}\right\rfloor^{\sharp_1}}, \sqrt[\angle -\propto]{\propto -\left\lfloor \propto -\lfloor\lfloor \exists Z_1^{iu}\rfloor\rfloor^{\angle -\propto}\right\rfloor^{\sharp_1}}, \\ \lfloor\lfloor\lfloor \exists Z_1^{fl}\rfloor\rfloor^{\angle -\propto}\rfloor^{\sharp_1}, \lfloor\lfloor\lfloor \exists Z_1^{fu}\rfloor\rfloor^{\angle -\propto}\rfloor^{\sharp_1} \end{array} \right]$$

$$\sharp_2\text{Ⓢ}_2 = \left[\begin{array}{c} \sqrt[\angle -\propto]{\propto -\left\lfloor \propto -\lfloor\lfloor \exists Z_2^{tl}\rfloor\rfloor^{\angle -\propto}\right\rfloor^{\sharp_2}}, \sqrt[\angle -\propto]{\propto -\left\lfloor \propto -\lfloor\lfloor \exists Z_2^{tu}\rfloor\rfloor^{\angle -\propto}\right\rfloor^{\sharp_2}}, \\ \sqrt[\angle -\propto]{\propto -\left\lfloor \propto -\lfloor\lfloor \exists Z_2^{il}\rfloor\rfloor^{\angle -\propto}\right\rfloor^{\sharp_2}}, \sqrt[\angle -\propto]{\propto -\left\lfloor \propto -\lfloor\lfloor \exists Z_2^{iu}\rfloor\rfloor^{\angle -\propto}\right\rfloor^{\sharp_2}}, \\ \lfloor\lfloor\lfloor \exists Z_2^{fl}\rfloor\rfloor^{\angle -\propto}\rfloor^{\sharp_2}, \lfloor\lfloor\lfloor \exists Z_2^{fu}\rfloor\rfloor^{\angle -\propto}\rfloor^{\sharp_2} \end{array} \right].$$

Now, $\sharp_1\text{Ⓢ}_1 \vartriangle \sharp_2\text{Ⓢ}_2$

$$= \left[\begin{array}{c} \sqrt[\angle -\propto]{\begin{array}{l} \left\lfloor \propto -\left\lfloor \propto -\lfloor\lfloor \exists Z_1^{tl}\rfloor\rfloor^{\angle -\propto}\right\rfloor^{\sharp_1}\right\rfloor + \\ \left\lfloor \propto -\left\lfloor \propto -\lfloor\lfloor \exists Z_2^{tl}\rfloor\rfloor^{\angle -\propto}\right\rfloor^{\sharp_2}\right\rfloor \\ -\left\lfloor \propto -\left\lfloor \propto -\lfloor\lfloor \exists Z_1^{tl}\rfloor\rfloor^{\angle -\propto}\right\rfloor^{\sharp_1}\right\rfloor \cdot \\ \left\lfloor \propto -\left\lfloor \propto -\lfloor\lfloor \exists Z_2^{tl}\rfloor\rfloor^{\angle -\propto}\right\rfloor^{\sharp_2}\right\rfloor \end{array}}, \sqrt[\angle -\propto]{\begin{array}{l} \left\lfloor \propto -\left\lfloor \propto -\lfloor\lfloor \exists Z_1^{tu}\rfloor\rfloor^{\angle -\propto}\right\rfloor^{\sharp_1}\right\rfloor + \\ \left\lfloor \propto -\left\lfloor \propto -\lfloor\lfloor \exists Z_2^{tu}\rfloor\rfloor^{\angle -\propto}\right\rfloor^{\sharp_2}\right\rfloor \\ -\left\lfloor \propto -\left\lfloor \propto -\lfloor\lfloor \exists Z_1^{tu}\rfloor\rfloor^{\angle -\propto}\right\rfloor^{\sharp_1}\right\rfloor \cdot \\ \left\lfloor \propto -\left\lfloor \propto -\lfloor\lfloor \exists Z_2^{tu}\rfloor\rfloor^{\angle -\propto}\right\rfloor^{\sharp_2}\right\rfloor \end{array}}, \\ \sqrt[\angle -\propto]{\begin{array}{l} \left\lfloor \propto -\left\lfloor \propto -\lfloor\lfloor \exists Z_1^{il}\rfloor\rfloor^{\angle -\propto}\right\rfloor^{\sharp_1}\right\rfloor + \\ \left\lfloor \propto -\left\lfloor \propto -\lfloor\lfloor \exists Z_2^{il}\rfloor\rfloor^{\angle -\propto}\right\rfloor^{\sharp_2}\right\rfloor \\ -\left\lfloor \propto -\left\lfloor \propto -\lfloor\lfloor \exists Z_1^{il}\rfloor\rfloor^{\angle -\propto}\right\rfloor^{\sharp_1}\right\rfloor \cdot \\ \left\lfloor \propto -\left\lfloor \propto -\lfloor\lfloor \exists Z_2^{il}\rfloor\rfloor^{\angle -\propto}\right\rfloor^{\sharp_2}\right\rfloor \end{array}}, \sqrt[\angle -\propto]{\begin{array}{l} \left\lfloor \propto -\left\lfloor \propto -\lfloor\lfloor \exists Z_1^{iu}\rfloor\rfloor^{\angle -\propto}\right\rfloor^{\sharp_1}\right\rfloor + \\ \left\lfloor \propto -\left\lfloor \propto -\lfloor\lfloor \exists Z_2^{iu}\rfloor\rfloor^{\angle -\propto}\right\rfloor^{\sharp_2}\right\rfloor \\ -\left\lfloor \propto -\left\lfloor \propto -\lfloor\lfloor \exists Z_1^{iu}\rfloor\rfloor^{\angle -\propto}\right\rfloor^{\sharp_1}\right\rfloor \cdot \\ \left\lfloor \propto -\left\lfloor \propto -\lfloor\lfloor \exists Z_2^{iu}\rfloor\rfloor^{\angle -\propto}\right\rfloor^{\sharp_2}\right\rfloor \end{array}}, \\ \lfloor\lfloor\lfloor \exists Z_1^{fl}\rfloor\rfloor^{\angle -\propto}\rfloor^{\sharp_1} \cdot \lfloor\lfloor\lfloor \exists Z_2^{fl}\rfloor\rfloor^{\angle -\propto}\rfloor^{\sharp_2}, \lfloor\lfloor\lfloor \exists Z_1^{fu}\rfloor\rfloor^{\angle -\propto}\rfloor^{\sharp_1} \cdot \lfloor\lfloor\lfloor \exists Z_2^{fu}\rfloor\rfloor^{\angle -\propto}\rfloor^{\sharp_2} \end{array} \right]$$

$$= \begin{bmatrix} \sqrt[\angle -\propto]{\propto -\left\lfloor \propto -\lfloor\lfloor \exists Z_1^{tl} \rfloor\rfloor^{\angle -\propto} \right\rfloor^{\sharp_1} \left\lfloor \propto -\lfloor\lfloor \exists Z_2^{tl} \rfloor\rfloor^{\angle -\propto} \right\rfloor^{\sharp_2}}, \\ \sqrt[\angle -\propto]{\propto -\left\lfloor \propto -\lfloor\lfloor \exists Z_1^{tu} \rfloor\rfloor^{\angle -\propto} \right\rfloor^{\sharp_1} \left\lfloor \propto -\lfloor\lfloor \exists Z_2^{tu} \rfloor\rfloor^{\angle -\propto} \right\rfloor^{\sharp_2}}, \\ \sqrt[\angle -\propto]{\propto -\left\lfloor \propto -\lfloor\lfloor \exists Z_1^{il} \rfloor\rfloor^{\angle -\propto} \right\rfloor^{\sharp_1} \left\lfloor \propto -\lfloor\lfloor \exists Z_2^{il} \rfloor\rfloor^{\angle -\propto} \right\rfloor^{\sharp_2}}, \\ \sqrt[\angle -\propto]{\propto -\left\lfloor \propto -\lfloor\lfloor \exists Z_1^{iu} \rfloor\rfloor^{\angle -\propto} \right\rfloor^{\sharp_1} \left\lfloor \propto -\lfloor\lfloor \exists Z_2^{iu} \rfloor\rfloor^{\angle -\propto} \right\rfloor^{\sharp_2}}, \\ \lfloor\lfloor\lfloor \exists Z_1^{fl} \rfloor\rfloor^{\angle -\propto} \rfloor^{\sharp_1} \cdot \lfloor\lfloor\lfloor \exists Z_2^{fl} \rfloor\rfloor^{\angle -\propto} \rfloor^{\sharp_2}, \\ \lfloor\lfloor\lfloor \exists Z_1^{fu} \rfloor\rfloor^{\angle -\propto} \rfloor^{\sharp_1} \cdot \lfloor\lfloor\lfloor \exists Z_2^{fu} \rfloor\rfloor^{\angle -\propto} \rfloor^{\sharp_2} \end{bmatrix}$$

Hence, IVTNRFWA$\lfloor \circledS_1, \circledS_2 \rfloor$

$$= \begin{bmatrix} \sqrt[\angle -\propto]{\propto -\nabla_{i\to 1}^{2} \left\lfloor \propto -\lfloor\lfloor \exists Z_i^{tl} \rfloor\rfloor^{\angle -\propto} \right\rfloor^{\sharp_i}}, \sqrt[\angle -\propto]{\propto -\nabla_{i\to 1}^{2} \left\lfloor \propto -\lfloor\lfloor \exists Z_i^{tu} \rfloor\rfloor^{\angle -\propto} \right\rfloor^{\sharp_i}}, \\ \sqrt[\angle -\propto]{\propto -\nabla_{i\to 1}^{2} \left\lfloor \propto -\lfloor\lfloor \exists Z_i^{il} \rfloor\rfloor^{\angle -\propto} \right\rfloor^{\sharp_i}}, \sqrt[\angle -\propto]{\propto -\nabla_{i\to 1}^{2} \left\lfloor \propto -\lfloor\lfloor \exists Z_i^{iu} \rfloor\rfloor^{\angle -\propto} \right\rfloor^{\sharp_i}}, \\ \nabla_{i\to 1}^{2} \lfloor\lfloor\lfloor \exists Z_i^{fl} \rfloor\rfloor^{\angle -\propto} \rfloor^{\sharp_i}, \nabla_{i\to 1}^{2} \lfloor\lfloor\lfloor \exists Z_i^{fu} \rfloor\rfloor^{\angle -\propto} \rfloor^{\sharp_i} \end{bmatrix}.$$

It valid for $n \geq 3$,

Thus, IVTNRFWA$\lfloor \circledS_1, \circledS_2, ..., \circledS_l \rfloor$

$$= \begin{bmatrix} \sqrt[\angle -\propto]{\propto -\nabla_{i\to 1}^{l} \left\lfloor \propto -\lfloor\lfloor \exists Z_i^{tl} \rfloor\rfloor^{\angle -\propto} \right\rfloor^{\sharp_i}}, \sqrt[\angle -\propto]{\propto -\nabla_{i\to 1}^{l} \left\lfloor \propto -\lfloor\lfloor \exists Z_i^{tu} \rfloor\rfloor^{\angle -\propto} \right\rfloor^{\sharp_i}}, \\ \sqrt[\angle -\propto]{\propto -\nabla_{i\to 1}^{l} \left\lfloor \propto -\lfloor\lfloor \exists Z_i^{il} \rfloor\rfloor^{\angle -\propto} \right\rfloor^{\sharp_i}}, \sqrt[\angle -\propto]{\propto -\nabla_{i\to 1}^{l} \left\lfloor \propto -\lfloor\lfloor \exists Z_i^{iu} \rfloor\rfloor^{\angle -\propto} \right\rfloor^{\sharp_i}}, \\ \nabla_{i\to 1}^{l} \lfloor\lfloor\lfloor \exists Z_i^{fl} \rfloor\rfloor^{\angle -\propto} \rfloor^{\sharp_i}, \nabla_{i\to 1}^{l} \lfloor\lfloor\lfloor \exists Z_i^{fu} \rfloor\rfloor^{\angle -\propto} \rfloor^{\sharp_i} \end{bmatrix}.$$

If $n = l + 1$, then IVTNRFWA $\lfloor \circledS_1, \circledS_2, ..., \circledS_l, \circledS_{l+1} \rfloor$

$$= \left[\begin{array}{c} \begin{array}{c} \angle^{-\propto}\sqrt{\begin{array}{c} \triangle_{i\to 1}^{l}\left\lfloor \propto -\left\lfloor \propto -\left\lfloor\left\lfloor \exists Z_i^{tl}\right\rfloor\right\rfloor^{\angle^{-\propto}}\right\rfloor^{\sharp_i}\right\rfloor + \left\lfloor \propto -\left\lfloor \propto -\left\lfloor Z_{l+1}^{tl}\right\rfloor^{\angle^{-\propto}}\right\rfloor^{\sharp_{l+1}}\right\rfloor \\ -\nabla_{i\to 1}^{l}\left\lfloor \propto -\left\lfloor \propto -\left\lfloor\left\lfloor \exists Z_i^{tl}\right\rfloor\right\rfloor^{\angle^{-\propto}}\right\rfloor^{\sharp_i}\right\rfloor \cdot \left\lfloor \propto -\left\lfloor \propto -\left\lfloor Z_{l+1}^{tl}\right\rfloor^{\angle^{-\propto}}\right\rfloor^{\sharp_{l+1}}\right\rfloor \end{array}}, \\ \angle^{-\propto}\sqrt{\begin{array}{c} \triangle_{i\to 1}^{l}\left\lfloor \propto -\left\lfloor \propto -\left\lfloor\left\lfloor \exists Z_i^{tu}\right\rfloor\right\rfloor^{\angle^{-\propto}}\right\rfloor^{\sharp_i}\right\rfloor + \left\lfloor \propto -\left\lfloor \propto -\left\lfloor Z_{l+1}^{tu}\right\rfloor^{\angle^{-\propto}}\right\rfloor^{\sharp_{l+1}}\right\rfloor \\ -\nabla_{i\to 1}^{l}\left\lfloor \propto -\left\lfloor \propto -\left\lfloor\left\lfloor \exists Z_i^{tu}\right\rfloor\right\rfloor^{\angle^{-\propto}}\right\rfloor^{\sharp_i}\right\rfloor \cdot \left\lfloor \propto -\left\lfloor \propto -\left\lfloor Z_{l+1}^{tu}\right\rfloor^{\angle^{-\propto}}\right\rfloor^{\sharp_{l+1}}\right\rfloor \end{array}}, \\ \angle^{-\propto}\sqrt{\begin{array}{c} \triangle_{i\to 1}^{l}\left\lfloor \propto -\left\lfloor \propto -\left\lfloor\left\lfloor \exists Z_i^{il}\right\rfloor\right\rfloor^{\angle^{-\propto}}\right\rfloor^{\sharp_i}\right\rfloor + \left\lfloor \propto -\left\lfloor \propto -\left\lfloor Z_{l+1}^{il}\right\rfloor^{\angle^{-\propto}}\right\rfloor^{\sharp_{l+1}}\right\rfloor \\ -\nabla_{i\to 1}^{l}\left\lfloor \propto -\left\lfloor \propto -\left\lfloor\left\lfloor \exists Z_i^{il}\right\rfloor\right\rfloor^{\angle^{-\propto}}\right\rfloor^{\sharp_i}\right\rfloor \cdot \left\lfloor \propto -\left\lfloor \propto -\left\lfloor Z_{l+1}^{il}\right\rfloor^{\angle^{-\propto}}\right\rfloor^{\sharp_{l+1}}\right\rfloor \end{array}}, \\ \angle^{-\propto}\sqrt{\begin{array}{c} \triangle_{i\to 1}^{l}\left\lfloor \propto -\left\lfloor \propto -\left\lfloor\left\lfloor \exists Z_i^{iu}\right\rfloor\right\rfloor^{\angle^{-\propto}}\right\rfloor^{\sharp_i}\right\rfloor + \left\lfloor \propto -\left\lfloor \propto -\left\lfloor Z_{l+1}^{iu}\right\rfloor^{\angle^{-\propto}}\right\rfloor^{\sharp_{l+1}}\right\rfloor \\ -\nabla_{i\to 1}^{l}\left\lfloor \propto -\left\lfloor \propto -\left\lfloor\left\lfloor \exists Z_i^{iu}\right\rfloor\right\rfloor^{\angle^{-\propto}}\right\rfloor^{\sharp_i}\right\rfloor, \left\lfloor \propto -\left\lfloor \propto -\left\lfloor Z_{l+1}^{iu}\right\rfloor^{\angle^{-\propto}}\right\rfloor^{\sharp_{l+1}}\right\rfloor \end{array}}, \\ \nabla_{i\to 1}^{l}\left\lfloor\left\lfloor\left\lfloor \exists Z_i^{fl}\right\rfloor\right\rfloor^{\angle^{-\propto}}\right\rfloor^{\sharp_i} \cdot \left\lfloor\left\lfloor Z_{l+1}^{fl}\right\rfloor^{\angle^{-\propto}}\right\rfloor^{\sharp_{l+1}}, \nabla_{i\to 1}^{l}\left\lfloor\left\lfloor\left\lfloor \exists Z_i^{fu}\right\rfloor\right\rfloor^{\angle^{-\propto}}\right\rfloor^{\sharp_i} \cdot \left\lfloor\left\lfloor Z_{l+1}^{fu}\right\rfloor^{\angle^{-\propto}}\right\rfloor^{\sharp_{l+1}} \end{array} \right]$$

$$= \left[\begin{array}{c} \angle^{-\propto}\sqrt{\propto -\nabla_{i\to 1}^{l+1}\left\lfloor \propto -\left\lfloor\left\lfloor \exists Z_i^{tl}\right\rfloor\right\rfloor^{\angle^{-\propto}}\right\rfloor^{\sharp_i}}, \angle^{-\propto}\sqrt{\propto -\nabla_{i\to 1}^{l+1}\left\lfloor \propto -\left\lfloor\left\lfloor \exists Z_i^{tu}\right\rfloor\right\rfloor^{\angle^{-\propto}}\right\rfloor^{\sharp_i}}, \\ \angle^{-\propto}\sqrt{\propto -\nabla_{i\to 1}^{l+1}\left\lfloor \propto -\left\lfloor\left\lfloor \exists Z_i^{il}\right\rfloor\right\rfloor^{\angle^{-\propto}}\right\rfloor^{\sharp_i}}, \angle^{-\propto}\sqrt{\propto -\nabla_{i\to 1}^{l+1}\left\lfloor \propto -\left\lfloor\left\lfloor \exists Z_i^{iu}\right\rfloor\right\rfloor^{\angle^{-\propto}}\right\rfloor^{\sharp_i}}, \\ \nabla_{i\to 1}^{l+1}\left\lfloor\left\lfloor\left\lfloor \exists Z_i^{fl}\right\rfloor\right\rfloor^{\angle^{-\propto}}\right\rfloor^{\sharp_i}, \nabla_{i\to 1}^{l+1}\left\lfloor\left\lfloor\left\lfloor \exists Z_i^{fu}\right\rfloor\right\rfloor^{\angle^{-\propto}}\right\rfloor^{\sharp_i} \end{array} \right].$$

Theorem 2.5. *Let* $\circledS_i = \left\langle \left\lfloor \left\lfloor\left\lfloor \exists Z_i^{tl}, \exists Z_i^{tu}\right\rfloor, \left\lfloor \exists Z_i^{il}, \exists Z_i^{iu}\right\rfloor, \left\lfloor \exists Z_i^{fl}, \exists Z_i^{fu}\right\rfloor\right\rfloor \right\rangle$ *be the IVTNRFNs. Then IVTNRFWA* $\left\lfloor \circledS_1, \circledS_2, ..., \circledS_n\right\rfloor = \circledS$ *(idempotency property).*

Proof Since $\left\lfloor \exists Z_i^{tl}\right\rfloor = \left\lfloor \exists Z^{tl}\right\rfloor$, $\left\lfloor \exists Z_i^{il}\right\rfloor = \left\lfloor \exists Z^{il}\right\rfloor$ and $\left\lfloor \exists Z_i^{fl}\right\rfloor = \left\lfloor \exists Z^{fl}\right\rfloor$ and $\left\lfloor \exists Z_i^{tu}\right\rfloor = \left\lfloor \exists Z^{tu}\right\rfloor$, $\left\lfloor \exists Z_i^{iu}\right\rfloor = \left\lfloor \exists Z^{iu}\right\rfloor$ and $\left\lfloor \exists Z_i^{fu}\right\rfloor = \left\lfloor \exists Z^{fu}\right\rfloor$ and $\triangle_{i\to 1}^{n}\sharp_i = 1$. Now, IVTNRFWA$\left\lfloor \circledS_1, \circledS_2, ..., \circledS_n\right\rfloor$

$$= \left[\begin{array}{c} \angle^{-\propto}\sqrt{\propto -\nabla_{i\to 1}^{n}\left\lfloor \propto -\left\lfloor\left\lfloor \exists Z_i^{tl}\right\rfloor\right\rfloor^{\angle^{-\propto}}\right\rfloor^{\sharp_i}}, \angle^{-\propto}\sqrt{\propto -\nabla_{i\to 1}^{n}\left\lfloor \propto -\left\lfloor\left\lfloor \exists Z_i^{tu}\right\rfloor\right\rfloor^{\angle^{-\propto}}\right\rfloor^{\sharp_i}}, \\ \angle^{-\propto}\sqrt{\propto -\nabla_{i\to 1}^{n}\left\lfloor \propto -\left\lfloor\left\lfloor \exists Z_i^{il}\right\rfloor\right\rfloor^{\angle^{-\propto}}\right\rfloor^{\sharp_i}}, \angle^{-\propto}\sqrt{\propto -\nabla_{i\to 1}^{n}\left\lfloor \propto -\left\lfloor\left\lfloor \exists Z_i^{iu}\right\rfloor\right\rfloor^{\angle^{-\propto}}\right\rfloor^{\sharp_i}}, \\ \nabla_{i\to 1}^{n}\left\lfloor\left\lfloor\left\lfloor \exists Z_i^{fl}\right\rfloor\right\rfloor^{\angle^{-\propto}}\right\rfloor^{\sharp_i}, \nabla_{i\to 1}^{n}\left\lfloor\left\lfloor\left\lfloor \exists Z_i^{fu}\right\rfloor\right\rfloor^{\angle^{-\propto}}\right\rfloor^{\sharp_i} \end{array} \right]$$

$$= \left[\begin{array}{c} \angle^{-\propto}\sqrt{\propto -\left\lfloor \propto -\left\lfloor\left\lfloor \exists Z^{tl}\right\rfloor^{\angle^{-\propto}}\right\rfloor^{\triangle_{i\to 1}^{n}\sharp_i}}, \angle^{-\propto}\sqrt{\propto -\left\lfloor \propto -\left\lfloor\left\lfloor \exists Z^{tu}\right\rfloor^{\angle^{-\propto}}\right\rfloor^{\triangle_{i\to 1}^{n}\sharp_i}}, \\ \angle^{-\propto}\sqrt{\propto -\left\lfloor \propto -\left\lfloor\left\lfloor \exists Z^{il}\right\rfloor^{\angle^{-\propto}}\right\rfloor^{\triangle_{i\to 1}^{n}\sharp_i}}, \angle^{-\propto}\sqrt{\propto -\left\lfloor \propto -\left\lfloor\left\lfloor \exists Z^{iu}\right\rfloor^{\angle^{-\propto}}\right\rfloor^{\triangle_{i\to 1}^{n}\sharp_i}}, \\ \left\lfloor\left\lfloor\left\lfloor \exists Z^{fl}\right\rfloor^{\angle^{-\propto}}\right\rfloor^{\triangle_{i\to 1}^{n}\sharp_i}, \left\lfloor\left\lfloor\left\lfloor \exists Z^{fu}\right\rfloor^{\angle^{-\propto}}\right\rfloor^{\triangle_{i\to 1}^{n}\sharp_i} \end{array} \right]$$

$$= \begin{bmatrix} \sqrt[\angle -\propto]{\propto -\left\lfloor \propto -\lfloor\lfloor \exists Z^{tl}\rfloor^{\angle -\propto}\right\rfloor}, \sqrt[\angle -\propto]{\propto -\left\lfloor \propto -\lfloor\lfloor \exists Z^{tu}\rfloor^{\angle -\propto}\right\rfloor}, \\ \sqrt[\angle -\propto]{\propto -\left\lfloor \propto -\lfloor\lfloor \exists Z^{il}\rfloor^{\angle -\propto}\right\rfloor}, \sqrt[\angle -\propto]{\propto -\left\lfloor \propto -\lfloor\lfloor \exists Z^{iu}\rfloor^{\angle -\propto}\right\rfloor}, \\ \lfloor\lfloor \exists Z^{fl}\rfloor^{\angle -\propto}, \lfloor\lfloor \exists Z^{fu}\rfloor^{\angle -\propto} \end{bmatrix}$$
$$= \text{Ⓢ}$$

Theorem 2.6. *Let* $\text{Ⓢ}_i = \left\langle \left\lfloor \lfloor\lfloor \exists Z_i^{tl}, \exists Z_i^{tu}\rfloor, \lfloor \exists Z_i^{il}, \exists Z_i^{iu}\rfloor, \lfloor \exists Z_i^{fl}, \exists Z_i^{fu}\rfloor\rfloor \right\rfloor \right\rangle$ *be the IVTNRFNs. Then* $IVTNRFWA\lfloor \text{Ⓢ}_1, \text{Ⓢ}_2, ..., \text{Ⓢ}_n\rfloor$
where $\underline{\lfloor \exists Z^{tl}\rfloor} = \min\lfloor \exists Z_{ij}^{tl}\rfloor$, $\overline{\lfloor \exists Z^{tl}\rfloor} = \max\lfloor \exists Z_{ij}^{tl}\rfloor$, $\underline{\lfloor \exists Z^{il}\rfloor} = \min\lfloor \exists Z_{ij}^{il}\rfloor$, $\overline{\lfloor \exists Z^{il}\rfloor} = \max\lfloor \exists Z_{ij}^{il}\rfloor$, $\underline{\lfloor \exists Z^{fl}\rfloor} = \min\lfloor \exists Z_{ij}^{fl}\rfloor$, $\overline{\lfloor \exists Z^{fl}\rfloor} = \max\lfloor \exists Z_{ij}^{fl}\rfloor$
and $\underline{\lfloor \exists Z^{tu}\rfloor} = \min\lfloor \exists Z_{ij}^{tu}\rfloor$, $\overline{\lfloor \exists Z^{tu}\rfloor} = \max\lfloor \exists Z_{ij}^{tu}\rfloor$, $\underline{\lfloor \exists Z^{iu}\rfloor} = \min\lfloor \exists Z_{ij}^{iu}\rfloor$, $\overline{\lfloor \exists Z^{iu}\rfloor} = \max\lfloor \exists Z_{ij}^{iu}\rfloor$, $\underline{\lfloor \exists Z^{fu}\rfloor} = \min\lfloor \exists Z_{ij}^{fu}\rfloor$, $\overline{\lfloor \exists Z^{fu}\rfloor} = \max\lfloor \exists Z_{ij}^{fu}\rfloor$ *and where* $1 \le i \le n,\ j = 1, 2, ..., i_j$. *Then,*
$\left\langle \underline{\lfloor \exists Z^{tl}\rfloor}, \underline{\lfloor \exists Z^{tu}\rfloor}, \underline{\lfloor \exists Z^{il}\rfloor}, \underline{\lfloor \exists Z^{iu}\rfloor}, \overline{\lfloor \exists Z^{fl}\rfloor}, \overline{\lfloor \exists Z^{fu}\rfloor} \right\rangle$

$$\le IVTNRFWA\lfloor \text{Ⓢ}_1, \text{Ⓢ}_2, ..., \text{Ⓢ}_n\rfloor$$
$$\le \left\langle \overline{\lfloor \exists Z^{tl}, \exists Z^{tu}\rfloor}, \overline{\lfloor \exists Z^{il}, \exists Z^{iu}\rfloor}, \underline{\lfloor \exists Z^{fl}, \exists Z^{fu}\rfloor} \right\rangle.$$

(Boundedness property).

Proof Since, $\underline{\lfloor \exists Z^{tl}\rfloor} = \min\lfloor \exists Z_{ij}^{tl}\rfloor$, $\overline{\lfloor \exists Z^{tl}\rfloor} = \max\lfloor \exists Z_{ij}^{tl}\rfloor$ and $\underline{\lfloor \exists Z^{tl}\rfloor} \le \lfloor \exists Z_{ij}^{tl}\rfloor \le \overline{\lfloor \exists Z^{tl}\rfloor}$ and $\underline{\lfloor \exists Z^{tu}\rfloor} = \min\lfloor \exists Z_{ij}^{tu}\rfloor$, $\overline{\lfloor \exists Z^{tu}\rfloor} = \max\lfloor \exists Z_{ij}^{tu}\rfloor$ and $\underline{\lfloor \exists Z^{tu}\rfloor} \le \lfloor \exists Z_{ij}^{tu}\rfloor \le \overline{\lfloor \exists Z^{tu}\rfloor}$.

Now $\underline{\lfloor \exists Z^{tl}\rfloor}, \underline{\lfloor \exists Z^{tu}\rfloor}$

$$= \sqrt[\angle -\propto]{\propto -\nabla_{i\to 1}^{n}\left\lfloor \propto -\lfloor \underline{\lfloor \exists Z^{tl}\rfloor}\rfloor^{\angle -\propto}\right\rfloor^{\sharp_i}}, \sqrt[\angle -\propto]{\propto -\nabla_{i\to 1}^{n}\left\lfloor \propto -\lfloor \underline{\lfloor \exists Z^{tu}\rfloor}\rfloor^{\angle -\propto}\right\rfloor^{\sharp_i}}$$
$$\le \sqrt[\angle -\propto]{\propto -\nabla_{i\to 1}^{n}\left\lfloor \propto -\lfloor\lfloor\lfloor \exists Z_{ij}^{tl}\rfloor\rfloor\rfloor^{\angle -\propto}\right\rfloor^{\sharp_i}}, \sqrt[\angle -\propto]{\propto -\nabla_{i\to 1}^{n}\left\lfloor \propto -\lfloor\lfloor\lfloor \exists Z_{ij}^{tu}\rfloor\rfloor\rfloor^{\angle -\propto}\right\rfloor^{\sharp_i}}$$
$$\le \sqrt[\angle -\propto]{\propto -\nabla_{i\to 1}^{n}\left\lfloor \propto -\lfloor \overline{\lfloor \exists Z^{tl}\rfloor}\rfloor^{\angle -\propto}\right\rfloor^{\sharp_i}}, \sqrt[\angle -\propto]{\propto -\nabla_{i\to 1}^{n}\left\lfloor \propto -\lfloor \overline{\lfloor \exists Z^{tu}\rfloor}\rfloor^{\angle -\propto}\right\rfloor^{\sharp_i}}$$
$$= \overline{\lfloor \exists Z^{tl}\rfloor}.$$

Since, $\underline{\lfloor \exists Z^{il}\rfloor} = \min\lfloor \exists Z_{ij}^{il}\rfloor$, $\overline{\lfloor \exists Z^{il}\rfloor} = \max\lfloor \exists Z_{ij}^{il}\rfloor$ and $\underline{\lfloor \exists Z^{il}\rfloor} \le \lfloor \exists Z_{ij}^{il}\rfloor \le \overline{\lfloor \exists Z^{il}\rfloor}$ and $\underline{\lfloor \exists Z^{iu}\rfloor} = \min\lfloor \exists Z_{ij}^{iu}\rfloor$, $\overline{\lfloor \exists Z^{iu}\rfloor} = \max\lfloor \exists Z_{ij}^{iu}\rfloor$ and $\underline{\lfloor \exists Z^{iu}\rfloor} \le \lfloor \exists Z_{ij}^{iu}\rfloor \le \overline{\lfloor \exists Z^{iu}\rfloor}$.

Now, $\underline{\lfloor \exists Z^{il}\rfloor}, \underline{\lfloor \exists Z^{iu}\rfloor}$

$$
\begin{aligned}
&= \left[{}^{\angle-\propto}\sqrt{\propto -\nabla_{i\to 1}^{n}\left\lfloor \propto -\underline{\lfloor\lfloor \exists Z^{il}\rfloor\rfloor^{\angle-\propto}}\right\rfloor^{\sharp_i}},\ {}^{\angle-\propto}\sqrt{\propto -\nabla_{i\to 1}^{n}\left\lfloor \propto -\underline{\lfloor\lfloor \exists Z^{iu}\rfloor\rfloor^{\angle-\propto}}\right\rfloor^{\sharp_i}}\right] \\
&\leq \left[{}^{\angle-\propto}\sqrt{\propto -\nabla_{i\to 1}^{n}\left\lfloor \propto -\lfloor\lfloor \exists Z_{ij}^{il}\rfloor\rfloor^{\angle-\propto}\right\rfloor^{\sharp_i}},\ {}^{\angle-\propto}\sqrt{\propto -\nabla_{i\to 1}^{n}\left\lfloor \propto -\lfloor\lfloor \exists Z_{ij}^{iu}\rfloor\rfloor^{\angle-\propto}\right\rfloor^{\sharp_i}}\right] \\
&\leq \left[{}^{\angle-\propto}\sqrt{\propto -\nabla_{i\to 1}^{n}\left\lfloor \propto -\overline{\lfloor\lfloor \exists Z^{il}\rfloor\rfloor^{\angle-\propto}}\right\rfloor^{\sharp_i}},\ {}^{\angle-\propto}\sqrt{\propto -\nabla_{i\to 1}^{n}\left\lfloor \propto -\overline{\lfloor\lfloor \exists Z^{iu}\rfloor\rfloor^{\angle-\propto}}\right\rfloor^{\sharp_i}}\right] \\
&= \overline{\lfloor \exists Z^{il}\rfloor}, \overline{\lfloor \exists Z^{iu}\rfloor}.
\end{aligned}
$$

Since, $\underline{\lfloor\lfloor \exists Z^{fl}\rfloor^{\angle-\propto}\rfloor} = \min\lfloor\lfloor \exists Z_{ij}^{fl}\rfloor\rfloor^{\angle-\propto}$, $\overline{\lfloor\lfloor \exists Z^{fl}\rfloor^{\angle-\propto}\rfloor} = \max\lfloor\lfloor \exists Z_{ij}^{fl}\rfloor\rfloor^{\angle-\propto}$ and $\underline{\lfloor\lfloor \exists Z^{fl}\rfloor^{\angle-\propto}\rfloor} \leq \lfloor\lfloor \exists Z_{ij}^{fl}\rfloor\rfloor^{\angle-\propto} \leq \overline{\lfloor\lfloor \exists Z^{fl}\rfloor^{\angle-\propto}\rfloor}$ and $\underline{\lfloor\lfloor \exists Z^{fu}\rfloor^{\angle-\propto}\rfloor} = \min\lfloor\lfloor \exists Z_{ij}^{fu}\rfloor\rfloor^{\angle-\propto}$, $\overline{\lfloor\lfloor \exists Z^{fu}\rfloor^{\angle-\propto}\rfloor} = \max\lfloor\lfloor \exists Z_{ij}^{fu}\rfloor\rfloor^{\angle-\propto}$ and $\underline{\lfloor\lfloor \exists Z^{fu}\rfloor^{\angle-\propto}\rfloor} \leq \lfloor\lfloor \exists Z_{ij}^{fu}\rfloor\rfloor^{\angle-\propto} \leq \overline{\lfloor\lfloor \exists Z^{fu}\rfloor^{\angle-\propto}\rfloor}$.

We have,

$$
\begin{aligned}
\underline{\lfloor\lfloor \exists Z^{fl}\rfloor^{\angle-\propto}} &= \nabla_{i\to 1}^{n}\underline{\lfloor\lfloor \exists Z^{fl}\rfloor^{\angle-\propto}\rfloor^{\sharp_i}}, \nabla_{i\to 1}^{n}\underline{\lfloor\lfloor \exists Z^{fu}\rfloor^{\angle-\propto}\rfloor^{\sharp_i}} \\
&\leq \nabla_{i\to 1}^{n}\lfloor\lfloor\lfloor \exists Z_{ij}^{fl}\rfloor\rfloor^{\angle-\propto}\rfloor^{\sharp_i}, \nabla_{i\to 1}^{n}\lfloor\lfloor\lfloor \exists Z_{ij}^{fu}\rfloor\rfloor^{\angle-\propto}\rfloor^{\sharp_i} \\
&\leq \nabla_{i\to 1}^{n}\overline{\lfloor\lfloor \exists Z^{fl}\rfloor^{\angle-\propto}\rfloor^{\sharp_i}}, \nabla_{i\to 1}^{n}\overline{\lfloor\lfloor \exists Z^{fu}\rfloor^{\angle-\propto}\rfloor^{\sharp_i}} \\
&= \overline{\lfloor\lfloor \exists Z^{fl}\rfloor^{\angle-\propto}\rfloor}, \overline{\lfloor\lfloor \exists Z^{fu}\rfloor^{\angle-\propto}\rfloor}.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&\frac{1}{2}\times\left[\begin{array}{c}\left[\begin{array}{c}\left\lfloor {}^{\angle-\propto}\sqrt{\propto -\nabla_{i\to 1}^{n}\left\lfloor \propto -\underline{\lfloor\lfloor \exists Z^{tl}\rfloor\rfloor^{\angle-\propto}}\right\rfloor^{\sharp_i}}\right\rfloor^2 - \left\lfloor {}^{\angle-\propto}\sqrt{\propto -\nabla_{i\to 1}^{n}\left\lfloor \propto -\underline{\lfloor\lfloor \exists Z^{il}\rfloor\rfloor^{\angle-\propto}}\right\rfloor^{\sharp_i}}\right\rfloor^2 \\ +1-\left\lfloor \nabla_{i\to 1}^{n}\lfloor\lfloor\overline{\lfloor \exists Z^{fl}\rfloor}\rfloor^{\angle-\propto}\rfloor^{\sharp_i}\right\rfloor^2\end{array}\right] \\ +\left[\begin{array}{c}\left\lfloor {}^{\angle-\propto}\sqrt{\propto -\nabla_{i\to 1}^{n}\left\lfloor \propto -\underline{\lfloor\lfloor \exists Z^{tu}\rfloor\rfloor^{\angle-\propto}}\right\rfloor^{\sharp_i}}\right\rfloor^2 - \left\lfloor {}^{\angle-\propto}\sqrt{\propto -\nabla_{i\to 1}^{n}\left\lfloor \propto -\underline{\lfloor\lfloor \exists Z^{iu}\rfloor\rfloor^{\angle-\propto}}\right\rfloor^{\sharp_i}}\right\rfloor^2 \\ -\left\lfloor \nabla_{i\to 1}^{n}\lfloor\lfloor\overline{\lfloor \exists Z^{fu}\rfloor}\rfloor^{\angle-\propto}\rfloor^{\sharp_i}\right\rfloor^2\end{array}\right]\end{array}\right] \\
\leq\ &\frac{1}{2}\times\left[\begin{array}{c}\left[\begin{array}{c}\left\lfloor {}^{\angle-\propto}\sqrt{\propto -\nabla_{i\to 1}^{n}\left\lfloor \propto -\lfloor\lfloor\lfloor \exists Z_{ij}^{tl}\rfloor\rfloor\rfloor^{\angle-\propto}\right\rfloor^{\sharp_i}}\right\rfloor^2 - \left\lfloor {}^{\angle-\propto}\sqrt{\propto -\nabla_{i\to 1}^{n}\left\lfloor \propto -\lfloor\lfloor \exists Z_{ij}^{il}\rfloor\rfloor^{\angle-\propto}\right\rfloor^{\sharp_i}}\right\rfloor^2 + \\ +1-\left\lfloor \nabla_{i\to 1}^{n}\lfloor\lfloor\lfloor \exists Z_{ij}^{fl}\rfloor\rfloor^{\angle-\propto}\rfloor^{\sharp_i}\right\rfloor^2\end{array}\right] \\ +\left[\begin{array}{c}\left\lfloor {}^{\angle-\propto}\sqrt{\propto -\nabla_{i\to 1}^{n}\left\lfloor \propto -\lfloor\lfloor\lfloor \exists Z_{ij}^{tu}\rfloor\rfloor\rfloor^{\angle-\propto}\right\rfloor^{\sharp_i}}\right\rfloor^2 - \left\lfloor {}^{\angle-\propto}\sqrt{\propto -\nabla_{i\to 1}^{n}\left\lfloor \propto -\lfloor\lfloor \exists Z_{ij}^{iu}\rfloor\rfloor^{\angle-\propto}\right\rfloor^{\sharp_i}}\right\rfloor^2 + \\ -\left\lfloor \nabla_{i\to 1}^{n}\lfloor\lfloor\lfloor \exists Z_{ij}^{fu}\rfloor\rfloor^{\angle-\propto}\rfloor^{\sharp_i}\right\rfloor^2\end{array}\right]\end{array}\right] \\
\leq\ &\frac{1}{2}\times\left[\begin{array}{c}\left[\begin{array}{c}\left\lfloor {}^{\angle-\propto}\sqrt{\propto -\nabla_{i\to 1}^{n}\left\lfloor \propto -\lfloor\lfloor\overline{\lfloor \exists Z^{tl}\rfloor}\rfloor^{\angle-\propto}\right\rfloor^{\sharp_i}}\right\rfloor^2 - \left\lfloor {}^{\angle-\propto}\sqrt{\propto -\nabla_{i\to 1}^{n}\left\lfloor \propto -\lfloor\lfloor\overline{\lfloor \exists Z^{il}\rfloor}\rfloor^{\angle-\propto}\right\rfloor^{\sharp_i}}\right\rfloor^2 + \\ +1-\left\lfloor \nabla_{i\to 1}^{n}\lfloor\lfloor\underline{\lfloor \exists Z^{fl}\rfloor}\rfloor^{\angle-\propto}\rfloor^{\sharp_i}\right\rfloor^2\end{array}\right] \\ +\left[\begin{array}{c}\left\lfloor {}^{\angle-\propto}\sqrt{\propto -\nabla_{i\to 1}^{n}\left\lfloor \propto -\lfloor\lfloor\overline{\lfloor \exists Z^{tu}\rfloor}\rfloor^{\angle-\propto}\right\rfloor^{\sharp_i}}\right\rfloor^2 - \left\lfloor {}^{\angle-\propto}\sqrt{\propto -\nabla_{i\to 1}^{n}\left\lfloor \propto -\lfloor\lfloor\overline{\lfloor \exists Z^{iu}\rfloor}\rfloor^{\angle-\propto}\right\rfloor^{\sharp_i}}\right\rfloor^2 + \\ -\left\lfloor \nabla_{i\to 1}^{n}\lfloor\lfloor\underline{\lfloor \exists Z^{fu}\rfloor}\rfloor^{\angle-\propto}\rfloor^{\sharp_i}\right\rfloor^2\end{array}\right]\end{array}\right].
\end{aligned}
$$

Hence,

$\left\langle \underline{\lfloor \exists Z^{tl} \rfloor, \lfloor \exists Z^{tu} \rfloor}, \underline{\lfloor \exists Z^{il} \rfloor, \lfloor \exists Z^{iu} \rfloor}, \overline{\lfloor \exists Z^{fl} \rfloor, \lfloor \exists Z^{fu} \rfloor} \right\rangle$
$\leq IVTNRFWA\lfloor \text{Ⓢ}_1, \text{Ⓢ}_2, ..., \text{Ⓢ}_n \rfloor$
$\leq \langle \overline{\lfloor \exists Z^{tl} \rfloor, \lfloor \exists Z^{tu} \rfloor}, \overline{\lfloor \exists Z^{il} \rfloor, \lfloor \exists Z^{iu} \rfloor}, \underline{\lfloor \exists Z^{fl} \rfloor, \lfloor \exists Z^{fu} \rfloor} \rangle.$

Theorem 2.7. *Let* $\text{Ⓢ}_i = \left\langle \left\lfloor \lfloor \exists Z_i^{tl}, \exists Z_i^{tu} \rfloor, \lfloor \exists Z_i^{il}, \exists Z_i^{iu} \rfloor, \lfloor \exists Z_i^{fl}, \exists Z_i^{fu} \rfloor \right\rfloor \right\rangle$ *and* $W_i = \langle \lfloor \lfloor \exists Z_{h_{ij}}^{tl}, \exists Z_{h_{ij}}^{tu} \rfloor, \lfloor \exists Z_{h_{ij}}^{il}, \exists Z_{h_{ij}}^{iu} \rfloor, \lfloor \exists Z_{h_{ij}}^{fl}, \exists Z_{h_{ij}}^{fu} \rfloor \rfloor \rangle$, *be the IVTNRFWAs. For any* i, *if there is* $\lfloor \exists Z_{t_{ij}}^{tu} \rfloor^2 \leq \lfloor \exists Z_{h_{ij}}^{tu} \rfloor^2$ *and* $\lfloor \exists Z_{t_{ij}}^{iu} \rfloor^2 \leq \lfloor \exists Z_{h_{ij}}^{iu} \rfloor^2$ *and* $\lfloor \exists Z_{t_{ij}}^{fu} \rfloor^2 \geq \lfloor \exists Z_{h_{ij}}^{fu} \rfloor^2$ *or* $\text{Ⓢ}_i \leq W_i$. *Prove that* $IVTNRFWA\lfloor \text{Ⓢ}_1, \text{Ⓢ}_2, ..., \text{Ⓢ}_n \rfloor \leq IVTNRFWA\lfloor W_1, W_2, ..., W_n \rfloor$, *where* $\lfloor i = 1, 2, ..., n \rfloor; \lfloor j = 1, 2, ..., i_j \rfloor$ *(monotonicity property).*

Proof For any i, $\lfloor \exists Z_{t_{ij}}^{tl} \rfloor^2 \leq \lfloor \exists Z_{h_{ij}}^{tl} \rfloor^2$.
Therefore, $\propto - \lfloor \lfloor \exists Z_{t_i}^{tl} \rfloor \rfloor^2 \geq \propto - \lfloor \lfloor \exists Z_{h_i}^{tl} \rfloor \rfloor^2$.
Hence, $\nabla_{i\to 1}^n \left\lfloor \propto - \left\lfloor \lfloor \exists Z_{t_i}^{tl} \rfloor \right\rfloor^2 \right\rfloor^{\sharp_i} \geq \nabla_{i\to 1}^n \left\lfloor \propto - \left\lfloor \lfloor \exists Z_{h_i}^{tl} \rfloor \right\rfloor^2 \right\rfloor^{\sharp_i}$
and $\sqrt[\angle -\propto]{\propto - \nabla_{i\to 1}^n \left\lfloor \propto - \left\lfloor \lfloor \exists Z_{t_i}^{tl} \rfloor \right\rfloor^{\angle -\propto} \right\rfloor^{\sharp_i}} \leq \sqrt[\angle -\propto]{\propto - \nabla_{i\to 1}^n \left\lfloor \propto - \left\lfloor \lfloor \exists Z_{h_i}^{tl} \rfloor \right\rfloor^{\angle -\propto} \right\rfloor^{\sharp_i}}$.
Similarly, $\lfloor \exists Z_{t_{ij}}^{tu} \rfloor^2 \leq \lfloor \exists Z_{h_{ij}}^{tu} \rfloor^2$.
Therefore, $\propto - \lfloor \lfloor \exists Z_{t_i}^{tu} \rfloor \rfloor^2 \geq \propto - \lfloor \lfloor \exists Z_{h_i}^{tu} \rfloor \rfloor^2$.
Hence, $\nabla_{i\to 1}^n \left\lfloor \propto - \left\lfloor \lfloor \exists Z_{t_i}^{tu} \rfloor \right\rfloor^2 \right\rfloor^{\sharp_i} \geq \nabla_{i\to 1}^n \left\lfloor \propto - \left\lfloor \lfloor \exists Z_{h_i}^{tu} \rfloor \right\rfloor^2 \right\rfloor^{\sharp_i}$
and $\sqrt[\angle -\propto]{\propto - \nabla_{i\to 1}^n \left\lfloor \propto - \left\lfloor \lfloor \exists Z_{t_i}^{tu} \rfloor \right\rfloor^{\angle -\propto} \right\rfloor^{\sharp_i}} \leq \sqrt[\angle -\propto]{\propto - \nabla_{i\to 1}^n \left\lfloor \propto - \left\lfloor \lfloor \exists Z_{h_i}^{tu} \rfloor \right\rfloor^{\angle -\propto} \right\rfloor^{\sharp_i}}$.
For any i, $\lfloor \exists Z_{t_{ij}}^{il} \rfloor^{\angle -\propto} \leq \left\lfloor \lfloor \exists Z_{h_{ij}}^{il} \rfloor \right\rfloor^{\angle -\propto}$.
Therefore, $\propto - \left\lfloor \lfloor \exists Z_{t_i}^{il} \rfloor \right\rfloor^{\angle -\propto} \geq \propto - \left\lfloor \lfloor \exists Z_{h_i}^{il} \rfloor \right\rfloor^{\angle -\propto}$.
Hence, $\nabla_{i\to 1}^n \left\lfloor \propto - \left\lfloor \lfloor \exists Z_{t_i}^{il} \rfloor \right\rfloor^{\angle -\propto} \right\rfloor^{\sharp_i} \geq \nabla_{i\to 1}^n \left\lfloor \propto - \left\lfloor \lfloor \exists Z_{h_i}^{il} \rfloor \right\rfloor^{\angle -\propto} \right\rfloor^{\sharp_i}$.
This implies that $\sqrt[\angle -\propto]{\propto - \nabla_{i\to 1}^n \left\lfloor \propto - \left\lfloor \lfloor \exists Z_{t_i}^{il} \rfloor \right\rfloor^{\angle -\propto} \right\rfloor^{\sharp_i}} \leq \sqrt[\angle -\propto]{\propto - \nabla_{i\to 1}^n \left\lfloor \propto - \left\lfloor \lfloor \exists Z_{h_i}^{il} \rfloor \right\rfloor^{\angle -\propto} \right\rfloor^{\sharp_i}}$.

Similarly, for any i, $\lfloor \exists Z_{t_{ij}}^{iu} \rfloor^{\angle -\propto} \leq \left\lfloor \lfloor \exists Z_{h_{ij}}^{iu} \rfloor \right\rfloor^{\angle -\propto}$.
Therefore, $\propto - \left\lfloor \lfloor \exists Z_{t_i}^{iu} \rfloor \right\rfloor^{\angle -\propto} \geq \propto - \left\lfloor \lfloor \exists Z_{h_i}^{iu} \rfloor \right\rfloor^{\angle -\propto}$.
Hence, $\nabla_{i\to 1}^n \left\lfloor \propto - \left\lfloor \lfloor \exists Z_{t_i}^{iu} \rfloor \right\rfloor^{\angle -\propto} \right\rfloor^{\sharp_i} \geq \nabla_{i\to 1}^n \left\lfloor \propto - \left\lfloor \lfloor \exists Z_{h_i}^{iu} \rfloor \right\rfloor^{\angle -\propto} \right\rfloor^{\sharp_i}$.
This implies that $\sqrt[\angle -\propto]{\propto - \nabla_{i\to 1}^n \left\lfloor \propto - \left\lfloor \lfloor \exists Z_{t_i}^{iu} \rfloor \right\rfloor^{\angle -\propto} \right\rfloor^{\sharp_i}} \leq \sqrt[\angle -\propto]{\propto - \nabla_{i\to 1}^n \left\lfloor \propto - \left\lfloor \lfloor \exists Z_{h_i}^{iu} \rfloor \right\rfloor^{\angle -\propto} \right\rfloor^{\sharp_i}}$.

For any i, $\left\lfloor \lfloor \exists Z_{t_{ij}}^{fl} \rfloor \right\rfloor^2 \geq \left\lfloor \lfloor \exists Z_{h_{ij}}^{fl} \rfloor \right\rfloor^2$ and $\left\lfloor \lfloor \exists Z_{t_{ij}}^{fl} \rfloor \right\rfloor^{\angle -\propto} \geq \left\lfloor \lfloor \exists Z_{h_{ij}}^{fl} \rfloor \right\rfloor^{\angle -\propto}$.
Therefore, $\propto - \frac{\left\lfloor \nabla_{i\to 1}^n \lfloor \exists Z_{t_{ij}}^{fl} \rfloor \right\rfloor^{\angle -\propto}}{2} \leq \propto - \frac{\left\lfloor \nabla_{i\to 1}^n \lfloor \exists Z_{h_{ij}}^{fl} \rfloor \right\rfloor^{\angle -\propto}}{2}$.
Similarly, for any i,

$\left\lfloor \lfloor \exists Z_{t_{ij}}^{fu} \rfloor \right\rfloor^2 \geq \left\lfloor \lfloor \exists Z_{h_{ij}}^{fu} \rfloor \right\rfloor^2$ and $\left\lfloor \lfloor \exists Z_{t_{ij}}^{fu} \rfloor \right\rfloor^{\angle -\propto} \geq \left\lfloor \lfloor \exists Z_{h_{ij}}^{fu} \rfloor \right\rfloor^{\angle -\propto}$.

Therefore, $-\left\lfloor \nabla_{i\to 1}^{n} \lfloor \exists Z_{t_{ij}}^{fu} \rfloor \right\rfloor^{\angle -\propto} \leq -\left\lfloor \nabla_{i\to 1}^{n} \lfloor \exists Z_{h_{ij}}^{fu} \rfloor \right\rfloor^{\angle -\propto}$.

Hence,

$$\frac{1}{2}\times\left[\begin{array}{c}\left[\begin{array}{c}\left[\left\lfloor \sqrt[\angle -\propto]{\propto -\nabla_{i\to 1}^{n}\left\lfloor \propto -\lfloor\lfloor \exists Z_{ti}^{tl}\rfloor\rfloor^{\angle -\propto}\right\rfloor^{\sharp_i}}\right\rfloor^2-\left\lfloor \sqrt[\angle -\propto]{\propto -\nabla_{i\to 1}^{n}\left\lfloor \propto -\lfloor\lfloor \exists Z_{t_i}^{il}\rfloor\rfloor^{\angle -\propto}\right\rfloor^{\sharp_i}}\right\rfloor^2\right]\\ +\propto -\left\lfloor \nabla_{i\to 1}^{n}\lfloor\lfloor \exists Z_{t_i}^{fl}\rfloor\rfloor^{\angle -\propto}\right\rfloor^2\end{array}\right]\\ +\left[\begin{array}{c}\left[\left\lfloor \sqrt[\angle -\propto]{\propto -\nabla_{i\to 1}^{n}\left\lfloor \propto -\lfloor\lfloor \exists Z_{ti}^{tu}\rfloor\rfloor^{\angle -\propto}\right\rfloor^{\sharp_i}}\right\rfloor^2-\left\lfloor \sqrt[\angle -\propto]{\propto -\nabla_{i\to 1}^{n}\left\lfloor \propto -\lfloor\lfloor \exists Z_{t_i}^{iu}\rfloor\rfloor^{\angle -\propto}\right\rfloor^{\sharp_i}}\right\rfloor^2\right]\\ -\left\lfloor \nabla_{i\to 1}^{n}\lfloor\lfloor \exists Z_{t_i}^{fu}\rfloor\rfloor^{\angle -\propto}\right\rfloor^2\end{array}\right]\end{array}\right]$$

$$\leq \frac{1}{2}\times\left[\begin{array}{c}\left[\begin{array}{c}\left[\left\lfloor \sqrt[\angle -\propto]{\propto -\nabla_{i\to 1}^{n}\left\lfloor \propto -\lfloor\lfloor \exists Z_{hi}^{tl}\rfloor\rfloor^{\angle -\propto}\right\rfloor^{\sharp_i}}\right\rfloor^2-\left\lfloor \sqrt[\angle -\propto]{\propto -\nabla_{i\to 1}^{n}\left\lfloor \propto -\lfloor\lfloor \exists Z_{h_i}^{il}\rfloor\rfloor^{\angle -\propto}\right\rfloor^{\sharp_i}}\right\rfloor^2\right]\\ +\propto -\left\lfloor \nabla_{i\to 1}^{n}\lfloor\lfloor \exists Z_{h_i}^{fl}\rfloor\rfloor^{\angle -\propto}\right\rfloor^2\end{array}\right]\\ +\left[\begin{array}{c}\left[\left\lfloor \sqrt[\angle -\propto]{\propto -\nabla_{i\to 1}^{n}\left\lfloor \propto -\lfloor\lfloor \exists Z_{hi}^{tu}\rfloor\rfloor^{\angle -\propto}\right\rfloor^{\sharp_i}}\right\rfloor^2-\left\lfloor \sqrt[\angle -\propto]{\propto -\nabla_{i\to 1}^{n}\left\lfloor \propto -\lfloor\lfloor \exists Z_{h_i}^{iu}\rfloor\rfloor^{\angle -\propto}\right\rfloor^{\sharp_i}}\right\rfloor^2\right]\\ -\left\lfloor \nabla_{i\to 1}^{n}\lfloor\lfloor \exists Z_{h_i}^{fu}\rfloor\rfloor^{\angle -\propto}\right\rfloor^2\end{array}\right]\end{array}\right].$$

Hence, IVTNRFWA$\lfloor \circledS_1, \circledS_2, ..., \circledS_n\rfloor \leq IVTNRFWA\lfloor W_1, W_2, ..., W_n\rfloor$.

2.2. *IVTNRFWG*

Definition 2.8. Let $\circledS_i = \left\langle \left\lfloor \lfloor \exists Z_i^{tl}, \exists Z_i^{tu}\rfloor, \lfloor \exists Z_i^{il}, \exists Z_i^{iu}\rfloor, \lfloor \exists Z_i^{fl}, \exists Z_i^{fu}\rfloor \right\rfloor \right\rangle$ be the IVTNRFNs. Then NRFWG $\lfloor \circledS_1, \circledS_2, ..., \circledS_n\rfloor = \nabla_{i\to 1}^{n} \circledS_i^{\sharp_i}$.

Corollary 2.9. *Let* $\circledS_i = \left\langle \left\lfloor \lfloor \exists Z_i^{tl}, \exists Z_i^{tu}\rfloor, \lfloor \exists Z_i^{il}, \exists Z_i^{iu}\rfloor, \lfloor \exists Z_i^{fl}, \exists Z_i^{fu}\rfloor \right\rfloor \right\rangle$ *be the IVTNRFNs. Then IVTNRFWG* $\lfloor \circledS_1, \circledS_2, ..., \circledS_n\rfloor$

$$= \left[\begin{array}{c}\nabla_{i\to 1}^{n}\lfloor\lfloor\lfloor \exists Z_i^{tl}\rfloor\rfloor^{\angle -\propto}\rfloor^{\sharp_i}, \nabla_{i\to 1}^{n}\lfloor\lfloor\lfloor \exists Z_i^{tu}\rfloor\rfloor^{\angle -\propto}\rfloor^{\sharp_i}\\ \sqrt[\angle -\propto]{\propto -\nabla_{i\to 1}^{n}\left\lfloor \propto -\lfloor\lfloor \exists Z_i^{il}\rfloor\rfloor^{\angle -\propto}\right\rfloor^{\sharp_i}}, \sqrt[\angle -\propto]{\propto -\nabla_{i\to 1}^{n}\left\lfloor \propto -\lfloor\lfloor \exists Z_i^{iu}\rfloor\rfloor^{\angle -\propto}\right\rfloor^{\sharp_i}}\\ \sqrt[\angle -\propto]{\propto -\nabla_{i\to 1}^{n}\left\lfloor \propto -\lfloor\lfloor \exists Z_i^{fl}\rfloor\rfloor^{\angle -\propto}\right\rfloor^{\sharp_i}}, \sqrt[\angle -\propto]{\propto -\nabla_{i\to 1}^{n}\left\lfloor \propto -\lfloor\lfloor \exists Z_i^{fu}\rfloor\rfloor^{\angle -\propto}\right\rfloor^{\sharp_i}}\end{array}\right].$$

Corollary 2.10. *Let* $\circledS_i = \left\langle \left\lfloor \lfloor \exists Z_i^{tl}, \exists Z_i^{tu}\rfloor, \lfloor \exists Z_i^{il}, \exists Z_i^{iu}\rfloor, \lfloor \exists Z_i^{fl}, \exists Z_i^{fu}\rfloor \right\rfloor \right\rangle$ *be the IVTNRFNs and all are equal. Then* $NRFWG\lfloor \circledS_1, \circledS_2, ..., \circledS_n\rfloor = \circledS$.

2.3. *Generalized IVTNRFWA (GIVTNRFWA)*

Definition 2.11. Let $\circledS_i = \left\langle \left\lfloor \lfloor \exists Z_i^{tl}, \exists Z_i^{tu}\rfloor, \lfloor \exists Z_i^{il}, \exists Z_i^{iu}\rfloor, \lfloor \exists Z_i^{fl}, \exists Z_i^{fu}\rfloor \right\rfloor \right\rangle$ be the IVTNRFN. Then GIVTNRFWA $\lfloor \circledS_1, \circledS_2, ..., \circledS_n\rfloor = \left\lfloor \triangle_{i\to 1}^{n} \sharp_i \circledS_i^{\varrho} \right\rfloor^{1/\varrho}$.

Theorem 2.12. *Let* $\circledS_i = \left\langle \left\lfloor \lfloor \exists Z_i^{tl}, \exists Z_i^{tu} \rfloor, \lfloor \exists Z_i^{il}, \exists Z_i^{iu} \rfloor, \lfloor \exists Z_i^{fl}, \exists Z_i^{fu} \rfloor \rfloor \right\rangle$ *be the IVTNRFNs. Then GIVTNRFWA* $\lfloor \circledS_1, \circledS_2, ..., \circledS_n \rfloor$

$$= \left[\begin{array}{l} \left\lfloor \sqrt[\angle -\propto]{\propto -\nabla_{i\to 1}^{n} \left\lfloor \propto -\left\lfloor \lfloor \lfloor \exists Z_i^{tl} \rfloor \rfloor^{\angle -\propto} \right\rfloor^{\angle -\propto} \right\rfloor^{\sharp_i}} \right\rfloor^{1/\angle -\propto}, \left\lfloor \sqrt[\angle -\propto]{\propto -\nabla_{i\to 1}^{n} \left\lfloor \propto -\left\lfloor \lfloor \lfloor \exists Z_i^{tu} \rfloor \rfloor^{\angle -\propto} \right\rfloor^{\angle -\propto} \right\rfloor^{\sharp_i}} \right\rfloor^{1/\angle -\propto}, \\ \left\lfloor \sqrt[\angle -\propto]{\propto -\nabla_{i\to 1}^{n} \left\lfloor \propto -\left\lfloor \lfloor \lfloor \exists Z_i^{il} \rfloor \rfloor^{\angle -\propto} \right\rfloor^{\angle -\propto} \right\rfloor^{\sharp_i}} \right\rfloor^{1/\angle -\propto}, \left\lfloor \sqrt[\angle -\propto]{\propto -\nabla_{i\to 1}^{n} \left\lfloor \propto -\left\lfloor \lfloor \lfloor \exists Z_i^{iu} \rfloor \rfloor^{\angle -\propto} \right\rfloor^{\angle -\propto} \right\rfloor^{\sharp_i}} \right\rfloor^{1/\angle -\propto}, \\ \sqrt[\angle -\propto]{\propto -\left\lfloor \propto -\left\lfloor \nabla_{i\to 1}^{n} \left\lfloor \sqrt[\angle -\propto]{\propto -\left\lfloor \propto -\lfloor \lfloor \exists Z_i^{fl} \rfloor \rfloor^{\angle -\propto} \right\rfloor^{\angle -\propto}} \right\rfloor^{\sharp_i} \right\rfloor^{\angle -\propto} \right\rfloor^{1/\angle -\propto}}, \\ \sqrt[\angle -\propto]{\propto -\left\lfloor \propto -\left\lfloor \nabla_{i\to 1}^{n} \left\lfloor \sqrt[\angle -\propto]{\propto -\left\lfloor \propto -\lfloor \lfloor \exists Z_i^{fu} \rfloor \rfloor^{\angle -\propto} \right\rfloor^{\angle -\propto}} \right\rfloor^{\sharp_i} \right\rfloor^{\angle -\propto} \right\rfloor^{1/\angle -\propto}} \end{array} \right].$$

Proof To illustrate this, we may first show that,

$$\triangle_{i\to 1}^{n} \sharp_i \circledS_i^{\angle -\propto} = \left[\begin{array}{ll} \sqrt[\angle -\propto]{\propto -\nabla_{i\to 1}^{n} \left\lfloor \propto -\left\lfloor \lfloor \lfloor \exists Z_i^{tl} \rfloor \rfloor^{\angle -\propto} \right\rfloor^{\angle -\propto} \right\rfloor^{\sharp_i}}, & \sqrt[\angle -\propto]{\propto -\nabla_{i\to 1}^{n} \left\lfloor \propto -\left\lfloor \lfloor \lfloor \exists Z_i^{tu} \rfloor \rfloor^{\angle -\propto} \right\rfloor^{\angle -\propto} \right\rfloor^{\sharp_i}} \\ \sqrt[\angle -\propto]{\propto -\nabla_{i\to 1}^{n} \left\lfloor \propto -\left\lfloor \lfloor \lfloor \exists Z_i^{il} \rfloor \rfloor^{\angle -\propto} \right\rfloor^{\angle -\propto} \right\rfloor^{\sharp_i}}, & \sqrt[\angle -\propto]{\propto -\nabla_{i\to 1}^{n} \left\lfloor \propto -\left\lfloor \lfloor \lfloor \exists Z_i^{iu} \rfloor \rfloor^{\angle -\propto} \right\rfloor^{\angle -\propto} \right\rfloor^{\sharp_i}} \\ \nabla_{i\to 1}^{n} \left\lfloor \sqrt[\angle -\propto]{\propto -\left\lfloor \propto -\lfloor \lfloor \exists Z_i^{fl} \rfloor \rfloor^{\angle -\propto} \right\rfloor^{\angle -\propto}} \right\rfloor^{\sharp_i}, & \nabla_{i\to 1}^{n} \left\lfloor \sqrt[\angle -\propto]{\propto -\left\lfloor \propto -\lfloor \lfloor \exists Z_i^{fu} \rfloor \rfloor^{\angle -\propto} \right\rfloor^{\angle -\propto}} \right\rfloor^{\sharp_i} \end{array} \right].$$

Put $n = 2$, $\sharp_1 Ⓢ_1 \vartriangle \sharp_2 Ⓢ_2$

$$= \left[\begin{array}{c} \sqrt[\angle-\infty]{\left[\sqrt[\angle-\infty]{\propto - \left[\propto - \left\lfloor \lfloor \exists Z_1^{tl} \rfloor \rfloor^{\angle-\infty}\right]^{\angle-\infty}}\right]^{\sharp_1}\right]^{\angle-\infty} + \left[\sqrt[\angle-\infty]{\propto - \left[\propto - \left\lfloor \lfloor \exists Z_2^{tl} \rfloor \rfloor^{\angle-\infty}\right]^{\angle-\infty}}\right]^{\sharp_1}\right]^{\angle-\infty}} \\ - \left[\sqrt[\angle-\infty]{\propto - \left[\propto - \left\lfloor \lfloor \exists Z_1^{tl} \rfloor \rfloor^{\angle-\infty}\right]^{\angle-\infty}}\right]^{\sharp_1}\right]^{\angle-\infty} \cdot \left[\sqrt[\angle-\infty]{\propto - \left[\propto - \left\lfloor \lfloor \exists Z_2^{tl} \rfloor \rfloor^{\angle-\infty}\right]^{\angle-\infty}}\right]^{\sharp_1}\right]^{\angle-\infty}} \\ , \sqrt[\angle-\infty]{\left[\sqrt[\angle-\infty]{\propto - \left[\propto - \left\lfloor \lfloor \exists Z_1^{tu} \rfloor \rfloor^{\angle-\infty}\right]^{\angle-\infty}}\right]^{\sharp_1}\right]^{\angle-\infty} + \left[\sqrt[\angle-\infty]{\propto - \left[\propto - \left\lfloor \lfloor \exists Z_2^{tu} \rfloor \rfloor^{\angle-\infty}\right]^{\angle-\infty}}\right]^{\sharp_1}\right]^{\angle-\infty}} \\ - \left[\sqrt[\angle-\infty]{\propto - \left[\propto - \left\lfloor \lfloor \exists Z_1^{tu} \rfloor \rfloor^{\angle-\infty}\right]^{\angle-\infty}}\right]^{\sharp_1}\right]^{\angle-\infty} \cdot \left[\sqrt[\angle-\infty]{\propto - \left[\propto - \left\lfloor \lfloor \exists Z_2^{tu} \rfloor \rfloor^{\angle-\infty}\right]^{\angle-\infty}}\right]^{\sharp_1}\right]^{\angle-\infty}}, \\ \sqrt[\angle-\infty]{\left[\sqrt[\angle-\infty]{\propto - \left[\propto - \left\lfloor \lfloor \exists Z_1^{il} \rfloor \rfloor^{\angle-\infty}\right]^{\angle-\infty}}\right]^{\sharp_1}\right]^{\angle-\infty} + \left[\sqrt[\angle-\infty]{\propto - \left[\propto - \left\lfloor \lfloor \exists Z_2^{il} \rfloor \rfloor^{\angle-\infty}\right]^{\angle-\infty}}\right]^{\sharp_1}\right]^{\angle-\infty}} \\ - \left[\sqrt[\angle-\infty]{\propto - \left[\propto - \left\lfloor \lfloor \exists Z_1^{il} \rfloor \rfloor^{\angle-\infty}\right]^{\angle-\infty}}\right]^{\sharp_1}\right]^{\angle-\infty} \cdot \left[\sqrt[\angle-\infty]{\propto - \left[\propto - \left\lfloor \lfloor \exists Z_2^{il} \rfloor \rfloor^{\angle-\infty}\right]^{\angle-\infty}}\right]^{\sharp_1}\right]^{\angle-\infty}} \\ , \sqrt[\angle-\infty]{\left[\sqrt[\angle-\infty]{\propto - \left[\propto - \left\lfloor \lfloor \exists Z_1^{iu} \rfloor \rfloor^{\angle-\infty}\right]^{\angle-\infty}}\right]^{\sharp_1}\right]^{\angle-\infty} + \left[\sqrt[\angle-\infty]{\propto - \left[\propto - \left\lfloor \lfloor \exists Z_2^{iu} \rfloor \rfloor^{\angle-\infty}\right]^{\angle-\infty}}\right]^{\sharp_1}\right]^{\angle-\infty}} \\ - \left[\sqrt[\angle-\infty]{\propto - \left[\propto - \left\lfloor \lfloor \exists Z_1^{iu} \rfloor \rfloor^{\angle-\infty}\right]^{\angle-\infty}}\right]^{\sharp_1}\right]^{\angle-\infty} \cdot \left[\sqrt[\angle-\infty]{\propto - \left[\propto - \left\lfloor \lfloor \exists Z_2^{iu} \rfloor \rfloor^{\angle-\infty}\right]^{\angle-\infty}}\right]^{\sharp_1}\right]^{\angle-\infty}} \\ \left[\sqrt[\angle-\infty]{\propto - \left[\propto - \left\lfloor \lfloor \exists Z_1^{fl} \rfloor \rfloor^{\angle-\infty}\right]^{\angle-\infty}}\right]^{\sharp_1}, \left[\sqrt[\angle-\infty]{\propto - \left[\propto - \left\lfloor \lfloor \exists Z_2^{fl} \rfloor \rfloor^{\angle-\infty}\right]^{\angle-\infty}}\right]^{\sharp_1} \\ , \left[\sqrt[\angle-\infty]{\propto - \left[\propto - \left\lfloor \lfloor \exists Z_1^{fu} \rfloor \rfloor^{\angle-\infty}\right]^{\angle-\infty}}\right]^{\sharp_1}, \left[\sqrt[\angle-\infty]{\propto - \left[\propto - \left\lfloor \lfloor \exists Z_2^{fu} \rfloor \rfloor^{\angle-\infty}\right]^{\angle-\infty}}\right]^{\sharp_1} \end{array} \right]$$

$$= \left[\begin{array}{c} \sqrt[\angle-\infty]{\propto - \nabla_{i\to 1}^2 \left[\propto - \left\lfloor \lfloor \exists Z_1^{tl} \rfloor \rfloor^{\angle-\infty}\right]^{\angle-\infty}}\right]^{\sharp_i}, \sqrt[\angle-\infty]{\propto - \nabla_{i\to 1}^2 \left[\propto - \left\lfloor \lfloor \exists Z_1^{tu} \rfloor \rfloor^{\angle-\infty}\right]^{\angle-\infty}}\right]^{\sharp_i} \\ \sqrt[\angle-\infty]{\propto - \nabla_{i\to 1}^2 \left[\propto - \left\lfloor \lfloor \exists Z_1^{il} \rfloor \rfloor^{\angle-\infty}\right]^{\angle-\infty}}\right]^{\sharp_i}, \sqrt[\angle-\infty]{\propto - \nabla_{i\to 1}^2 \left[\propto - \left\lfloor \lfloor \exists Z_1^{iu} \rfloor \rfloor^{\angle-\infty}\right]^{\angle-\infty}}\right]^{\sharp_i} \\ \nabla_{i\to 1}^2 \left[\sqrt[\angle-\infty]{\propto - \left[\propto - \left\lfloor \lfloor \exists Z_i^{fl} \rfloor \rfloor^{\angle-\infty}\right]^{\angle-\infty}}\right]^{\sharp_i}, \nabla_{i\to 1}^2 \left[\sqrt[\angle-\infty]{\propto - \left[\propto - \left\lfloor \lfloor \exists Z_i^{fu} \rfloor \rfloor^{\angle-\infty}\right]^{\angle-\infty}}\right]^{\sharp_i} \end{array} \right].$$

Hence,

$$\triangle_{i\to 1}^{l}\,\sharp_i\circledS_i^{\varrho} = \begin{bmatrix} \sqrt[\angle-\propto]{\propto-\nabla_{i\to 1}^{l}\left[\propto-\left\lfloor\lfloor\dashv Z_1^{tl}\rfloor\rfloor^{\angle-\propto}\right\rfloor^{\angle-\propto}\right]^{\sharp_i}}, \sqrt[\angle-\propto]{\propto-\nabla_{i\to 1}^{l}\left[\propto-\left\lfloor\lfloor\dashv Z_1^{tu}\rfloor\rfloor^{\angle-\propto}\right\rfloor^{\angle-\propto}\right]^{\sharp_i}} \\ \sqrt[\angle-\propto]{\propto-\nabla_{i\to 1}^{l}\left[\propto-\left\lfloor\lfloor\dashv Z_1^{il}\rfloor\rfloor^{\angle-\propto}\right\rfloor^{\angle-\propto}\right]^{\sharp_i}}, \sqrt[\angle-\propto]{\propto-\nabla_{i\to 1}^{l}\left[\propto-\left\lfloor\lfloor\dashv Z_1^{iu}\rfloor\rfloor^{\angle-\propto}\right\rfloor^{\angle-\propto}\right]^{\sharp_i}} \\ \nabla_{i\to 1}^{l}\left[\sqrt[\angle-\propto]{\propto-\left\lfloor\propto-\lfloor\lfloor\dashv Z_i^{fl}\rfloor\rfloor^{\angle-\propto}\right\rfloor^{\angle-\propto}}\right]^{\sharp_i}, \nabla_{i\to 1}^{l}\left[\sqrt[\angle-\propto]{\propto-\left\lfloor\propto-\lfloor\lfloor\dashv Z_i^{fu}\rfloor\rfloor^{\angle-\propto}\right\rfloor^{\angle-\propto}}\right]^{\sharp_i} \end{bmatrix}.$$

If $n = l+1$, then $\triangle_{i\to 1}^{l}\,\sharp_i\circledS_i^{\varrho} + \sharp_{l+1}\circledS_{l+1}^{\varrho} = \triangle_{i\to 1}^{l+1}\,\sharp_i\circledS_i^{\varrho}$.

Now, $\triangle_{i\to 1}^{l}\,\sharp_i\circledS_i^{\varrho} + \sharp_{l+1}\circledS_{l+1}^{\varrho} = \sharp_1\circledS_1^{\varrho} \triangle \sharp_2\circledS_2^{\varrho} \triangle \ldots \triangle \sharp_l\circledS_l^{\varrho} \triangle \sharp_{l+1}\circledS_{l+1}^{\varrho}$

$$= \begin{bmatrix} \sqrt[\angle-\propto]{\left[\sqrt[\angle-\propto]{\propto-\nabla_{i\to 1}^{l}\left[\propto-\left\lfloor\lfloor\dashv Z_i^{tl}\rfloor\rfloor^{\angle-\propto}\right\rfloor^{\angle-\propto}\right]^{\sharp_i}}\right]^{\angle-\propto}+\left[\sqrt[\angle-\propto]{\propto-\left\lfloor\propto-\left\lfloor\lfloor Z_{l+1}^{tl}\rfloor^{\angle-\propto}\right\rfloor^{\angle-\propto}\right\rfloor^{\sharp_1}}\right]^{\angle-\propto}-\left[\sqrt[\angle-\propto]{\propto-\nabla_{i\to 1}^{l}\left[\propto-\left\lfloor\lfloor\dashv Z_i^{tl}\rfloor\rfloor^{\angle-\propto}\right\rfloor^{\angle-\propto}\right]^{\sharp_i}}\right]^{\angle-\propto}\cdot\left[\sqrt[\angle-\propto]{\propto-\left\lfloor\propto-\left\lfloor\lfloor Z_{l+1}^{tl}\rfloor^{\angle-\propto}\right\rfloor^{\angle-\propto}\right\rfloor^{\sharp_1}}\right]^{\angle-\propto}}, \\ \sqrt[\angle-\propto]{\left[\sqrt[\angle-\propto]{\propto-\nabla_{i\to 1}^{l}\left[\propto-\left\lfloor\lfloor\dashv Z_i^{tu}\rfloor\rfloor^{\angle-\propto}\right\rfloor^{\angle-\propto}\right]^{\sharp_i}}\right]^{\angle-\propto}+\left[\sqrt[\angle-\propto]{\propto-\left\lfloor\propto-\left\lfloor\lfloor Z_{l+1}^{tu}\rfloor^{\angle-\propto}\right\rfloor^{\angle-\propto}\right\rfloor^{\sharp_1}}\right]^{\angle-\propto}-\left[\sqrt[\angle-\propto]{\propto-\nabla_{i\to 1}^{l}\left[\propto-\left\lfloor\lfloor\dashv Z_i^{tu}\rfloor\rfloor^{\angle-\propto}\right\rfloor^{\angle-\propto}\right]^{\sharp_i}}\right]^{\angle-\propto}\cdot\left[\sqrt[\angle-\propto]{\propto-\left\lfloor\propto-\left\lfloor\lfloor Z_{l+1}^{tu}\rfloor^{\angle-\propto}\right\rfloor^{\angle-\propto}\right\rfloor^{\sharp_1}}\right]^{\angle-\propto}}, \\ \sqrt[\angle-\propto]{\left[\sqrt[\angle-\propto]{\propto-\nabla_{i\to 1}^{l}\left[\propto-\left\lfloor\lfloor\dashv Z_i^{il}\rfloor\rfloor^{\angle-\propto}\right\rfloor^{\angle-\propto}\right]^{\sharp_i}}\right]^{\angle-\propto}+\left[\sqrt[\angle-\propto]{\propto-\left\lfloor\propto-\left\lfloor\lfloor Z_{l+1}^{il}\rfloor^{\angle-\propto}\right\rfloor^{\angle-\propto}\right\rfloor^{\sharp_1}}\right]^{\angle-\propto}-\left[\sqrt[\angle-\propto]{\propto-\nabla_{i\to 1}^{l}\left[\propto-\left\lfloor\lfloor\dashv Z_i^{il}\rfloor\rfloor^{\angle-\propto}\right\rfloor^{\angle-\propto}\right]^{\sharp_i}}\right]^{\angle-\propto}\cdot\left[\sqrt[\angle-\propto]{\propto-\left\lfloor\propto-\left\lfloor\lfloor Z_{l+1}^{il}\rfloor^{\angle-\propto}\right\rfloor^{\angle-\propto}\right\rfloor^{\sharp_1}}\right]^{\angle-\propto}}, \\ \sqrt[\angle-\propto]{\left[\sqrt[\angle-\propto]{\propto-\nabla_{i\to 1}^{l}\left[\propto-\left\lfloor\lfloor\dashv Z_i^{iu}\rfloor\rfloor^{\angle-\propto}\right\rfloor^{\angle-\propto}\right]^{\sharp_i}}\right]^{\angle-\propto}+\left[\sqrt[\angle-\propto]{\propto-\left\lfloor\propto-\left\lfloor\lfloor Z_{l+1}^{iu}\rfloor^{\angle-\propto}\right\rfloor^{\angle-\propto}\right\rfloor^{\sharp_1}}\right]^{\angle-\propto}-\left[\sqrt[\angle-\propto]{\propto-\nabla_{i\to 1}^{l}\left[\propto-\left\lfloor\lfloor\dashv Z_i^{iu}\rfloor\rfloor^{\angle-\propto}\right\rfloor^{\angle-\propto}\right]^{\sharp_i}}\right]^{\angle-\propto}\cdot\left[\sqrt[\angle-\propto]{\propto-\left\lfloor\propto-\left\lfloor\lfloor Z_{l+1}^{iu}\rfloor^{\angle-\propto}\right\rfloor^{\angle-\propto}\right\rfloor^{\sharp_1}}\right]^{\angle-\propto}}, \\ \nabla_{i\to 1}^{l}\left[\sqrt[\angle-\propto]{\propto-\left\lfloor\propto-\lfloor\lfloor\dashv Z_i^{fl}\rfloor\rfloor^{\angle-\propto}\right\rfloor^{\angle-\propto}}\right]^{\sharp_i}\cdot\left[\sqrt[\angle-\propto]{\propto-\left\lfloor\propto-\lfloor Z_{l+1}^{fl}\rfloor^{\angle-\propto}\right\rfloor^{\angle-\propto}}\right]^{\sharp_1} \\ , \nabla_{i\to 1}^{l}\left[\sqrt[\angle-\propto]{\propto-\left\lfloor\propto-\lfloor\lfloor\dashv Z_i^{fu}\rfloor\rfloor^{\angle-\propto}\right\rfloor^{\angle-\propto}}\right]^{\sharp_i}\cdot\left[\sqrt[\angle-\propto]{\propto-\left\lfloor\propto-\lfloor Z_{l+1}^{fu}\rfloor^{\angle-\propto}\right\rfloor^{\angle-\propto}}\right]^{\sharp_1} \end{bmatrix}$$

$$\triangle_{i\to 1}^{l+1}\,\sharp_i\circledS_i^{\angle-\propto} = \begin{bmatrix} \sqrt[\angle-\propto]{\propto-\nabla_{i\to 1}^{l+1}\left[\propto-\left\lfloor\lfloor\dashv Z_1^{tl}\rfloor\rfloor^{\angle-\propto}\right\rfloor^{\angle-\propto}\right]^{\sharp_i}}, \sqrt[\angle-\propto]{\propto-\nabla_{i\to 1}^{l+1}\left[\propto-\left\lfloor\lfloor\dashv Z_1^{tu}\rfloor\rfloor^{\angle-\propto}\right\rfloor^{\angle-\propto}\right]^{\sharp_i}} \\ \sqrt[\angle-\propto]{\propto-\nabla_{i\to 1}^{l+1}\left[\propto-\left\lfloor\lfloor\dashv Z_1^{il}\rfloor\rfloor^{\angle-\propto}\right\rfloor^{\angle-\propto}\right]^{\sharp_i}}, \sqrt[\angle-\propto]{\propto-\nabla_{i\to 1}^{l+1}\left[\propto-\left\lfloor\lfloor\dashv Z_1^{iu}\rfloor\rfloor^{\angle-\propto}\right\rfloor^{\angle-\propto}\right]^{\sharp_i}} \\ \nabla_{i\to 1}^{l+1}\left[\sqrt[\angle-\propto]{\propto-\left\lfloor\propto-\lfloor\lfloor\dashv Z_i^{fl}\rfloor\rfloor^{\angle-\propto}\right\rfloor^{\angle-\propto}}\right]^{\sharp_i}, \nabla_{i\to 1}^{l+1}\left[\sqrt[\angle-\propto]{\propto-\left\lfloor\propto-\lfloor\lfloor\dashv Z_i^{fu}\rfloor\rfloor^{\angle-\propto}\right\rfloor^{\angle-\propto}}\right]^{\sharp_i} \end{bmatrix}.$$

$$\left\lfloor \triangle_{i\to 1}^{l+1} \sharp_i \circledS_i^{\varrho} \right\rfloor^{1/\varrho} =$$

$$\left[\begin{array}{c} \left[\sqrt[\angle^{-\propto}]{\propto -\nabla_{i\to 1}^{l+1}\left[\propto -\left\lfloor \lfloor\lfloor \exists Z_i^{tl}\rfloor\rfloor^{\angle^{-\propto}}\right\rfloor^{\angle^{-\propto}}\right]^{\sharp_i}}\right]^{1/\angle^{-\propto}}, \left[\sqrt[\angle^{-\propto}]{\propto -\nabla_{i\to 1}^{l+1}\left[\propto -\left\lfloor \lfloor\lfloor \exists Z_i^{tu}\rfloor\rfloor^{\angle^{-\propto}}\right\rfloor^{\angle^{-\propto}}\right]^{\sharp_i}}\right]^{1/\angle^{-\propto}} \\ \left[\sqrt[\angle^{-\propto}]{\propto -\nabla_{i\to 1}^{l+1}\left[\propto -\left\lfloor \lfloor\lfloor \exists Z_i^{il}\rfloor\rfloor^{\angle^{-\propto}}\right\rfloor^{\angle^{-\propto}}\right]^{\sharp_i}}\right]^{1/\angle^{-\propto}}, \left[\sqrt[\angle^{-\propto}]{\propto -\nabla_{i\to 1}^{l+1}\left[\propto -\left\lfloor \lfloor\lfloor \exists Z_i^{iu}\rfloor\rfloor^{\angle^{-\propto}}\right\rfloor^{\angle^{-\propto}}\right]^{\sharp_i}}\right]^{1/\angle^{-\propto}} \\ \sqrt[\angle^{-\propto}]{\propto -\left[\propto -\left[\nabla_{i\to 1}^{l+1}\left[\sqrt[\angle^{-\propto}]{\propto -\left[\propto -\lfloor\lfloor \exists Z_i^{fl}\rfloor\rfloor^{\angle^{-\propto}}\right]^{\angle^{-\propto}}}\right]^{\sharp_i}\right]^2\right]^{1/\angle^{-\propto}}}, \\ \sqrt[\angle^{-\propto}]{\propto -\left[\propto -\left[\nabla_{i\to 1}^{l+1}\left[\sqrt[\angle^{-\propto}]{\propto -\left[\propto -\lfloor\lfloor \exists Z_i^{fu}\rfloor\rfloor^{\angle^{-\propto}}\right]^{\angle^{-\propto}}}\right]^{\sharp_i}\right]^2\right]^{1/\angle^{-\propto}}} \end{array}\right]$$

Theorem 2.13. *If all* $\circledS_i = \left\langle \left\lfloor \lfloor\lfloor \exists Z_i^{tl}, \exists Z_i^{tu}\rfloor, \lfloor \exists Z_i^{il}, \exists Z_i^{iu}\rfloor, \lfloor \exists Z_i^{fl}, \exists Z_i^{fu}\rfloor\rfloor \right\rfloor \right\rangle$ *and all are equal. Then* $GIVTNRFWA\lfloor \circledS_1, \circledS_2, ..., \circledS_n\rfloor = \circledS$.

2.4. *Generalized IVT NRFWG (GIVT NRFWG)*

Definition 2.14. Let $\circledS_i = \left\langle \left\lfloor \lfloor\lfloor \exists Z_i^{tl}, \exists Z_i^{tu}\rfloor, \lfloor \exists Z_i^{il}, \exists Z_i^{iu}\rfloor, \lfloor \exists Z_i^{fl}, \exists Z_i^{fu}\rfloor\rfloor \right\rfloor \right\rangle$ be the IVT-NRFNs. Then GIVTNRFWG $\lfloor \circledS_1, \circledS_2, ..., \circledS_n\rfloor = \frac{1}{\varrho}\left\lfloor \nabla_{i\to 1}^{n} \lfloor \varrho \circledS_i\rfloor^{\sharp_i}\right\rfloor$.

Corollary 2.15. *Let* $\circledS_i = \left\langle \left\lfloor \lfloor\lfloor \exists Z_i^{tl}, \exists Z_i^{tu}\rfloor, \lfloor \exists Z_i^{il}, \exists Z_i^{iu}\rfloor, \lfloor \exists Z_i^{fl}, \exists Z_i^{fu}\rfloor\rfloor \right\rfloor \right\rangle$ *be the IVTNRFNs. Then* $GIVTNRFWG\lfloor \circledS_1, \circledS_2, ..., \circledS_n\rfloor$

$$= \left[\begin{array}{c} \sqrt[\angle^{-\propto}]{\propto -\left[\propto -\left[\nabla_{i\to 1}^{n}\left[\sqrt[\angle^{-\propto}]{\propto -\left[\propto -\lfloor\lfloor \exists Z_i^{tl}\rfloor\rfloor^{\angle^{-\propto}}\right]^{\angle^{-\propto}}}\right]^{\sharp_i}\right]^{\angle^{-\propto}}\right]^{1/\angle^{-\propto}}}, \\ \sqrt[\angle^{-\propto}]{\propto -\left[\propto -\left[\nabla_{i\to 1}^{n}\left[\sqrt[\angle^{-\propto}]{\propto -\left[\propto -\lfloor\lfloor \exists Z_i^{tu}\rfloor\rfloor^{\angle^{-\propto}}\right]^{\angle^{-\propto}}}\right]^{\sharp_i}\right]^{\angle^{-\propto}}\right]^{1/\angle^{-\propto}}} \\ \left[\sqrt[\angle^{-\propto}]{\propto -\nabla_{i\to 1}^{n}\left[\propto -\left\lfloor \lfloor\lfloor \exists Z_i^{il}\rfloor\rfloor^{\angle^{-\propto}}\right\rfloor^{\angle^{-\propto}}\right]^{\sharp_i}}\right]^{1/\angle^{-\propto}}, \left[\sqrt[\angle^{-\propto}]{\propto -\nabla_{i\to 1}^{n}\left[\propto -\left\lfloor \lfloor\lfloor \exists Z_i^{iu}\rfloor\rfloor^{\angle^{-\propto}}\right\rfloor^{\angle^{-\propto}}\right]^{\sharp_i}}\right]^{1/\angle^{-\propto}} \\ \left[\sqrt[\angle^{-\propto}]{\propto -\nabla_{i\to 1}^{n}\left[\propto -\left\lfloor \lfloor\lfloor \exists Z_i^{fl}\rfloor\rfloor^{\angle^{-\propto}}\right\rfloor^{\angle^{-\propto}}\right]^{\sharp_i}}\right]^{1/\angle^{-\propto}}, \left[\sqrt[\angle^{-\propto}]{\propto -\nabla_{i\to 1}^{n}\left[\propto -\left\lfloor \lfloor\lfloor \exists Z_i^{fu}\rfloor\rfloor^{\angle^{-\propto}}\right\rfloor^{\angle^{-\propto}}\right]^{\sharp_i}}\right]^{1/\angle^{-\propto}} \end{array}\right].$$

Corollary 2.16. *If all* $\circledS_i = \left\langle \left\lfloor \lfloor\lfloor \exists Z_i^{tl}, \exists Z_i^{tu}\rfloor, \lfloor \exists Z_i^{il}, \exists Z_i^{iu}\rfloor, \lfloor \exists Z_i^{fl}, \exists Z_i^{fu}\rfloor\rfloor \right\rfloor \right\rangle$ *are equal. Then* $GIVTNRFWG\lfloor \circledS_1, \circledS_2, ..., \circledS_n\rfloor = \circledS$.

3. **Conclusion:**

This work presents novel weighted operators, such as geometric and averaging operators. Boundedness, idempotency, commutativity, associativity, and monotonicity are some of the

characteristics of these operators. In order to describe the weighted vector, we looked at a number of conventional measures. A wide range of aggregation operator conditions have been investigated. Some findings have been made after a few aggregating techniques for these IVT-NRFNs have been examined.

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